In the degenerate case when $y = 0$ the general result (4), (5), becomes

$$\int_{-\infty}^{\infty} [c^{2k} + (x - x^{-1})^{2k}]^{-1}\, dx = \pi k^{-1} c^{1-2k} \csc(\pi/2k). \tag{8}$$

Special results for $k = 1$, $c = \frac{1}{2}$ and $k = 2$, $c = 1/\sqrt{2}$ are

$$\int_0^{\infty} \frac{dx}{\frac{1}{4} + (x - x^{-1})^2} = \int_0^{\infty} \frac{dx}{\frac{1}{4} + (x - x^{-1})^4} = \pi, \tag{9}$$

which may be verified independently.

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On the Evaluation of Certain Complex Elliptic Integrals

By H. A. Lang and D. F. Stevens

1. Introduction. Elliptic integrals of the third kind are occasionally encountered in the form

$$a\Pi(\phi, \alpha^2, k) + \bar{a}\Pi(\phi, \bar{\alpha}^2, k)$$

or

$$a\Pi(\phi, \alpha^2, k) - \bar{a}\Pi(\phi, \bar{\alpha}^2, k)$$

where a, $\bar{a}$ and α^2, $\bar{\alpha}^2$ are complex conjugates and the modulus k is real such that $0 < k^2 < 1.0$. It is usually desirable to rewrite these integrals using only real coefficients and parameters. This paper gives an elementary procedure for the evaluation of these expressions, together with the correction of some existing formulas.

2. Modified Development. We follow the development suggested by Hoüel [1], but use the notation of Byrd and Friedman [2] wherever applicable. Since the modulus k is the same in all of the elliptic integrals considered here, it will be omitted throughout.

For convenience, set

$$\begin{aligned} 2\Pi_1 &= \Pi(\phi, \alpha^2) + \Pi(\phi, \bar{\alpha}^2) = \int_0^{\phi} \frac{d\phi}{(1 - \alpha^2 \sin^2 \phi)\Delta} + \int_0^{\phi} \frac{d\phi}{(1 - \bar{\alpha}^2 \sin^2 \phi)\Delta} \\ 2i\Pi_2 &= \Pi(\phi, \alpha^2) - \Pi(\phi, \bar{\alpha}^2) = \int_0^{\phi} \frac{d\phi}{(1 - \alpha^2 \sin^2 \phi)\Delta} - \int_0^{\phi} \frac{d\phi}{(1 - \bar{\alpha}^2 \sin^2 \phi)\Delta} \end{aligned} \tag{1}$$

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where

$$\Delta = \sqrt{1 - k^2 \sin^2 \phi}$$

Then, if $a = a_1 + ib_1$, we have

$$(2) \qquad \begin{aligned} a\Pi(\phi, \alpha^2) + \bar{a}\Pi(\phi, \bar{\alpha}^2) &= 2a_1\Pi_1 - 2b_1\Pi_2 \\ a\Pi(\phi, \alpha^2) - \bar{a}\Pi(\phi, \bar{\alpha}^2) &= 2ia_1\Pi_2 + 2ib_1\Pi_1 . \end{aligned}$$

Π_1 and Π_2 will be given by the equations

$$(3) \qquad \begin{aligned} s_1 \Pi_1 + t_1 \Pi_2 &= -\frac{1}{m_1} F(\phi) - n_1 \Pi(\phi, \alpha_1^2) + \tau_1 \\ s_2 \Pi_1 + t_2 \Pi_2 &= -\frac{1}{m_2} F(\phi) - n_2 \Pi(\phi, \alpha_2^2) + \tau_2 \end{aligned}$$

where

$$\tau_i = \int_0^{p_i} \frac{dx}{1 + h_i x^2}, \qquad p_i = \frac{\sin \phi \cos \phi}{(1 + m_i \sin^2 \phi)\Delta},$$

and the other subsidiary quantities are obtained as follows. We first set $\alpha^2 = -\gamma_1 - i\gamma_2$, $r^2 = \gamma_1^2 + \gamma_2^2$, $z = \sin^2 \phi$ in equations (1) and rationalize:

$$\Pi_1 = \int_0^\phi \frac{(1 + \gamma_1 z)\, d\phi}{(1 + 2\gamma_1 z + r^2 z^2)\Delta}$$

$$\Pi_2 = \int_0^\phi \frac{-\gamma_2 z\, d\phi}{(1 + 2\gamma_1 z + r^2 z^2)\Delta}$$

We substitute these forms in a general equation (3) to obtain*

$$(4) \qquad \int_0^\phi \frac{s(1 + \gamma_1 z)\, d\phi}{(1 + 2\gamma_1 z + r^2 z^2)\Delta} - \int_0^\phi \frac{t\gamma_2 z\, d\phi}{(1 + 2\gamma_1 z + r^2 z^2)\Delta} + \frac{1}{m}\int_0^\phi \frac{d\phi}{\Delta} + n \int_0^\phi \frac{d\phi}{(1 - \alpha^2 z)\Delta} = \int_0^p \frac{dx}{1 + hx^2}$$

Differentiating both sides of (4) with respect to ϕ we obtain

$$(5) \qquad \frac{s(1 + \gamma_1 z) - tz\gamma_2}{(1 + 2\gamma_1 z + r^2 z^2)\Delta} + \frac{1}{m\Delta} + \frac{n}{(1 - \alpha^2 z)\Delta} = \frac{1}{1 + hp^2} \frac{dp}{d\phi} .$$

The left side of this equation reduces to

$$\frac{a_0 + a_1 z + a_2 z^2 + a_3 z^3}{b_0 + b_1 z + b_2 z^2 + b_3 z^3} \cdot \frac{1}{\Delta}$$

where

$$a_0 = s + \frac{1}{m} + n \qquad\qquad b_0 = 1$$

* The two sets of subsidiary quantities arise from the two roots of a quadratic equation in m which results from this general equation. Note that the α in equation (4) is a generic expression for the α_1 and α_2 of equations (3), and is therefore distinct from the α of equations (1) and (2).

$$a_1 = s\gamma_1 - t\gamma_2 - \frac{\alpha^2}{m} + \frac{2\gamma_1}{m} - s\alpha^2 + 2n\gamma_1 \qquad b_1 = 2\gamma_1 - \alpha^2$$

$$a_2 = -s\alpha^2\gamma_1 - t\alpha^2\gamma_2 + nr^2 - \frac{2\gamma_1\alpha^2}{m} + \frac{r^2}{m} \qquad b_2 = r^2 - 2\gamma_1\alpha^2$$

$$a_3 = -\frac{\alpha^2 r^2}{m} \qquad b_3 = -\alpha^2 r^2.$$

The right side of (5) reduces to

$$\frac{c_0 + c_1 z + c_2 z^2 + c_3 z^3}{d_0 + d_1 z + d_2 z^2 + d_3 z^3} \cdot \frac{1}{\Delta}$$

where

$$c_0 = 1 \qquad d_0 = 1$$

$$c_1 = -(2 + m) \qquad d_1 = 2m - k^2 + h$$

$$c_2 = k^2(2m + 1) \qquad d_2 = m^2 - 2mk^2 - h$$

$$c_3 = -k^2 m \qquad d_3 = -k^2 m^2.$$

Setting $a_i = c_i$, $b_i = d_i$ we obtain the following relations:

$$\text{(I)} \quad s + \frac{1}{m} + n = 1$$

$$\text{(II)} \quad s\gamma_1 - t\gamma_2 - \frac{\alpha^2}{m} + \frac{2\gamma_1}{m} - s\alpha^2 + 2n\gamma_1 = -(2 + m)$$

$$(6) \qquad \text{(III)} \quad -s\alpha^2\gamma_1 + t\alpha^2\gamma_2 + nr^2 - \frac{2\gamma_1\alpha^2}{m} + \frac{r^2}{m} = k^2(2m + 1)$$

$$\text{(IV)} \quad \alpha^2 r^2 = m^2 k^2$$

$$\text{(V)} \quad 2\gamma_1 - \alpha^2 = 2m - k^2 + h$$

$$\text{(VI)} \quad r^2 - 2\gamma_1\alpha^2 = m^2 - 2mk^2 - h.$$

The elimination of h from (V) and (VI) and the use of (IV) results in a quadratic in m with real roots:

$$(r^2 + 2k^2\gamma_1 + k^2)m^2 + 2(1 - k^2)r^2 m - r^2(r^2 + 2\gamma_1 + k^2) = 0.$$

Letting m_1, m_2 be these roots, we solve equations (6) for all subsidiary quantities as follows

$$\alpha_i^2 = \frac{m_i^2 k^2}{r^2}$$

$$s_i m_i = m_i - m_i n_i - 1$$

$$n_i m_i = \frac{m_i[\alpha_i^4 - (2 + m_i)\alpha_i^2 + (1 + 2m)k^2] - r^2}{r^2 + 2\gamma_1\alpha_i^2 + \alpha_i^4}$$

$$t_i m_i \gamma_2 = m_i^2 + (\gamma_1 + 2 - \alpha_i^2)m_i + n_i m_i(\gamma_1 + \alpha_i^2) + \gamma_1$$

$$h_i = \frac{(\alpha_i^2 - k^2)(k^4 + 2k^2\gamma_1 + r^2)}{k^2(1 - k^2)}$$

If we take $m_1^2 < m_2^2$ then $0 < \alpha_1^2 < k^2 < \alpha_2^2 < 1$ and $h_1 < 0 < h_2$. In the special case where $r^2 + 2\gamma_1 + k^2 = 0$, we have $m_1 = 0 = \alpha_1^2$ and $h_1 = -r^2$. Then the first equation of (3) is replaced by

$$s_1 \Pi_1 + t_1 \Pi_2 = -\frac{k^2}{r^2} F(\phi) + \frac{1}{r} \tanh^{-1} \frac{r \cos \phi \sin \phi}{\Delta}, \tag{8}$$

the second remaining unchanged.

3. Formulas to be Corrected. It will be seen that the definitions (7) are those given by Byrd and Friedman under formulas 417.00 and 438.00 [2, p. 231, 232, 238] with the exceptions of s_i and t_i. Thus the formulas depending upon these quantities are incorrect. They are 416.00, 417.00, 418.00, 419.00, 437.00, 438.00, 439.00 and 440.00.

We also note that these same errors occurred in Hoüel [1, equations 119, p. LIII]. He gives, in his own notation

$$A = (1 - C)g - 1$$

$$Cg(\mu^2 - 2m\mu \cos \nu + m^2) = g[m^2 + (2 + g)m + (1 + 2m)k^2] - \mu^2.$$

These equations should read respectively

$$Ag = (1 - C)g - 1$$

$$Cg(\mu^2 - 2m\mu \cos \nu + m^2) = g[m^2 + (2 + g)m + (1 + 2g)k^2] - u^2.$$

To correct Byrd and Friedman formulas 416.00, 417.00, 419.00, 437.00, 438.00, and 440.00 it is necessary only to modify the definitions of s_i and t_i to those listed in (7), since as listed they are formal consequences of equations (3) and equation (8). In formulas 418.00 and 439.00, however, we must not only modify s_i and t_i, but also change the sign of the term involving $\Pi(\alpha_i^2, k)$ (in both 418.00 and 439.00), and the sign of the term involving $\tanh^{-1}(rsnu_1cdu_1)$ (in 439.00); these signs should all be negative. (See the end of the next section for a modified, corrected version of 439.00.)

4. Simplication of Results. The form of definitions (7) suggests the following simplifications in the Byrd and Friedman formulas under consideration. Let us multiply equations (3) by m_i to obtain

$$s_i\Pi_1 + t_i\Pi_2 = -F(\phi) - n_i\Pi(\phi, \alpha_i^2) + \tau_i \qquad i = 1, 2$$

where now

$$\tau_i = \int_0^{p_i} \frac{m_i\, dx}{1 + h_i x^2},$$

the p_i are defined as before, and definitions (7) are replaced by:

$$\alpha_i^2 = \frac{m_i^2 k^2}{r^2}$$

$$s_i = m_i - n_i - 1$$

$$n_i(r^2 + 2\gamma_1\alpha_i^2 + \alpha_i^4) = m_i[\alpha_i^4 - (2 + m_i)\alpha_i^2 + (1 + 2m)k^2] - r^2$$

$$t_i\gamma_2 = m_i^2 + m_i(\gamma_1 + 2 - \alpha_i^2) + n_i(\gamma_1 + \alpha_i^2) + \gamma_1$$

$$h_i k^2(1 - k^2) = (\alpha_i^2 - k^2)(k^4 + 2k^2\gamma_1 + r^2).$$

This permits us to write formulas 416.00, 417.00, 437.00 and 438.00 with no explicit appearances of m_1 or m_2. For example, 416.00 becomes:

$$416.00'\text{:}\ a\Pi(\alpha^2) + \bar{a}\Pi(\bar{\alpha}^2) = \frac{2}{s_1 t_2 - s_2 t_1}\{[a_1(t_1 - t_2) + b_1(s_1 - s_2)]K + n_2(a_1 t_1 + b_1 s_1)\Pi(\alpha_2^2) - n_1(a_1 t_2 + b_1 s_2)\Pi(\alpha_1^2)\}$$

A similar simplification is possible in the special case when $m_1 = 0 = \alpha_1^2$. In this case we multiply the second equation of (3) by m_2 as above, and multiply equation (8) by r^2 to obtain

$$s_1 \Pi_1 + t_1 \Pi_2 = -k^2F(\phi) + r \tanh^{-1} \frac{r \cos\phi \sin\phi}{\Delta}$$

where

$$s_1 = r^2 - k^2$$

$$t_1\gamma_2 = 2r^2 + r^2\gamma_1 + k^2$$

(n_2, s_2 and t_2 are defined as in (7').) This permits us to write formulas 418.00, 419.00, 439.00, and 440.00 with no explicit appearances of m_2 or r_2 (But note that the two occurrences of r in formulas 439.00 and 440.00 are preserved.) For example, 439.00 (with the sign changes mentioned in Section 4 incorporated) becomes

$$439.00'\text{:}\ a\Pi(u_1, \alpha^2) + \bar{a}\Pi(u_1, \bar{\alpha}^2) = \frac{2}{s_2 t_1 - t_2 s_1}\left\{[a_1(k^2t_2 - t_1) + b_1(k^2s_2 - s_1)]u_1 - n_2(a_1 t_1 + b_1 s_1)\Pi(u_1, \alpha_2^2) + (a_1 t_1 + b_1 s_1)r^2 - r(a_1 t_2 + b_1 s_2) \tanh^{-1} \frac{r \cos\phi \sin\phi}{\Delta}\right\}.$$

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The Numerical Evaluation of the Eighteenth Perfect Number

By D. Scheffler and R. Ondrejka

On November 17, 1959 the IBM 709 installation at the National Aviation Facilities Experimental Center in Atlantic City, New Jersey computed the largest known perfect number, corresponding to the eighteenth Mersenne prime [1]. The result was checked by recomputation one week later. Running time for this cal-

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