Tables of Abscissas and Weights for Numerical Evaluation of Integrals of the Form $\int_0^\infty e^{-x}x^n f(x)\,dx$

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There are many physical problems [1, 2] whose solutions involve integrals of the form

$$(1) \qquad \int_0^\infty x^n e^{-x} f(x)\,dx,$$

hence it is desirable to have a table of the weights and abscissas suitable for use in the numerical integration formula of Gaussian type

$$(2) \qquad \int_0^\infty x^n e^{-x} f(x)\,dx = \sum_{k=0}^{N} a_{kN}^n f(x_{kN}^n) + E\,;$$

where

$$E = \frac{N!(N+n)!}{(2N)!} f^{(2m)}(\xi)\,.$$

It is known that if $L_N^n(x)$ is the generalized Laguerre polynomial defined by

$$(3) \qquad L_N^n(x) = \sum_{m=0}^{N} \binom{N+n}{N-m} \frac{(-x)^m}{m!}$$

then [3] the abscissas x_{kN}^n appearing in Eq. (2) are the roots of $L_N^n(x) = 0$, while the coefficients a_{kN}^n are given by

$$(4) \qquad a_{kN}^n = \frac{1}{x_{kN}^n} \frac{1}{[L_N^{n\prime}(x_{kN}^n)]^2}$$

where L' denotes the derivative of L.

The most extensive tables of a_{kN}^0 and x_{kN}^0 have been given by Salzer and Zucker [4] who tabulated the weights and roots to at most 14 places for $N = 2(1)15$. A very short table of the a_{kN}^n and x_{kN}^n to four places has been given by Burnett [5] for $n = 2, 3, 4$ and $N = 2, 3$. The present tables contain the roots, weights, and the weights multiplied by $\exp(x_{kN}^n)$ for use in the numerical integration formula, Eq. (2). The tables are good to 18 significant figures for $n = 0(1)5$. For $n = 0$ we give the appropriate numbers for $N = 4(4)32$ and for $n > 0$ we give the numbers for $N = 4(4)16$.

As an illustration of a possible use of these tables we cite the calculation of such lattice sums as [6]

$$(5) \qquad T_{r,s,t} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{n^r m^s}{(m^2 + n^2)^t}\,.$$

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By using the identity

$$z^{-t} = \{\Gamma(t)\}^{-1} \int_0^\infty e^{-zu} u^{t-1}\, du$$

we may convert the expressions for $T_{r,s,t}$ into an integral form

$$T_{r,s,t} = \frac{1}{\Gamma(t)} \int_0^\infty u^{t-1} \sigma_r(u) \sigma_s(u)\, du \tag{6}$$

where

$$\sigma_r(u) = \sum_{n=1}^{\infty} n^r e^{-n^2 u} \tag{7}$$

which has an exponential behavior. Previous tabulations of lattice sums have relied on the use of the Poisson summation formula to find the behavior [6] of $\sigma_r(u)$ for small u or on direct summation of a large number of terms of Eq. (5) by electronic computer [7]. The present tables enable one to evaluate the integral in Eq. (6) by direct numerical integration. To facilitate this method we are now computing tables of theta functions evaluated at the zeros of Laguerre polynomials.

The values in the tables were computed on WEIZAC using double precision floating operations which are accurate to approximately 21 significant figures. The zeros of $L_N^n(x)$ were computed using Newton's method. If $x_{kN}^{n(i)}$ is the ith approximation to the kth root of $L_N^n(x)$, then

$$x_{kN}^{n(i+1)} = x_{kN}^{n(i)} - L_N^n(x_{kN}^{n(i)}) / L_N^{n'}(x_{kN}^{n(i)}) = x_{kN}^{n(i)} - \Delta \tag{8}$$

where $L_N^n(x)$ is calculated from the recurrence relation [3]

$$\begin{gathered} NL_N^n(x) - (2N - 1 + n - x) L_{N-1}^n(x) + (N - 1 + n) L_{N-2}^n(x) = 0, \\ L_{-1}^n(x) = 0, \qquad L_0^n(x) = 1, \end{gathered} \tag{9}$$

and

$$L_N^{n'}(x) = \frac{1}{x} [NL_N^n(x) - (N + n) L_{N-1}^n(x)]. \tag{10}$$

The iteration was terminated when $|\Delta| < 2^{-61}$. For the initial guesses for the iteration, various sources were used. For $n = 0$, $N = 4, 8, 12$ the values given by Salzer and Zucker [4] were used. For $n = 0$, $N = 16, 20$, the values given by Head and Wilson [8] were used. In all other cases, we made use of the following inequality given by Szego [9]

$$\begin{aligned} A &= \frac{j_{kn}^2}{4N + 2(n+1)} < x_{kN}^n < B \\ &= \frac{(2k + n + 1)^2 + (2k + n + 1)[(2k + n + 1)^2 + \frac{1}{4} - n^2]^{\frac{1}{2}}}{2N + n + 1} \end{aligned} \tag{11}$$

where j_{kn} is the kth zero of $J_n(x)$ and is tabulated in Watson [10] for $n = 0(1)5$, $k = 1(1)40$. The way this was done is as follows. First we determined $\lambda_{kN}^n = (x_{kN}^n - A)/(B - A)$ for $n = 0$, $N = 15, 16$ using the values given by Head and Wilson [8]. From the two graphs of $\lambda_{k,15}^0$ and $\lambda_{k,16}^0$ we estimated that $4(k/(2N - 1))^3$ was a good approximation to λ_{kN}^n, independent of n. Hence our initial guess for x_{kN}^n was

$$x_{kN}^{n(0)} = \frac{j_{kn}^2 + 8\left(\frac{k}{2N-1}\right)^3 [(2k+n+1)^2 - \frac{1}{2}j_{kn}^2 + (2k+n+1) \cdot \{(2k+n+1)^2 + \frac{1}{4} - n^2\}^{\frac{1}{2}}]}{4N + 2n + 2}. \tag{12}$$

This choice of $x_{kN}^{n(0)}$ was a rather good approximation to x_{kN}^n, differing from it by at most 10% and usually by less than 1%, and always being closer to it than to any other root of $L_N^n(x)$. Convergence to x_{kN}^n occurred in almost every case in at most 10 iterations, and usually in much less. However, in some cases in which $k = N$, Newton's method favored convergence to $x_{N-1,N}^n$. This caused very little trouble for an approximate value for $x_{N,N}^n$ was computed from the relation derived from Eq. (3)

$$\sum_{k=1}^{N} x_{k,N}^n = N(N + n) \tag{13}$$

and used as an initial value in the iteration. In general, this relation together with a second relation derived from Eq. (3)

$$\prod_{k=1} x_{k,N}^n = \frac{(N+n)!}{n!} \tag{14}$$

were used as checks on the computation. They always checked to 18 significant figures. In addition the following quantities were computed by the machine and these also checked to 18 significant figures (all summations for $k = 1, \cdots, N$).

$$\sum a_{kN}^n = n!$$
$$\sum a_{kN}^n x_{kN}^n = (n+1)!$$
$$\sum a_{kN}^n (x_{kN}^n)^7 = (n+7)!$$
$$\sum a_{kN}^n (x_{kN}^n)^{15} = (n+15)! \qquad N \neq 4$$
$$\sum a_{kN}^n (x_{kN}^n)^{23} = (n+23)! \qquad N \neq 4, 8$$
$$\begin{cases} \sum a_{kN}^n (x_{kN}^n)^{31} = (n+31)! & n = 0, 1, 2; N \geqq 16 \\ \sum a_{kN}^n (x_{kN}^n)^{27} = (n+27)! & n = 3, 4, 5; N = 16. \end{cases}$$

The values for $n = 0$, $N = 32$ were not checked by the machine and were only checked by hand using Eq. (13), while for $n = 0$, $N = 28$, only Eqs. (13) and (14) were checked by the computer. This modification was necessary because the values $a_{28,28}^0$, $a_{31,32}^0$, and $a_{32,32}^0$ were out of the range of the double precision floating routine and had to be computed by hand.

TABLE

Zeros of Generalized Laguerre Polynomials	Weights	Weight times Exponential of Zero $= a^n_{kN} \exp(x^n_{kN})$
	$n = 0, N = 4$	
3.22547689619392312 (−1)	6.03154104341633602 (−1)	8.32739123837889247 (−1)
1.74576110115834658 (0)	3.57418692437799687 (−1)	2.04810243845429682 (0)
4.53662029692112798 (0)	3.88879085150053843 (−2)	3.63114630582151786 (0)
9.39507091230113313 (0)	5.39294705561327450 (−4)	6.48714508440766227 (0)
	$n = 0, N = 8$	
1.70279632305101000 (−1)	3.69188589341637530 (−1)	4.37723410492911373 (−1)
9.03701776799379912 (−1)	4.18786780814342956 (−1)	1.03386934766559764 (0)
2.25108662986613069 (0)	1.75794986637171806 (−1)	1.66970976565877575 (0)
4.26670017028765879 (0)	3.33434922612156515 (−2)	2.37692470175859948 (0)
7.04590540239346570 (0)	2.79453623522567252 (−3)	3.20854091334792628 (0)
1.07585160101809952 (1)	9.07650877335821310 (−5)	4.26857551082513220 (0)
1.57406786412780046 (1)	8.48574671627253154 (−7)	5.81808336867192193 (0)
2.28631317368892641 (1)	1.04800117487151038 (−9)	8.90622621529221140 (0)
	$n = 0, N = 12$	
1.15722117358020675 (−1)	2.64731371055443190 (−1)	2.97209636044410798 (−1)
6.11757484515130665 (−1)	3.77759275873137982 (−1)	6.96462980430597231 (−1)
1.51261026977641879 (0)	2.44082011319877564 (−1)	1.10778139461575215 (0)
2.83375133774350723 (0)	9.04492222116809307 (−2)	1.53846423904282912 (0)
4.59922763941834848 (0)	2.01023811546340965 (−2)	1.99832760627424276 (0)
6.84452545311517735 (0)	2.66397354186531588 (−3)	2.50074576910086687 (0)
9.62131684245686704 (0)	2.03231592662999392 (−4)	3.06532151828238624 (0)
1.30060549933063477 (1)	8.36505585681979875 (−6)	3.72328911078277160 (0)
1.71168551874622557 (1)	1.66849387654091026 (−7)	4.52981402998173506 (0)
2.21510903793970057 (1)	1.34239103051500415 (−9)	5.59725846183532109 (0)
2.84879672509840003 (1)	3.06160163503502078 (−12)	7.21299546092587700 (0)
3.70991210444669203 (1)	8.14807746742624168 (−16)	1.05438374619100811 (1)
	$n = 0, N = 16$	
8.76494104789278403 (−2)	2.06151714957800994 (−1)	2.25036314864247252 (−1)
4.62696328915080832 (−1)	3.31057854950884166 (−1)	5.25836052762342454 (−1)
1.14105777483122686 (0)	2.65795777644214153 (−1)	8.31961391687087088 (−1)
2.12928364509838062 (0)	1.36296934296377540 (−1)	1.14609924096375170 (0)
3.43708663389320665 (0)	4.73289286941252190 (−2)	1.47175131696680859 (0)
5.07801861454976791 (0)	1.12999000803394532 (−2)	1.81313468738134816 (0)
7.07033853504823413 (0)	1.84907094352631086 (−3)	2.17551751969460745 (0)
9.43831433639193878 (0)	2.04271915308278460 (−4)	2.56576275016502921 (0)
1.22142233688661587 (1)	1.48445868739812988 (−5)	2.99321508637137516 (0)
1.54415273687816171 (1)	6.82831933087119956 (−7)	3.47123448310209029 (0)
1.91801568567531349 (1)	1.88102484107967321 (−8)	4.02004408644466886 (0)
2.35159056939919085 (1)	2.86235024297388162 (−10)	4.67251660773285426 (0)
2.85787297428821404 (1)	2.12707903322410297 (−12)	5.48742065798615247 (0)
3.45833987022866258 (1)	6.29796700251786779 (−15)	6.58536123328921366 (0)
4.19404526476883326 (1)	5.05047370003551282 (−18)	8.27635798436423448 (0)
5.17011603395433184 (1)	4.16146237037285519 (−22)	1.18242775516584348 (1)
	$n = 0, N = 20$	
7.05398896919887534 (−2)	1.68746801851113862 (−1)	1.81080062418989255 (−1)
3.72126818001611444 (−1)	2.91254362006068282 (−1)	4.22556767878563975 (−1)
9.16582102483273565 (−1)	2.66686102867001289 (−1)	6.66909546701848151 (−1)
1.70730653102834388 (0)	1.66002453269506840 (−1)	9.15352372783073673 (−1)
2.74919925530943213 (0)	7.48260646687923705 (−2)	1.16953970719554597 (0)
4.04892531385088692 (0)	2.49644173092832211 (−2)	1.43135498592820599 (0)
5.61517497086161651 (0)	6.20255084457223685 (−3)	1.70298113798502272 (0)
7.45901745367106331 (0)	1.14496238647690824 (−3)	1.98701589079274721 (0)
9.59439286958109677 (0)	1.55741773027811975 (−4)	2.28663578125343079 (0)
1.20388025469643163 (1)	1.54014408652249157 (−5)	2.60583472755383333 (0)
1.48142934426307400 (1)	1.08648636651798235 (−6)	2.94978373421395086 (0)
1.79488955205193760 (1)	5.33012090955671475 (−8)	3.32539578200931955 (0)
2.14787882402850110 (1)	1.75798117905058200 (−9)	3.74225547058981092 (0)
2.54517027931869055 (1)	3.72550240251232087 (−11)	4.21423671025188042 (0)
2.99325546317006120 (1)	4.76752925157819052 (−13)	4.76251846149020929 (0)
3.50134342404790000 (1)	3.37284424336243841 (−15)	5.42172604424557430 (0)
4.08330570567285711 (1)	1.15501433950039883 (−17)	6.25401235693242129 (0)
4.76199940473465021 (1)	1.53952214058234355 (−20)	7.38731438905443455 (0)
5.58107957500638989 (1)	5.28644272556915783 (−24)	9.15132873098747960 (0)
6.65244165256157538 (1)	1.65645661249902330 (−28)	1.28933886459399966 (1)

Zeros of Generalized Laguerre Polynomials	Weights	Weight times Exponential of Zero = $a^n_{kN} \exp(x^n_{kN})$
	$n = 0,\ N = 24$	
5.90198521815079770 (−2)	1.42811973334781851 (−1)	1.51494412859509452 (−1)
3.11239146198483727 (−1)	2.58774107517423903 (−1)	3.53256582529923847 (−1)
7.66096905545936646 (−1)	2.58806707272869802 (−1)	5.56784563288152601 (−1)
1.42559759080361309 (0)	1.83322688977778025 (−1)	7.62685317697309109 (−1)
2.29256205863219029 (0)	9.81662726299188922 (−2)	9.71872632246547580 (−1)
3.37077426420899772 (0)	4.07324781514086460 (−2)	1.18535789303780109 (0)
4.66508370346717079 (0)	1.32260194051201567 (−2)	1.40426562728441853 (0)
6.18153511873676541 (0)	3.36934905847830355 (−3)	1.62986861575704148 (0)
7.92753924717215218 (0)	6.72162564093547890 (−4)	1.86363505533207295 (0)
9.91209801507770602 (0)	1.04461214659275180 (−4)	2.10729115108148019 (0)
1.21461027117297656 (1)	1.25447219779933332 (−5)	2.36290589104193500 (0)
1.46427322895966743 (1)	1.15131581273727992 (−6)	2.63300875316385675 (0)
1.74179926465089787 (1)	7.96081295913363026 (−8)	2.92075757972772467 (0)
2.04914600826164247 (1)	4.07285898754999966 (−9)	3.23018513349235362 (0)
2.38873298481697332 (1)	1.50700822629258492 (−10)	3.56657337736875657 (0)
2.76359371743327174 (1)	3.91773651505845138 (−12)	3.93704375545516030 (0)
3.17760413523747233 (1)	6.89418105295808569 (−14)	4.35153118886351197 (0)
3.6358[illegible]058016516217 (1)	7.81980038245944847 (−16)	4.82448185489803577 (0)
4.14517204848707670 (1)	5.35018881301003760 (−18)	5.37802207978918232 (0)
4.71531064451563230 (1)	2.01051746455550347 (−20)	6.04841781261996523 (0)
5.36085745446950698 (1)	3.60576586455295904 (−23)	6.90089835218049651 (0)
6.10585314472187616 (1)	2.45181884587840269 (−26)	8.06996515614695609 (0)
6.99622400351050304 (1)	4.08830159368065782 (−30)	9.90279331948422546 (0)
8.14982792339488854 (1)	5.57534578832835675 (−34)	1.38205320947920057 (1)
	$n = 0,\ N = 28$	
5.07346248498738876 (−2)	1.23778843954286428 (−1)	1.30220749260603611 (−1)
2.67487268640741084 (−1)	2.32279276900901161 (−1)	3.03513987299129915 (−1)
6.58136628354791519 (−1)	2.47511896036477212 (−1)	4.77992610899299887 (−1)
1.22397180838490772 (0)	1.92307113132382827 (−1)	6.53972597062647847 (−1)
1.96676761247377770 (0)	1.16405361721130006 (−1)	8.32011455708276465 (−1)
2.88888332603032189 (0)	5.63459053644773065 (−2)	1.01271781273780542 (0)
3.99331165925011414 (0)	2.20663643262588079 (−2)	1.19675156048109225 (0)
5.28373606284344256 (0)	7.02588763558386773 (−3)	1.38483686392073190 (0)
6.76460340424350515 (0)	1.82060789269585487 (−3)	1.57777965578712887 (0)
8.44121632827132449 (0)	3.83344303857123177 (−4)	1.77648966944358961 (0)
1.03198504629932601 (1)	6.53508708069439831 (−5)	1.98200839256100634 (0)
1.24079034144606717 (1)	8.97136205341076834 (−6)	2.19554512993098825 (0)
1.47140851641357488 (1)	9.84701225624928887 (−7)	2.41852439028379250 (0)
1.72486634156080563 (1)	8.56407585267304245 (−8)	2.65264936115550918 (0)
2.00237833299517127 (1)	5.83683876313834429 (−9)	2.89998868599490885 (0)
2.30538901350302960 (1)	3.07563887784230228 (−10)	3.16309775591259801 (0)
2.63562973744013176 (1)	1.23259095272442282 (−11)	3.44519246556198147 (0)
2.99519668335961821 (1)	3.68217367410831200 (−13)	3.75040515240709831 (0)
3.38666055165844592 (1)	7.99879057596890965 (−15)	4.08417387436021254 (0)
3.81322544101946468 (1)	1.22492250032408341 (−16)	4.45385716433002825 (0)
4.27896723707725763 (1)	1.27112429503067374 (−18)	4.86974939776545707 (0)
4.78920716336227437 (1)	8.48859336768654320 (−21)	5.34685205218439829 (0)
5.35112979596642942 (1)	3.40245537942551185 (−23)	5.90818195463222266 (0)
5.97487960846412408 (1)	7.42015658886748513 (−26)	6.59151903072765714 (0)
6.67569772839064696 (1)	7.60041320580173769 (−29)	7.46490124681590107 (0)
7.47867781523391618 (1)	2.87391031794039581 (−32)	8.66878601010575288 (0)
8.43178371072270431 (1)	2.54182290388931800 (−36)	1.05661475342792109 (1)
9.65824206275273191 (1)	1.66137587802903396 (−41)	1.46446915172544714 (1)
	$n = 0,\ N = 32$	
4.44893658332670184 (−2)	1.09218341952384971 (−1)	1.14187105768104849 (−1)
2.34526109519618537 (−1)	2.10443107938813234 (−1)	2.66065216897615217 (−1)
5.76884629301886426 (−1)	2.35213229669848005 (−1)	4.18793137324852994 (−1)
1.07244875381781763 (0)	1.95903335972881043 (−1)	5.72532846499804707 (−1)
1.72240877644464544 (0)	1.29983786286071761 (−1)	7.27648788380971311 (−1)
2.52833670642579488 (0)	7.05786238657174415 (−2)	8.84536719340249717 (−1)
3.49221327302199449 (0)	3.17609125091750703 (−2)	1.04361887589207697 (0)
4.61645676974976739 (0)	1.19182148348385571 (−2)	1.20534927415235258 (0)
5.90395850417424395 (0)	3.73881629461152479 (−3)	1.37022133852178119 (0)
7.35812673318624111 (0)	9.80803306614955132 (−4)	1.53877725646864475 (0)
8.98294092421259610 (0)	2.14864918801364188 (−4)	1.71161935268645726 (0)
1.07830186325399721 (1)	3.92034196798794720 (−5)	1.88942406344948410 (0)
1.27636979867427251 (1)	5.93454161286863288 (−6)	2.07295934024653367 (0)

Zeros of Generalized Laguerre Polynomials	Weights	Weight times Exponential of Zero = $a^n_{kN} \exp(x^n_{kN})$
1.49311397555225573 (1)	7.41640457866755222 (−7)	2.26310663399696361 (0)
1.72924543367153148 (1)	7.60456787912078148 (−8)	2.46088907248823613 (0)
1.98558609403360547 (1)	6.35060222662580674 (−9)	2.66750812639711715 (0)
2.26308890131967745 (1)	4.28138297104092888 (−10)	2.88439209292204179 (0)
2.56286360224592478 (1)	2.30589949189133608 (−11)	3.11326132703958617 (0)
2.88621018163234747 (1)	9.79937928872709406 (−13)	3.35621769259580256 (0)
3.23466291539647370 (1)	3.23780165772926646 (−14)	3.61586985648426880 (0)
3.61004948057519738 (1)	8.17182344342071943 (−16)	3.89551304494854956 (0)
4.01457197715394415 (1)	1.54213383339382337 (−17)	4.19939410471158549 (0)
4.45092079957549380 (1)	2.11979229016361861 (−19)	4.53311497853436176 (0)
4.92243949873086392 (1)	2.05442967378804543 (−21)	4.90427028761124484 (0)
5.43337213333969073 (1)	1.34698258663739516 (−23)	5.32350097202366611 (0)
5.98925091621340182 (1)	5.66129413039735937 (−26)	5.80633321423362136 (0)
6.59753772879350528 (1)	1.41856054546303691 (−28)	6.37661467415965254 (0)
7.26876280906627086 (1)	1.91337549445422431 (−31)	7.07352658070724211 (0)
8.01874469779135231 (1)	1.19224876009822236 (−34)	7.96769350929590065 (0)
8.87353404178923987 (1)	2.67151121924013699 (−38)	9.20504033127818968 (0)
9.88295428682839726 (1)	1.33861694210625628 (−42)	1.11630130907678735 (1)
1.11751398097937695 (2)	4.51053619389897424 (−48)	1.53901804152606398 (1)
	$n = 1, N = 4$	
7.43291927981431435 (−1)	4.46870593218776310 (−1)	9.39700286292286231 (−1)
2.57163500764627847 (0)	4.77635772363868313 (−1)	6.25091709508283381 (0)
5.73117875168909963 (0)	7.41777847310521364 (−2)	2.28714219375495781 (1)
1.09538943126831905 (1)	1.31584968630324014 (−3)	7.52353830776884715 (1)
	$n = 1, N = 8$	
4.09383573203185153 (−1)	1.87632541405723668 (−1)	2.82553823014304850 (−1)
1.38496318480313988 (0)	4.38985360731142368 (−1)	1.75360553036282958 (0)
2.95625455616886207 (0)	2.89996070781313351 (−1)	5.57541442577641522 (0)
5.18194310104007138 (0)	7.51413846166973434 (−2)	1.33773006498366668 (1)
8.16170968814581733 (0)	7.93264664870734492 (−3)	2.77973624568047784 (1)
1.20700551268371548 (1)	3.08642136813304225 (−4)	5.38782591536632858 (1)
1.72497355261489875 (1)	3.34895820979709614 (−6)	1.03842391519935203 (2)
2.45859552436527819 (1)	4.72139282319322132 (−9)	2.24706083439768244 (2)
	$n = 1, N = 12$	
2.82858348239914112 (−1)	1.01453765665592267 (−1)	1.34620745484392543 (−1)
9.52326041364613503 (−1)	3.17699483362918101 (−1)	8.23391640164269559 (−1)
2.01649213857768073 (0)	3.39986271374452795 (−1)	2.55395234400434764 (0)
3.49235406977802504 (0)	1.79532599439199819 (−1)	5.90001913843717294 (0)
5.40549102001572284 (0)	5.20319390353283915 (−2)	1.15836368199363577 (1)
7.79281394041241498 (0)	8.49161743576890271 (−3)	2.05762635029790113 (1)
1.07073886889890908 (1)	7.67135797278597991 (−4)	3.42793100498839410 (1)
1.42271523637899807 (1)	3.63620912038435440 (−5)	5.48810758737070439 (1)
1.84719966342299991 (1)	8.18456700779946096 (−7)	8.61552836575302900 (1)
2.36417837524008953 (1)	7.32315410040323253 (−9)	1.35579615855293272 (2)
3.01200586261063443 (1)	1.83970243442130593 (−11)	2.21678124758432626 (2)
3.88892843760953185 (1)	5.37762852227037045 (−15)	4.16862096414656053 (2)
	$n = 1, N = 16$	
2.16140305239452255 (−1)	6.32773328795394308 (−2)	7.85446678289795171 (−2)
7.26388243251803954 (−1)	2.31090461520719732 (−1)	4.77802508713723198 (−1)
1.53359316037354132 (0)	3.16933542163999097 (−1)	1.46892374608307634 (0)
2.64497099861191096 (0)	2.37894217875244607 (−1)	3.35027298906274298 (0)
4.07097816088019065 (0)	1.10272743359247882 (−1)	6.46355621447558663 (0)
5.82585551510560455 (0)	3.30895883563969837 (−2)	1.12157486249617326 (1)
7.92850418530666709 (0)	6.52571640779246630 (−3)	1.81106406690387663 (1)
1.04038082899510393 (1)	8.42711436792047207 (−4)	2.77968394748499728 (1)
1.32846610707070382 (1)	6.99619095556818742 (−5)	4.11449606829674792 (1)
1.66151732168666126 (1)	3.61208604213355474 (−6)	5.93794445636475115 (1)
2.04560060200272169 (1)	1.10154125147424469 (−7)	8.43201366918135643 (1)
2.48938470253519108 (1)	1.83568251273250421 (−9)	1.18866087257456150 (2)
3.00598629202025761 (1)	1.48148886588507814 (−11)	1.68085784287149817 (2)
3.61706945436791780 (1)	4.73484535903961320 (−14)	2.42124561031249231 (2)
4.36403651841768370 (1)	4.08414605500225599 (−17)	3.66328790805694462 (2)
5.35291511602684204 (1)	3.62395496247891406 (−21)	6.40615970837448162 (2)

Zeros of Generalized Laguerre Polynomials	Weights	Weight times Exponential of Zero = $a^n_{kN} \exp(x^n_{kN})$
	$n = 2, N = 4$	
1.22676326350030207 (0)	7.25524997698654378 (−1)	2.47416635265254532 (0)
3.41250735869694597 (0)	1.06342429197919458 (0)	3.22655960312239398 (1)
6.90269260585161340 (0)	2.06696131028353551 (−1)	2.05652340494177747 (2)
1.24580367719511386 (1)	4.35457929379748887 (−3)	1.12047669645817929 (3)
	$n = 2, N = 8$	
6.99330392297772446 (−1)	2.24797751904342970 (−1)	4.52384060238173233 (−1)
1.89881649533754637 (0)	7.99530958902891337 (−1)	5.33925681844090349 (0)
3.67761476834163509 (0)	7.13670609157191155 (−1)	2.82270577764238765 (1)
6.09929454816086345 (0)	2.31541145990131662 (−1)	1.03161617804640828 (2)
9.26742581328238619 (0)	2.91357867305206514 (−2)	3.08474054427306454 (2)
1.33607382722601106 (1)	1.30770014757761649 (−3)	8.29856650024944985 (2)
1.87281386688430816 (1)	1.60219339506628624 (−5)	2.17892467611196491 (3)
2.62686410414766043 (1)	2.52333939442792311 (−8)	6.46102645632067593 (3)
	$n = 2, N = 12$	
4.90239109177454061 (−1)	9.51482251459639793 (−2)	1.55349132168384772 (−1)
1.32377645578306233 (0)	4.77196653262217159 (−1)	1.79310697194985889 (0)
2.54213223523224643 (0)	7.20466902576669237 (−1)	9.15478262411010895 (0)
4.16451935324397066 (0)	4.93826609735938395 (−1)	3.17835397356631661 (1)
6.21800163077006985 (0)	1.75757334769862195 (−1)	8.81773930663432268 (1)
8.74078952492664456 (0)	3.38929300594179609 (−2)	2.11926747809213372 (2)
1.17872181154281359 (1)	3.51882418847753131 (−3)	4.62936424650790635 (2)
1.54365948600615973 (1)	1.87787474819631979 (−4)	9.49936530531789858 (2)
1.98104815433429476 (1)	4.68668021445018989 (−6)	1.88125704585776709 (3)
2.51111194768538770 (1)	4.59806360065300972 (−8)	3.69994780357075735 (3)
3.17262495017079404 (1)	1.25743502885190380 (−10)	7.55128122489510488 (3)
4.06488781934720538 (1)	3.99508663346059607 (−14)	1.79932534090427829 (4)
	$n = 2, N = 16$	
3.77613508344740734 (−1)	4.86064094670787025 (−2)	7.09069825088867147 (−2)
1.01749195760256570 (0)	2.93347390190440231 (−1)	8.11471686778248555 (−1)
1.94775802042424383 (0)	5.83219363383550989 (−1)	4.09008651869529020 (0)
3.17692724488986868 (0)	5.81874148596173479 (−1)	1.39492556822190547 (1)
4.71624006979179562 (0)	3.38180537473790217 (−1)	3.77907618869590957 (1)
6.58058826577491250 (0)	1.22105963944979891 (−1)	8.80339214718968970 (1)
8.78946527064707413 (0)	2.81146258006637302 (−2)	1.84564671246403063 (2)
1.13683230828333189 (1)	4.14314919248225670 (−3)	3.58534172771350011 (2)
1.43506267274370444 (1)	3.85648533767438300 (−4)	6.58551410235728057 (2)
1.77810957248416460 (1)	2.20158005631090712 (−5)	1.16135832614353863 (3)
2.17210847965713089 (1)	7.34236243815651751 (−7)	1.99151272949358401 (3)
2.62581386751110680 (1)	1.32646044204804137 (−8)	3.36092665129919429 (3)
3.15245960042758187 (1)	1.15266648290842947 (−10)	5.65798450641837801 (3)
3.77389210025289391 (1)	3.94706915124608697 (−13)	9.68455967415686438 (3)
4.53185461100898426 (1)	3.63797825636053360 (−16)	1.74765056820267912 (4)
5.53325835388358122 (1)	3.45457612313612400 (−20)	3.70706727320259718 (4)
	$n = 3, N = 4$	
1.75552164718549145 (0)	1.86033407414649995 (0)	1.07647588421845741 (1)
4.26560586565682345 (0)	3.35689101902892032 (0)	2.39037661575137080 (2)
8.05794068313800185 (0)	7.64453972843517610 (−1)	2.41474077829377203 (3)
1.39209318040196832 (1)	1.83209339810621145 (−2)	2.03578296138366883 (4)
	$n = 3, N = 8$	
1.02996168735087822 (0)	4.47659268688580562 (−1)	1.25387504291512878 (0)
2.43991423401425973 (0)	2.12957515366677356 (0)	2.44306070979065246 (1)
4.41318676383851689 (0)	2.37008143827449573 (0)	1.95607691717435238 (2)
7.01921044285466378 (0)	9.13728082731325425 (−1)	1.02145993175804680 (3)
1.03653585615970610 (1)	1.32197368598833373 (−1)	4.19605176057880639 (3)
1.46343284911153273 (1)	6.66810091481878879 (−3)	1.51220362170062521 (4)
2.01808472740399451 (1)	9.04305893787150415 (−5)	5.25709678492352907 (4)
2.79171925451893479 (1)	1.56535793848369390 (−7)	2.08399336850716076 (5)
	$n = 3, N = 12$	
7.31333453524153573 (−1)	1.52730239391842707 (−1)	3.17350448025970021 (−1)
1.72200587771537790 (0)	1.07163463296299004 (0)	5.99659048258816783 (0)
3.08711820886747629 (0)	2.09369597825120887 (0)	4.58809098751416806 (1)
4.84907183643667852 (0)	1.75927719487292856 (0)	2.24522264593643932 (2)
7.03654192274923744 (0)	7.38876099420103765 (−1)	8.40432710605229296 (2)
9.68900419917576858 (0)	1.63538474605744857 (−1)	2.63937440838205532 (3)
1.28619916279440914 (1)	1.90887802863997279 (−2)	7.35648187888716898 (3)

TABLE—*Continued*

Zeros of Generalized Laguerre Polynomials	Weights	Weight times Exponential of Zero = $a^n_{kN} \exp(x^n_{kN})$
1.66361029842912300(1)	1.12749487761940805(−3)	1.89270227412433568(4)
2.11344882654905403(1)	3.07769783328946458(−5)	4.64320234766113947(4)
2.65616764973530448(1)	3.27387146853583194(−7)	1.12370201079018605(5)
3.33094793553748032(1)	9.65351819752401242(−10)	2.82363592457219416(5)
4.23811857710775980(1)	3.30506360143886766(−13)	8.41579093496417164(5)
	$n = 3$, $N = 16$	
5.67443458991574145(−1)	6.50981121009449315(−2)	1.14816936994675722(−1)
1.33290773275989334(0)	5.65273224236402383(−1)	2.14354639377120874(0)
2.38148248007005849(0)	1.48989313857906502(0)	1.61220335260503062(1)
3.72382664209342697(0)	1.86124704489653486(0)	7.70976931131275901(1)
5.37212395216187100(0)	1.30047554848272360(0)	2.80018019446226477(2)
7.34193662826135234(0)	5.47819646856915096(−1)	8.45668526756508810(2)
9.65333213726122905(0)	1.43843043208105701(−1)	2.24015331269839424(3)
1.23323014070182379(1)	2.37498472421622760(−2)	5.38903830867048069(3)
1.54128500654077155(1)	2.44244469350610863(−3)	1.20653652088452424(4)
1.89402755758602723(1)	1.52340599558850768(−4)	2.56137300941415792(4)
2.29765957156018692(1)	5.50126530501356105(−6)	5.23686431947102649(4)
2.76101814474260749(1)	1.06842629643597444(−7)	1.04639104770985088(5)
3.29745092032584955(1)	9.92524335897222214(−10)	2.07677073163230993(5)
3.92898232533641479(1)	3.61863342780076823(−12)	4.18695241024345828(5)
4.69768962767103126(1)	3.54373278073215328(−15)	8.93856486144031606(5)
5.71135140237534688(1)	3.58180355287779219(−19)	2.28131757123540439(6)
	$n = 4$, $N = 4$	
2.31915524835569955(0)	6.57222031198722368(0)	6.68203029533717859(1)
5.12867199360117916(0)	1.37936031844206074(1)	2.32826105813133801(3)
9.20089134897243093(0)	3.54028469036795246(0)	3.50699000772320933(4)
1.53512814090706903(1)	9.38918132242164933(−2)	4.36118522676941444(5)
	$n = 4$, $N = 8$	
1.39445874535841327(0)	1.27551104534833669(0)	5.14386973811936581(0)
3.00412262031589098(0)	7.55971394358976204(0)	1.52468188096643765(2)
5.16118127238140954(0)	9.98706281698755810(0)	1.74144726764228389(3)
7.94175644134279969(0)	4.42018619472331421(0)	1.24308710943862802(4)
1.14570496332457916(1)	7.17052700563446937(−1)	6.78085582369019424(4)
1.58935676714650236(1)	3.98824021635586082(−2)	3.18617808357226932(5)
2.16116099678724373(1)	5.89788576623485837(−4)	1.43383544036118813(6)
2.95362536480182341(1)	1.10804739994015747(−6)	7.44715365181727399(6)
	$n = 4$, $N = 12$	
1.00157108495383152(0)	3.59159548108807725(−1)	9.77831924001506431(−1)
2.14370613511225703(0)	3.26582820687688745(0)	2.78607678171097753(1)
3.64933321432680176(0)	7.82396142854537168(0)	3.00823651354474138(2)
5.54486813974946731(0)	7.75650688373216868(0)	1.98505168139673751(3)
7.86076259241096344(0)	3.73730972528196261(0)	9.69270306643015021(3)
1.06377654649934495(1)	9.29328933322615071(−1)	3.87339956748239464(4)
1.39325683951155931(1)	1.19937171658365840(−1)	1.34831503047832052(5)
1.78270063100842237(1)	7.73699187701495750(−3)	4.27309561007931643(5)
2.24457538203894399(1)	2.28490954516935571(−4)	1.27919289421063792(6)
2.79955513560437942(1)	2.61140180598097782(−6)	3.75999417306024579(6)
3.48721707324949280(1)	8.23746583872488555(−9)	1.14970110208989482(7)
4.40889427543252506(1)	3.01723739243414224(−12)	4.23832281907530249(7)
	$n = 4$, $N = 16$	
7.82339164085635910(−1)	1.29739702663052232(−1)	2.83686376560111066(−1)
1.67007183671874001(0)	1.49748665808338508(0)	7.95547187624795022(0)
2.83292171947257737(0)	4.94612614281207374(0)	8.40596269993120727(1)
4.28441836819021891(0)	7.43026705086785796(0)	5.39142636980744192(2)
6.03786871096889479(0)	6.05687666231872829(0)	2.53782571556701110(3)
8.10954933413974021(0)	2.90916131357755320(0)	9.67609960197269259(3)
1.05201017048502218(1)	8.55584651819837666(−1)	3.17018863503759819(4)
1.32960399123563184(1)	1.56009598652813629(−1)	9.28000231797978033(4)
1.64718887086961996(1)	1.75191359650891542(−2)	2.49553149851783729(5)
2.00934999730530394(1)	1.18227257300007126(−3)	6.29816124368594188(5)
2.42235299428062549(1)	4.58506311659349061(−5)	1.51876543553876336(6)
2.89511495806519383(1)	9.50616079827638761(−7)	3.55901373314319768(6)
3.44109430408020216(1)	9.38309759503402079(−9)	8.25712602731401404(6)
4.08248952713820481(1)	3.62281000770432509(−11)	1.94568396347320976(7)
4.86170549734729113(1)	3.75119190628700090(−14)	4.87851881245905145(7)
5.88737277583532394(1)	4.01670675392835967(−18)	1.48732275042273460(8)

TABLE—*Concluded*

Zeros of Generalized Laguerre Polynomials	Weights	Weight times Exponential of Zero = $a^n_{kN} \exp(x^n_{kN})$
	$n = 5, N = 4$	
2.91078636936770048 (0)	2.96843923155522145 (1)	5.45339079699888520 (2)
6.00000000000000000 (0)	7.00000000000000000 (1)	2.82400155444914586 (4)
1.03341103467862578 (1)	1.97469504802402056 (1)	6.07501194316356380 (5)
1.67551032838460417 (1)	5.68657204207579951 (−1)	1.07522522732126438 (7)
	$n = 5, N = 8$	
1.78786047157082745 (0)	4.78286518603309507 (0)	2.85855186820191489 (1)
3.58823885764338781 (0)	3.37613267971511042 (1)	1.22115795065788180 (3)
5.92009002612269490 (0)	5.12313725300345871 (1)	1.90808809485486296 (4)
8.86684640118214593 (0)	2.54081265226598562 (1)	1.80216730854013967 (5)
1.25435889444281568 (1)	4.53758898116486575 (0)	1.27185211458041413 (6)
1.71405288833306908 (1)	2.74338324361631772 (−1)	7.62647046314642504 (6)
2.30233426297284257 (1)	4.37283100034351893 (−3)	4.36187637191007579 (7)
3.11295037859936706 (1)	8.82759451633260886 (−6)	2.91886557731016559 (8)
	$n = 5, N = 12$	
1.29748784000685508 (0)	1.13334726805127857 (0)	4.14815342846979275 (0)
2.58620943634406719 (0)	1.27102752975832523 (1)	1.68784065874957923 (2)
4.22697995386950259 (0)	3.59951248234067119 (1)	2.46602657834341949 (3)
6.25087547037452539 (0)	4.09672458660868638 (1)	2.12401457309494012 (4)
8.69028125595059872 (0)	2.21781320782776474 (1)	1.31845757933745043 (5)
1.15872437894981120 (1)	6.09556014277879247 (0)	6.56583037728656770 (5)
1.49995922575396278 (1)	8.58547769806400297 (−1)	2.80546343478958589 (6)
1.90103582323823253 (1)	5.98481941810913418 (−2)	1.07930634493047940 (7)
2.37456904925278601 (1)	1.89538249456284785 (−3)	3.89331206067960144 (7)
2.94144761288863084 (1)	2.30998625486114025 (−5)	1.37452950475636027 (8)
3.64163482053466644 (1)	7.74406887601347796 (−8)	5.06277099810570424 (8)
4.57744569372735530 (1)	3.01616108514055821 (−11)	2.28585454857795443 (9)
	$n = 5, N = 16$	
1.01975621006729237 (0)	3.51651938860095021 (−1)	9.74961600786600682 (−1)
2.02686733891905119 (0)	5.12310712750064588 (0)	3.88857731425075226 (1)
3.30047946506096635 (0)	2.04008497899245158 (1)	5.53386139750131562 (2)
4.85757645713809794 (0)	3.57964501907962966 (1)	4.60742858155172212 (3)
6.71275559761595590 (0)	3.32890537160876807 (1)	2.73913958593091106 (4)
8.88305877614446853 (0)	1.79120868308842187 (1)	1.29124780054447953 (5)
1.13897106617410895 (1)	5.81751150687781879 (0)	5.14310915694565138 (5)
1.42597409258898484 (1)	1.15803031442409197 (0)	1.80570468896330916 (6)
1.75281787111426743 (1)	1.40641304118632739 (−1)	5.76108359319635732 (6)
2.12414122811650920 (1)	1.01865573672515247 (−2)	1.71023416056506905 (7)
2.54627162726269377 (1)	4.21355482124525534 (−4)	4.81909868948838928 (7)
3.02820392701258409 (1)	9.27055945773008095 (−6)	1.31349522839105173 (8)
3.58350465381012185 (1)	9.67230928638172189 (−8)	3.53584440429819067 (8)
4.23454277633176611 (1)	3.93647895812592163 (−10)	9.67149353332157447 (8)
5.02404476674638389 (1)	4.29130683419748559 (−13)	2.82968933922063744 (9)
6.06147860634799663 (1)	4.84789862852962661 (−17)	1.02381299844288556 (10)

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