A Note on Gaussian Twin Primes

By Daniel Shanks

If $m^2 + 1$ is a prime, then $m + i$ is a Gaussian prime and conversely. If $(n - 1)^2 + 1$ and $(n + 1)^2 + 1$ are both prime, then $n - 1 + i$ and $n + 1 + i$ form a pair of Gaussian twin primes [1, p. 82]. This is the case for $n = 3, 5, 15, 25, 55, \cdots, 184705, 184745, 184755, \cdots$; the corresponding (rational) primes being 5 and 17 for $n = 3$, and 34134040517 and 34134779537 for $n = 184755$. Let $g(N)$ be the number of such pairs for $4 \leqq n + 1 \leqq N$.

Similarly, let $z(N)$ be the number of pairs of rational twin primes, $n - 1$ and $n + 1$, (such as $n = 4, 6, 12, 18, \cdots$), for $5 \leqq n + 1 \leqq N$. Hardy and Littlewood [2] conjectured that

$$z(N) \sim 1.32032 \int_2^N \frac{dn}{(\log n)^2} \tag{1}$$

where

$$1.32032 \cdots = 2 \prod_{p=3}^{\infty} \left(1 - \frac{1}{(p-1)^2}\right), \tag{2}$$

the product being taken over all odd primes.

By the use of a sieve argument very similar to that recently presented [3] in support of another Hardy-Littlewood conjecture, the following asymptotic relation was obtained:

$$g(N) \sim 0.36932 z(N), \tag{3}$$

where

$$0.36932 \cdots = \frac{1}{4} \prod_{p=3}^{\infty} \left[1 - 2\left(\frac{-1}{p}\right)(p-2)^{-1}\right]. \tag{4}$$

Here $(-1/p)$ is the Legendre symbol. Assuming the truth of both (3) and (1), we have

$$g(N) \sim 0.48762 \int_2^N \frac{dn}{(\log n)^2}. \tag{5}$$

We may compute the constant in (5), and therefore also that in (4), from

$$0.48762 \cdots = \frac{\pi^2}{8} \prod_{p=4m+1} \left(1 - \frac{4}{p}\right)\left(\frac{p+1}{p-1}\right)^2, \tag{6}$$

the product being taken over all primes of the form $4m + 1$. The evaluation of the right side of (6) is facilitated by a transformation similar to that previously used [3] in computing the Hardy-Littlewood constants, h_a.

The number of Gaussian twin pairs, $g(N)$, was determined for $N = 500(500)185000$, by counting these pairs in a recently computed table [1, p. 81] of the greatest prime factor of $n^2 + 1$ for $n = 1(1)185000$. A short summary is shown in Table 1 together with a comparison of $g(N)$ and

---

Received December 31, 1959.

TABLE 1
Gaussian and Rational Twin Primes

N	$g(N)$	$\bar{g}(N)$	$g/\bar{g}$	$z(N)$	$\bar{z}(N)$	$z/\bar{z}$
10000	76	79.1	0.961	205	214.2	0.957
20000	127	132.1	0.961	342	357.8	0.956
30000	180	179.8	1.001	467	486.7	0.959
40000	234	224.3	1.043	591	607.4	0.973
50000	276	266.8	1.034	705	722.5	0.976
60000	321	307.8	1.043	811	833.4	0.973
70000	361	347.5	1.039	905	940.9	0.962
80000	403	386.2	1.044	1007	1045.7	0.963
90000	437	424.0	1.031	1116	1148.2	0.972
100000	463	461.2	1.004	1224	1248.7	0.980
110000	502	497.7	1.009			
120000	532	533.6	0.997			
130000	568	569.0	0.998			
140000	598	603.9	0.990			
150000	629	638.4-	0.985			
160000	660	672.6	0.981			
170000	696	706.4	0.985			
180000	734	739.8	0.992			
185000	762	756.4	1.007			

(7) $$\bar{g}(N) = 0.48762 \int_2^N \frac{dn}{(\log n)^2}.$$

Also shown are Glaisher's counts [4] of $z(N)$ to $N = 10^5$ and

(8) $$\bar{z}(N) = 1.32032 \int_2^N \frac{dn}{(\log n)^2}.$$

In this range of N the deviations from unity of $g(N)/\bar{g}(N)$ and $z(N)/\bar{z}(N)$ are about equal in magnitude, [5].

The slow oscillations of $g(N)/\bar{g}(N)$ around one have two significant consequences.

1. They make improbable any value of the *constant* in (5) which differs more than slightly from the theoretical value, (6).

2. They make possible a sensitive test for the correctness of the *function of* N assigned to the proposed asymptote, $\int_2^N dn/(\log n)^2$. For since $| g(N) - \bar{g}(N) | \ll \bar{g}(N)$, even small functional modifications in $\bar{g}(N)$ would greatly alter the phase, frequency, and amplitude of the corresponding oscillations of $g(N)/\bar{g}(N)$ around one. Now consider $P(N)$, the *total* number of Gaussian primes of the form $m + i$, [1, see Table on p. 78; p. 81]. The corresponding Hardy-Littlewood conjecture reads

(9) $$P(N) \sim \bar{P}(N) = 0.68641 \int_2^N dn/\log n,$$

and similar remarks are applicable to the function $\bar{P}(N)$ and to any oscillations of $P(N)/\bar{P}(N)$. But if (9) and (5) are both valid, we must expect that any slow oscillations of $g(N)/\bar{g}(N)$ and $P(N)/\bar{P}(N)$ will agree in phase and frequency. For, where

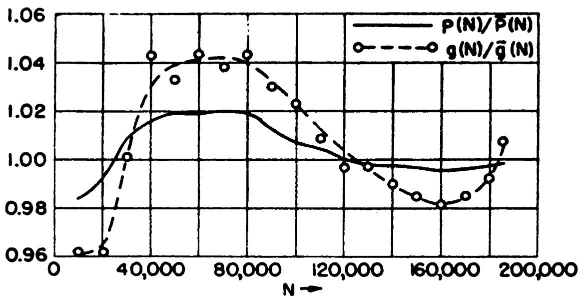

FIG. 1—Comparison of $g(N)/\bar{g}(N)$ with $P(N)/\bar{P}(N)$.

there is an excess of primes, there should generally also be an excess of twins, and if the oscillations are slow, then any complicating higher frequency fluctuations in the local density will largely disappear by integration. In Figure 1 we compare graphs of $g(N)/\bar{g}(N)$ and $P(N)/\bar{P}(N)$ for N to 185000. Very close agreement is seen in the phase and frequency of the slow oscillations. Since such an agreement would seem improbable if either or both of (5) and (9) were false, it may be regarded as providing further evidence in their favor.

The difficulties that stand in the way of a proof of (3), (assuming it to be true) are similar to those previously discussed for other problems [3]. Thus it is unlikely that (3) will be proven without a simultaneous solution of the long outstanding Goldbach, twin prime, and $n^2 + 1$ prime problems.

In conclusion it should be noted that the Gaussian twins on the line $n + i$ are by no means the only "twins" in the Gauss plane. On the line $n + 2i$, for instance, we not only have twins, $n = (179983, 179985)$, and triplets, $n = (423, 425, 427)$, but even one octuplet, $n = (-7, -5, -3, -1, +1, +3, +5, +7)$.

Applied Mathematics Laboratory
David Taylor Model Basin
Washington 7, District of Columbia

1. DANIEL SHANKS, "A sieve method for factoring numbers of the form $n^2 + 1$," *MTAC*, v. 13, 1959, p. 78–86.

2. G. H. HARDY & J. E. LITTLEWOOD, "Partitio numerorum III: On the expression of a number as a sum of primes," *Acta. Math.*, v. 44, 1923, p. 42.

3. DANIEL SHANKS, "On the conjecture of Hardy and Littlewood concerning the number of primes of the form $n^2 + a$," *Notices*, Amer. Math. Soc., v. 6, 1959, p. 417. Abstract 559–52. A forthcoming paper with the same title will give an expanded version of this report.

4. J. W. L. GLAISHER, "An enumeration of prime-pairs," *Messenger Math.*, v. 8, 1878, p. 28–33.

5. The empirical evidence for (1) is much more extensive. D. H. Lehmer has computed $z(N) = 183728$, $\bar{z}(N) = 183582$, and $z/z(N) = 1.0008$ for $N = 37 \cdot 10^6$. See the review, UMT 3, of D. H. LEHMER, "Tables concerning the distribution of primes up to 37 million," *MTAC*, v. 13, 1959, p. 56.