Skip-Term Summation of Sequences

By Irwin Roman

The literature on the summation of sequences and the finding of the sums of the corresponding series is voluminous and well known. For rapidly convergent sequences term-by-term evaluations furnish no great obstacle. Likewise, if the sequence can be matched with a sequence whose sum is known and the term-by-term difference of the two sequences converges rapidly, this difference-sequence can be evaluated by terms.

For series that converge slowly, various devices have been suggested. If the integral and derivatives of the general term, expressed as a function of the term index, can be evaluated, the Euler [3]-Maclaurin [7] formula or the extended-range form of these formulas [9] can be used to find the sum of the series, or of the residue after a selected term.

If a series converges sufficiently smoothly, the formulas of Lubbock [6], as modified by de Morgan [2], can be used to evaluate a partial sum of the terms of the series from those terms which are separated by a selected gap. Lubbock's formulas were revived by Sprague [12], have been discussed in Whittaker and Robinson [14], and were extended by Steffensen [13]. Lubbock's formulas all involve differences, and tables have been published by Davis [1] and others, to facilitate the evaluations.

The limit of the sum of a slowly convergent infinite series has been approximated by Salzer [10], who selected the reciprocal of the sequence index as the independent variable in Lagrangian interpolation and found the approximate limit as this variable approaches zero. He selects the last m members of the sequence of partial sums from the n members which have been calculated. Simple factors were published for $n = 5$, 10, 15, and 20, and for $m = 4$, 7, and 11, where $m < n$. Later, Salzer [11] published formulas and coefficients for the sum of a finite series, and Horgan [4] prepared decimal tables of coefficients for finite sums. These formulas involve no differences and are convenient for many purposes. However, they involve extrapolation. The present formulas involve only interpolation. Only a few coefficients are included, as examples of the method.

If a sequence of terms $\{\eta_k\}$ can be approximated by a polynomial of degree n in the index k, the values of $(n + 1)$ arbitrarily selected terms of the sequence can be used to determine the value of every other term. If $\{y_u\}$ is a subset of $\{\eta_k\}$, containing $(n + 1)$ elements, Lagrangian interpolation determines

$$\eta_k = \sum_{u=0}^{n} A_u^n(k) y_u ,$$

where $A_u^n(k)$ can be determined independently of the values of y_u. The sum of the sequence of m terms of $\{\eta_k\}$ is $S = \sum_{k=1}^{m} \eta_k = \sum_{k=1}^{m} \sum_{u=0}^{n} A_u^n(k) y_u$. As the number of terms is finite, the order of summation can be reversed, so that $S = \sum_{u=0}^{n} B_u^n y_u$, where $B_u^n = \sum_{k=1}^{m} A_u^n(k)$, can be determined independently of y_u.

Received June 28, 1957; in revised form March 18, 1958.

Although the preceding analysis is valid for every subset of the set, an application of special interest is that of skip-term summation in which the subset $\{y_u\}$ is determined by the relation $y_u = \eta_{gu}$. The first term of the set and of the subset is $y_0 = \eta_0$ and the final term is $y_n = \eta_{gn}$. For this choice $m = gn$ and the terms of the subset are taken from those of the set with a gap g in the index. The initial term is not included in the sum, but its value is needed to obtain the sum. This choice permits the summation of a sequence of terms in blocks, without including overlapping terms twice, and is convenient in evaluating the sum of a smooth sequence after selected terms have been computed and added. For this choice, the sum of the ng terms of the original sequence is $S = \sum_{u=0}^{n} B_u^{\ n}\eta_{gu}$, where $B_u^{\ n} = \sum_{k=1}^{gn} A_u^{\ n}(k)$.

The coefficient $A_u^{\ n}(k)$ is a polynomial of degree n in k, and depends on u. Accordingly, the summation involves sums of the type $\sum_{k=1}^{t} k^v$, which can be shown [9] to have the value

$$\sum_{k=1}^{t} k^v = v! \sum_{u=0}^{w} \frac{b_{2u}}{2^{2u}(v - 2u + 1)!}\left[\left(t + \frac{1}{2}\right)^{v-2u+1} - \left(\frac{1}{2}\right)^{v-2u+1}\right],$$

where w is the largest integer in $v/2$ and where $b_0 = 1$, $b_2 = -\frac{1}{6}$, $b_4 = \frac{7}{360}$, and

$$b_{2u} = -\sum_{i=1}^{u} b_{2u-2i}/(2i + 1)! \quad \text{for } u > 0.$$

The sum can be written

$$\sum_{k=1}^{t} k^v = \left(\frac{1}{2}\right)^{v+1} \sum_{u=0}^{w} b_{2u}[v!/(v - 2u + 1)!]\varphi_{v+1-2u}^{t},$$

where $\varphi_j^{\ t} = (2t + 1)^j - 1$. Specifically, for the sum from

$$k = 1 \quad \text{to} \quad k = t, \qquad \sum k = \varphi_2/8, \qquad \sum k^2 = (\varphi_3 - \varphi_1)/24,$$

$$\sum k^3 = (\varphi_4 - 2\varphi_2)/64, \quad \sum k^4 = (3\varphi_5 - 10\varphi_3 + 7\varphi_1)/480,$$

$$\sum k^5 = (\varphi_6 - 5\varphi_4 + 7\varphi_2)/384,$$

$$\sum k^6 = (3\varphi_7 - 21\varphi_5 + 49\varphi_3 - 31\varphi_1)/2688,$$

$$\sum k^7 = (3\varphi_8 - 28\varphi_6 + 98\varphi_4 - 124\varphi_2)/6144,$$

$$\sum k^8 = (5\varphi_9 - 60\varphi_7 + 294\varphi_5 - 620\varphi_3 + 381\varphi_1)/23040,$$

$$\sum k^9 = (\varphi_{10} - 15\varphi_8 + 98\varphi_6 - 310\varphi_4 + 381\varphi_2)/10240,$$

$$\sum k^{10} = (3\varphi_{11} - 55\varphi_9 + 462\varphi_7 - 2046\varphi_5 + 4191\varphi_3 - 2555\varphi_1)/67584,$$

where the value of t is implied. The customary forms of these sums are polynomials or products of polynomials in t. (See, e.g., Jolley [5] formulas 15 to 22 inclusive.)

The Lagrangian coefficients can be obtained in various manners. One method is that of expanding into polynomials the customary forms involving products. Another method involves the assumption of a polynomial of the proper degree and the determination of the coefficients by solving the system of linear equations corresponding to the various values of the terms of the subset used in the summation.

For $n = 2p$, the M.T.P. tables [8] of Lagrangian interpolation coefficients are given for the range $u \equiv [-p(1)p]$ so that the terms of the original set may be taken on the range $k \equiv [-gp(1)gp]$. For this choice $k = gu$. The subset is $\{y_u\}$ and, dropping the superscript n, the desired sum is

$$S = \sum_{k=1-gp}^{gp} \eta_k = \sum_{u=-p}^{p} B_u y_u \quad \text{where} \quad B_u = \sum_{k=1-gp}^{gp} A_u(k/g).$$

For n even, $A_{-u}(-x) = A_u(x)$ so that

$$B_u = A_u(0) + A_u(p) + \sum_{k=1}^{gp-1} [A_{-u}(k/g) + A_u(k/g)].$$

For integral values of u and v, $A_u(u) = 1$ and $A_u(v) = 0$ for $v \neq u$. Hence

$$B_{-p} = \sum_{k=1}^{gp-1} [A_{-u}(k/g) + A_u(k/g)],$$

$$B_p = 1 + B_{-p},$$

$$B_0 = 1 + 2\sum_{k=1}^{gp-1} A_0(k/g),$$

$$B_{-u} = B_u = \sum_{k=1}^{gp-1} [A_{-u}(k/g) + Au(k/g)] \quad \text{for} \quad 1 \leqq u \leqq p - 1.$$

Accordingly, the sum of the original set is

$$S = B_{-p}y_{-p} + B_0 y_0 + B_p y_p + \sum_{u=1}^{p-1} B_u(y_{-u} + y_u).$$

Values of B_k

Three Point

k \ g	5	10	25	50	100
−1	1.2	2.85	7.84	16.17	32.835
0	6.6	13.30	33.32	66.66	133.330
+1	2.2	3.85	8.84	17.17	33.835

Seven Point

k \ g	5	10	25	50	100
−3	1.007 808	2.450 365 875	6.830 150 082 56	14.147 218 165 08	28.787 894 830 158 75
±2	7.594 752	15.368 629 750	38.547 432 304 64	77.130 857 609 52	154.279 714 344 047 50
±1	1.150 720	2.022 188 125	4.858 920 038 40	9.661 606 076 20	19.295 089 152 381 25
0	9.493 440	19.317 632 500	48.526 995 148 80	97.120 636 298 40	194.274 603 346 825 00
+3	2.007 808	3.450 365 875	7.830 150 082 56	15.147 218 165 08	29.787 894 830 158 75

Eleven Point

k \ g	5	10
0	+34.028 105 216	+70.538 954 406 359 375 00
−5	+0.892 014 592	+2.208 634 246 136 718 75
+5	+1.892 014 592	+3.208 634 246 136 718 75
±1	−20.316 441 600	−42.788 478 982 539 062 50
±2	+21.805 521 920	+45.019 765 486 718 750 00
±3	−3.588 049 920	−7.870 561 162 597 656 25
±4	+8.692 902 400	+17.661 163 209 101 562 50

Values of B_u are shown for several values of n and g in the accompanying Table. The values are exact and have been checked by the relation $\sum_{u=-p}^{p} B_u = 2gp$ based on $y_k \equiv 1$. With allowance for rounding errors in the tenth decimal place, they have also been verified by addition of A_k from M.T.P.

If $m \neq gn$, the method can be used only when the sequence can be divided into blocks each of the selected form. The sum of the terms ahead of the first block must be calculated separately. For example, for sixth order summation of the block $k \equiv [-3g(1)3g]$, the sum from $1 - 3g$ to $3g$ is

$$S = B_{-3}y_{-3} + B_0 y_0 + B_3 y_3 + B_1(y_{-1} + y_1) + B_2(y_{-2} + y_2).$$

For two consecutive blocks $k \equiv [0(1)12g]$, the sum from 1 to $12g$ is

$$S = B_{-3}(y_0 + y_6) + B_0(y_3 + y_9) + B_3(y_6 + y_{12}) + B_1(y_2 + y_4 + y_8 + y_{10}) + B_2(y_1 + y_5 + y_7 + y_{11}).$$

For $n = 2p - 1$, the M.T.P. tables [8] use the subset range $u \equiv [1 - p(1)p]$, so that the original set must be taken on the range

$$k \equiv [g\{1 - p\} + 1(1)gp].$$

For this case, $A_{1-u}(1 - x) = A_u(x)$. Then

$$S = \sum_{k=1-(p-1)g}^{gp} \eta_k = \sum_{u=0}^{n} B_u y_u \quad \text{where} \quad Bu = \sum_{k=1-(p-1)g}^{gp} A_u(k/g).$$

The value of B_u can be written

$$B_u = \sum_{k=gp-g+1}^{gp} A_u(k/g) + \sum_{k=1}^{gp-g} [A_u(k/g) + A_{1-u}(\overline{q - 1 + k}/g)].$$

Accordingly the method is somewhat more involved for n odd than for n even.

Tests of this method on some of the examples given by Salzer [11] indicate a superiority of the latter method in summing slowly convergent series. However, the present method is convenient in summing a finite sequence of terms that are sufficiently smooth but the infinite series of which does not converge. It is also convenient when individual terms are not computed readily except for multiples of a constant, and where sequences of terms are simpler to compute for large index than for small.

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On a Conjecture Concerning the Primes

By R. B. Killgrove and K. E. Ralston

Consider the sequence $\{P_{0j}\}$, $j = 0, 1, 2, \cdots$, where P_{0j} is the jth prime number, $P_{00} = 2$, $P_{01} = 3$, $P_{02} = 5$, $\cdots$. Now define the absolute differences of the primes by the recursion relation

$$P_{ij} = | P_{i-1,j+1} - P_{i-1,j} | .$$

The conjecture (Norman L. Gilbreath, private communication, July 1958) is then that $P_{i0} = 1$ for all $i > 0$. The validity of the conjecture for the first few primes can be seen from the following table of their absolute differences.

$$\begin{array}{ccccccccccccc}
2 & & 3 & & 5 & & 7 & & 11 & & 13 & & 17 \\
& \mathit{1} & & 2 & & 2 & & 4 & & 2 & & 4 & \\
& & \mathit{1} & & 0 & & 2 & & 2 & & 2 & & \\
& & & \mathit{1} & & 2 & & 0 & & 0 & & & \\
& & & & \mathit{1} & & 2 & & 0 & & & & \\
& & & & & \mathit{1} & & 2 & & & & & \\
& & & & & & \mathit{1} & & & & & &
\end{array}$$

There are an uncountable number of sequences $\{b_{0j}\}$ with the property that their absolute differences b_{i0} defined as above are unity. In particular the sequences $\{k + 1, k, k, \cdots\}$ and any sequence of the form $\{b_{00} = 1; b_{0j} = 0 \text{ or } 2, j > 0\}$ have this property. Furthermore it can easily be verified that any sequence, $\{b_{0j}\}$, with the required property has its first absolute differences bounded by the sequence $\{2^j\}$, that is, $b_{1j} \leqq 2^j$.

Consider again the absolute differences of the primes. Since all primes greater than 2 are odd numbers it follows that all differences P_{ij}, $j > 0$, are even numbers. Now, if for some i and all j, $0 < j < M$, we have $P_{ij} = 0$ or 2 and $P_{i0} = 1$, then all of the differences that derive from them will be bounded by 2, from which it follows that

$$P_{i,0}, P_{i+1,0}, P_{i+2,0}, \cdots, P_{i+M-1,0} = 1.$$

We now define the function $P(i)$ to be the largest integer M such that $P_{ij} \leqq 2$ for all $j < M$. Thus we can say that $P_{k0} = 1$ for $i \leqq k < P(i) + i$.

A routine was coded for the SWAC to evaluate this function $P(i)$, using the primes less than 792,722 from a sieve prepared by D. H. Lehmer. The results of

Received Oct. 7, 1958. The preparation of this paper was sponsored by the Office of Naval Research.