29[B].—George E. Reynolds, *A New Method of Cube Root Extraction on Desk Calculators*, Electronics Research Directorate, Air Force Cambridge Research Center, Air Research and Development Command, USAF, April 1958, (ASTIA Doc. No. AD 133760), vi + 27 p., multilithed.

The method consists of the use of one iteration of a third-order iterative process for computing $\sqrt[3]{N}$, with the aid of a table from which certain functions of an approximate root A can be read. The iterative formula is used in the form

$$\frac{(2A)\left(\frac{N}{A^3}+\frac{1}{2}\right)}{\frac{N}{A^3}+2},$$

both $2A$ and A^3 being read from a table. Thus only one multiplication and two divisions, plus some minor auxiliary operations, are necessary.

Three tables are provided, these being so-called five-place, six-place, and seven-place cube root tables. The author states that a maximum error of one unit in the last significant digit of the rounded answer can be achieved. A few spot checks have revealed no cases in which the statement is false. It is worth noting that the only instance discovered by the reviewer in which the result obtained from the tables differed from the correct result by one unit in the last digit occurred in the beginning of the five-place table for the range in which the tabulated value of $2A$ has 5S instead of 6S. This agrees with the author's statements on p. 11–12 concerning the location of the maximum error in the tabulated values.

The instructions as to procedure are sufficiently explicit except in respect to the number of guard figures to be kept in the quotient N/A^3. One may infer from the corrected version (furnished with the review copy as an erratum) of the example on p. 15 that the author believes one guard figure will suffice, although there is no consideration of this issue in the paper.

Equation (21) is incorrect. Its right side should read

$$(2A)(N/A^3 + \tfrac{1}{2}) \div (N/A^3 + 2).$$

The reviewer is allergic to the use (p. 9) of the colloquial phrase "several times greater than" in lieu of "several times as large as". There is no such thing (p. 13, line 3 *fb*) as *the* Newton-Raphson Method for the cube root, since there are many equivalent equations with the cube root of N as a solution. The reviewer is unfamiliar with the term (p. 14, line 8) "increasing asymptotically". Is "monotonically" the adverb intended?

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30[G].—John L. Selfridge, *On Finite Semigroups*, Dissertation, University of California, Los Angeles, multilithed typescript, iv + 185 pp.

A *system* is a set of abstract elements, together with a binary operation [in this review called multiplication] defined from the cartesian product $S \times S$ to a set V. For $a, b \in S$, the value of the product is written ab. The system is *closed* whenever $V \subset S$. The *order* of the system is the number of elements in S.

Two systems S and T with sets of values U and V are called *isomorphic* whenever there is a one-to-one function φ from $S \cup U$ to $T \cup V$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in S$. They are *anti-isomorphic* whenever $\varphi(ab) = \varphi(b)\varphi(a)$ for all $a, b \in S$. "Using isomorphism as an equivalence relation $\cdots$ [systems] are divided into *classes*, all $\cdots$ [systems] in a class being isomorphic *copies*. If a pair of classes are anti-isomorphic (i.e., they have anti-isomorphic representatives) they are called a *type*. The remaining classes are types which are anti-isomorphic to themselves" (p. 3).

If a system is closed and multiplication is associative, it is called a *semigroup*. A semigroup in which, for some p, all products of p elements are equal is called *nilpotent*. The classification and enumeration of general systems of finite order is an exceedingly difficult problem, unsolved even for finite groups. In 1954 the reviewer (with Dr. Selfridge's able assistance) wrote a code [1] which caused SWAC to punch multiplication tables for all semigroups of order 4. One of the main tasks was to arrange the 3492 tables by types, and to select one table to represent each type. In order to represent the tables compactly and deal with them arithmetically, the elements of the semigroup were given the names 0, 1, 2, 3, and the sixteen entries in a multiplication table were written row after row in a single line of 16 base 4 digits. It was convenient to select as the representative of a type of semigroup that one whose table comes first in the lexicographic order by rows. Such a multiplication table is called a *row-normal table* by Selfridge. Similarly, *column-normal table* is defined.

The above considerations were purely matters of machine convenience, devised for a limited purpose. In his dissertation Selfridge has made a systematic study of multiplication tables, both for semigroups and for more general systems, and through an analysis of tables and row-normal tables has obtained new information about the structure of the algebraic systems. This is a good illustration of the feedback from digital computation to pure mathematics. Perhaps the greatest benefit of automatic computation for any problem is the resulting increased theoretical comprehension of the problem.

In the dissertation definitions and elementary properties of multiplication tables occupy p. 1–7. For example, it is proved that there are at least $(n^2 + n)^q/(n!)(n^2!)$ types of commutative systems of order n, where $q = n(n + 1)/2$. On p. 8–14 the author enumerates all possible initial rows of a row-normal semi-group table, and proves that there is an example of a semigroup corresponding to each possibility. Certain last rows are also enumerated.

Pages 15–19 contain a classification and enumeration of all tables corresponding to nilpotent semigroups in which every product of three elements if 0. As a result, it is proved that there are exactly Σ_n nilpotent semigroups of order n in which every triple product is 0, where

$$\Sigma_n = \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} (k-i)^{(n-k)^2}.$$

The author then tabulates Σ_n and certain auxiliary quantities for $n = 1(1)7$. It is startling to find that $\Sigma_7 = 5{,}944{,}080{,}072$, a low lower bound for the number of semigroups of order 7.

Pages 20–22 list the results of the enumeration of certain algebraic systems of orders 1, 2, 3, 4, and 5. The four properties of closure, associativity, commutativity, and having a multiplicative unit are considered separately, and the list gives the number of types, classes, and tables. Some of the numbers come from the earlier theory, some come from a list of semigroups of order 5, and some apparently come from other enumerations. We find, for example, that there are 183,732 semigroups of order 5, of which (by looking at Σ_5) we see that only 11,725 are of the types considered earlier. We find also that there are 720 types of closed commutatives systems of order 4 which have a unit.

After a three-page history of the use of high-speed computers to seek and enumerate various systems, the author devotes p. 26–33 to indicating how in 1955 he and Professor T. S. Motzkin found and classified all semigroups of order 5 on SWAC, using only 256 cells of rapid-access storage. To reduce the time, only normal tables were computed. There follows a two-page study of the feasibility of carrying on to order 6. Exclusive of punch time, on SWAC the principal search code took 3 minutes for order 4, 40 minutes for order 5, and might be expected to take about 25 hours for order 6. In all cases the editing and preparation of the tables for publication remains the major part of the work.

The greatest bulk of the dissertation is the appendix on pp. 41–185, where the semigroups of order 5 are tabulated in the form of multiplication tables. It is difficult (though not impossible) for the reader to determine just what is listed; the explanation (p. 28 and 29) refers to lists L_1 and L_2, which are not the lists in the appendix! It is to be hoped that Dr. Selfridge will add an explanation to any existing copies of this appendix.

Here is the reviewer's explanation of the appendix, based partly on personal copies of L_1 and L_2. There are 1915 distinct classes of these semigroups. Consider each multiplication table to be in row-normal form, and the resulting 1915 tables to be given *class numbers* [reviewer's term] from 0 to 1914 in lexicographic order by rows. Now 405 of the 1915 semigroups (including the 325 commutative semigroups) have the property that each is anti-isomorphic to itself, while the other 1510 semigroups form 755 anti-isomorphic pairs. The author omits the table in each anti-isomorphic pair that has the larger class number, and thus obtains an ordered set of 1160 row-normal tables, representing each of the 1160 distinct types of semigroups of order 5. These are given *type numbers* [reviewer's term] from 0 to 1159 in lexicographic order by rows.

The appendix lists each of these 1160 semigroups in the order of the type numbers. For each semigroup the following five items are given across the page, as listed by an IBM tabulator on multilith masters:

1) The type number.
2) The multiplication table in row-normal form (a 5-by-5 array of digits 0 through 4).
3) The class number.

That is all the information for the commutative semigroups. For the non-commutative semigroups, there are two more pieces of information:

4) The class number of the semigroup to which the listed semigroup is anti-isomorphic. (This is omitted if the class number is the same as item 3. That a noncommutative semigroup can be anti-isomorphic to itself had not been observed by all workers with semigroups.)

5) The column-normal isomorph of the same multiplication table (omitted if the row-normal table is itself column-normal). The semigroup is anti-isomorphic to itself if and only if item 4 is omitted and item 5 is included.

There is a bibliography of 34 titles, including work of a Japanese group including Professor Takayuki Tamura. It is notable that the Japanese group, working by hand, obtained all 126 semigroup types of order 4 prior to SWAC (without error), and that they finished obtaining the semigroups of order 5 almost simultaneously with the American group of Motzkin, Selfridge, and SWAC (but with at least one error discovered by the American group). It appears that in computing semigroups it has been a reasonably fair match between Japanese without an abacus and Americans armed with an electronic digital computer! The text of the dissertation reviewed here does not refer to the Japanese computation for order 5.

Minor criticisms: On page 8, line 8, for "... an element of a semigroup ..." read "... an element a of a semigroup ... ". In the footnote on page 8, for m read a. In reference [A2] the authors' names are permuted. In this and other references the authors' first names have been carefully omitted [why is this done so often?].

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1. George E. Forsythe, "SWAC computes 126 distinct semigroups of order 4," Amer. Math. Soc., *Proc.*, v. 6, 1955, p. 443–447.

31[K].—E. L. Lehmann, *Testing Statistical Hypotheses*, John Wiley & Sons, Inc. New York, 1959, xiii + 369 p., 24 cm. Price $11.00.

This long-awaited book is a welcome addition to Wiley's fine series of texts in the different areas of modern mathematical statistics, and the author is to be thanked and congratulated for a difficult but needed job exceedingly well done. The title does not convey an adequate idea of the scope of the book, which includes material from many branches of statistics, nor of the exceedingly helpful devices used that make it possible to catch up with the most recent advances. Going through it is a refreshing and stimulating experience.

The book provides "a systematic account of the theory of hypothesis testing and of the closely related theory of estimation by confidence sets. The principal applications of these theories are given, including the one- and two-sample problems concerning normal, binomial and Poisson distributions. There is also a treatment of permutation tests and of some aspects of the analysis of variance and of regression analysis. Introductions to multivariate and sequential analysis, and to non-parametric tests are offered. Methods based on large sample considerations (χ^2 and likelihood ratio tests) are sketched. The emphasis throughout is on the various optimum properties of the procedures. These are discussed in terms of the Neyman-Pearson formulation, but against a background of decision theory which frequently permits a broader justification of the results."

The level of the treatment is set by the fact that "the natural framework for a systematic treatment of hypothesis testing is the theory of measure in abstract spaces." By unrestricted use of the abstract approach, the author is enabled to bring the prepared reader abreast of the very latest developments. For this, one should have an appreciation, if not a knowledge, of concepts in measure theory, as well as a considerable background in "classical" mathematical statistics. It should be stated, however, that all necessary theorems are stated and discussed, if not always proved.

The text is greatly enriched by some very valuable time-saving features, such as footnotes that immediately relate an outside source to the point under discussion, and annotated lists of references, appearing at the close of each chapter and frequently summarizing a paper in a single sentence, thereby giving the reader a bird's-eye view of the latest pertinent papers as well as of the earlier literature. Furthermore, for each section a set of substantial exercises, totalling over 200, is provided. Many of these are accompanied with outlines of solutions, and provide introductions to additional topics.

Attention to estimation as a special case of hypothesis testing is essentially limited to confidence sets, point estimation receiving very little consideration. For the sake of completeness one would like to have seen some treatment of the Cramèr-Rao inequality and its modern development by Bhattacharya and others, as well as minimum-variance estimation on which so much practical work is based. This, however, is hardly a criticism, since treatment of estimation is not a main purpose.

While not designed as a "cookbook" in the analysis of actual data, because of its advanced nature the book does give a deep understanding of the many tests and their relationships to a unified theory. As such, and in view of its time-saving features, the book is well worth the price, and should be in the possession of the advanced worker in mathematical statistics and others having the requisite background.

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32[L].—L. N. Karmazina & É. A. Čhistova, *Tablitsy funktsiĭ Besselia ot mnimogo argumenta i integralov ot nikh* (*Tables of Bessel functions of imaginary argument and of integrals involving them*). Izdatel'stvo Akademii Nauk SSSR (Press of the Academy of Sciences of the USSR), Moscow, 1958, 328 p., 27 cm. Price 37 rubles 15 kopecks.

This volume in the series of Mathematical Tables from the Computational Center of the Academy of Sciences is a continuation of earlier work [1] on Bessel functions of real argument.

The present tables were prepared on the electronic computer STRELA. In the main table (pages 19–328), the values of the following seven functions are given for $x = 0(.001)5(.005)15(.01)100$:

$$e^{-x}I_0(x), \qquad e^{-x}I_1(x), \qquad e^{-x}\int_0^x I_0(u)\,du,$$

$$e^{x}K_0(x), \qquad e^{x}K_1(x), \qquad e^{x}\int_x^\infty K_0(u)\,du, \quad \text{and} \quad e^{x}.$$

The values are given here to 7D except near the origin, where they are to 7S; no differences are given. Near the origin, the following auxiliary functions are given:

$$I_0(x) \quad \text{and} \quad E_0(x) = K_0(x) + \ln x\, I_0(x)$$

for $x = 0(.001).15$ to 7S and 7D, respectively;

$$I_1(x) \quad \text{and} \quad E_1(x) = x[K_1(x) - \ln x\, I_1(x)]$$

for $x = 0(.001).2$ to 7D. By means of the formulas given on page 11, a number of related integrals can be evaluated by using the present tables.

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1. É. A. ČHistova, *Tablitsy funktsii Besselia ot deistvitel'nogo argumenta i integralov ot nikh* (*Tables of Bessel functions of real argument and of integrals involving them*). Izdatel'stvo Akademii Nauk SSSR (Press of the Academy of Sciences of the USSR), Moscow, 1958. *Math. Comp.*, v. 14, 1960, p. 79–80.

33[L].—F. A. Paxton & J. E. Rollin, *Tables of the Incomplete Elliptic Integrals of the First and Third Kind*, Curtiss-Wright Corporation, Research Division, Quehanna, Pennsylvania, June 1959, 436 p., 28 cm.

This large table gives values of the elliptic integral of the third kind, which in the notation of the authors is

$$\Pi(\phi, \alpha^2, k) = \int_0^\phi \frac{d\theta}{(1 - \alpha^2 \sin^2 \theta)\sqrt{1 - k^2 \sin^2 \theta}}$$

including, as a special case, the elliptic integral of the first kind, $F(\phi, k) = \Pi(\phi, 0, k)$. Values are tabulated to 7D without differences for $\phi = 0(1°)90°$, $\alpha^2 = 0(.02)1$, $k^2 = 0(.02)1$. The values were computed on an IBM 704 by Simpson's rule. The authors appear to claim general accuracy within 10 final units, except possibly for $\sin \phi$, α^2, and k^2 near unity. The reviewer has encountered nothing which invalidates the general claim, but the exception is certainly to be noticed.

There are several ways in which certain values, especially of the complete integrals ($\phi = 90°$), may be checked from existing tables with little or no arithmetic. Using for brevity the references of *MTAC*, v. 3, 1948, p. 250, let us consider four checks:

(i) Values of $F(90°, k) = K$ with argument k^2 are given to 10–12D in Hayashi 1. They show that the values of Paxton and Rollin are systematically too small; the error rises from 4 final units at $k^2 = .02$ to 11 final units at $k^2 = .98$. These errors are practically within the claimed limits. The machine value at $k^2 = 0$, which should equal $\frac{1}{2}\pi$, is 6 final units too small, but has been corrected by hand.

(ii) Values of $F(\phi, k)$ for $\phi = 0(1°)90°$, $k^2 = \frac{1}{2}$ (modular angle $= 45°$) are given to 10D in Legendre 3, 5, 6, 7, 8 (also to 12D in Legendre 3, 5). They show no errors in Paxton and Rollin exceeding 5 final units.

(iii) Values of

$$\Pi(\phi, 0, 1) = \int_0^\phi \sec \phi \, d\phi,$$

the inverse gudermannian, are given to 9D in Legendre 3, 5, 6, 7, 8 (also to 12D in Legendre 3, 5). They show that the later values of Paxton and Rollin are systematically too small, for example by about 1, 11, 45, 141 final units at $\phi = 45°$, 85°, 88°, 89°, respectively. It may be added that values of $F(\phi, k)$ for $\phi = 0(1°)90°$, $k^2 = 0(.01)1$ are given in Samoĭlova-Îakhontova 1, but to only 5D.

(iv) When $\alpha^2 = k^2$, we have

$$\Pi(90°, \alpha^2, k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \theta)^{-3/2} \, d\theta,$$

which is known to equal $E/(1 - k^2)$, where E is the complete elliptic integral of the second kind, given to 10D with argument k^2 in Hayashi 3. Evaluation shows that the values of Paxton and Rollin are systematically too small, for example by 4, 15, 31, 177, and almost 590 units at $k^2 = .2, .8, .9, .96, .98$, respectively.

As far as the reviewer's examination has gone, it seems likely that the table is correct everywhere to about 4D, almost everywhere to 5D, and in large regions to 6D and 7D. Even with this limitation, the table (which, as far as the reviewer is aware, is a corporation research report rather than a published work) must be regarded as epoch-making in the history of the tabulation of the elliptic integral of the third kind. It should be added that another sizable table of this integral, by Selfridge and Maxfield [1], appeared in 1958, but with a different argument system.

A. F.

1. R. G. Selfridge & J. E. Maxfield, *A Table of the Incomplete Elliptic Integral of the Third Kind*, Dover Publications, New York, 1958 (also Constable, London).

34[P, W].—Armour Research Foundation, *Proceedings of the Fourth Annual Computer Applications Symposium*, 1957, sponsored by the Armour Research Foundation of Illinois Institute of Technology, 1958, x + 126 p., 23 cm. Price $3.00.

This is a collection of 15 papers based on 12 talks, two luncheon addresses and a panel discussion. Practically all the symposium concerned itself with applications of digital computers.

The program of the symposium covered two days, one devoted to a session on "Business and Management Applications," the other to a session on "Engineering and Research Applications."

The following seven papers concerned with the first subject appear in the *Proceedings*:

An Extensive Hospital and Surgical Insurance Record-Keeping System—R. J. Koch,

A Central Computer Installation as a Part of an Air-Line Reservations System—R. A. McAvoy,

Fitting a Computer into an Inventory-Control Problem—O. A. Kral,

The Problems of Planning New Metropolitan Transportation Facilities and Some Computer Applications—J. D. Carroll, Jr.,

Data-Processing Tasks for the 1960 Census—D. H. Heiser & Dorothy P. Armstrong,

The Handling of Retail Requisitions from a General Warehouse—M. J. Stoughton,

Automatic Programming for Business Applications—Grace M. Hopper.

The papers generated at the session on "Engineering and Research Applications" are as follows:

Digital Simulation of Active Air Defense Systems—R. P. RICH,

Statistical Calculations in Product-Development Research—E. B. GASSER,

Progress in Computer Application to Electrical Machine and System Design—E. L. HARDER,

How Lazy Can You Get?—A. L. SAMUEL,

The Solution of Certain Problems Occurring in the Study of Fluid Flow—L. U. ALBERS,

A Dual-Use Digital Computer for Dynamic System Analysis—E. H. CLAMONS & R. D. ADAMS,

The Status of Automatic Programming for Scientific Problems—R. W. BEMER,

Panel Discussion.

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35[S].—VAL J. ASHBY & HENRY C. CATRON, *Tables of Nuclear Reaction Q Values*, UCRL-5419, Lawrence Radiation Laboratory, University of California, Livermore, California, 1959, 330 p., 28 cm. Price \$5.00. Available from the Office of Technical Services, Department of Commerce, Washington 25, D. C.

Tables of nuclear reaction Q values have been calculated from nuclide masses, when possible, for those 42 reactions involving γ, n, p, d, t, He^3, or He^4 as either incident or product particle for about 650 target nuclides. Approximately 8000 Q values are tabulated.

AUTHORS' SUMMARY

36[S, T].—JAMES MILLER, JOHN M. GERHAUSER & F. A. MATSEN, *Quantum Chemistry Integrals and Tables*, University of Texas Press, Austin, Texas, 1959, 1224 p., 26 cm. Price \$15.00.

One of the major difficulties in making quantum-mechanical calculations of the properties of atoms and molecules is the evaluation of the large number of difficult integrals which appear. This volume contains tables for the evaluation of the one- and two-center $1s$, $2s$, and $2p_\sigma$ integrals involved in energy and dipole moment calculations. No $2p\pi$ integrals are included. The tables are based on the usual Slater-type atomic orbitals. Molecular integrals are not tabulated directly, but rather auxiliary functions (A, B, G, and W in the usual Kotani notation). Some computation is therefore still necessary to arrive at a desired molecular integral, but it is within the reach of a desk calculator.

The tables were computed on an IBM 650 and reproduced by a photo-offset process to avoid introduction of errors. They appear quite clear and legible. The short textual parts of the book contain formulas as well as recommended interpolation procedures and their expected accuracy—a most welcome feature. Over 90 per cent of the pages are devoted to the difficult W functions.

The present tables naturally invite comparison with previous tables of molecular integrals, particularly those by Kotani, Amemiya, Ishiguro, and Kimura

[1], and by Preuss [2]. These both contain a wider variety of functions over a somewhat greater range of arguments but they are much briefer. Interpolation is consequently often a major difficulty in these tables, whereas interpolation is relatively easy in the present tables. This can be illustrated roughly by a comparison of the numbers of pages: 230 in Kotani, 305 in Preuss (2 vols.), and 1224 in the volume under review.

A number of research groups interested in quantum chemistry have access to their own high-speed computers, and the present tables will probably not be used much by them except to check out new codes, but a large number of workers whose computational aids are limited, for the most part, to desk calculators will be very happy to see this book. The authors and publisher are to be commended for their efforts in making these tables available.

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1. M. Kotani, A. Amemiya, E. Ishiguro & T. Kimura, *Table of Molecular Integrals*, Maruzen Co., Ltd., Tokyo, 1955.
2. H. Preuss, *Integraltafeln zur Quantenchemie*, Springer-Verlag, Berlin, vol. I, 1956, vol. II, 1957.

37[X].—L. Ivan Epstein, *Nomography*, Interscience Publishers, Inc., New York, 1958, x + 134 p., 24 cm. Price $4.50.

The contents are: Ch. I, Determinants; Ch. II, Nomograms; Ch. III, Projective Transformations; Ch. IV, Matrix Multiplication; Ch. V, More Than Three Variables; Ch. VI, Empirical Nomography; Ch. VII, Kellogg's Method; Ch. VIII, Nonprojective Transformations; Bibliography and Index.

According to the preface, Chs. I, II, III, V, and VI form an elementary text, for which only a knowledge of the elements of analytical geometry is required. Omission of Ch. IV, it is said, will not cause a loss of continuity. For Chs. VII and VIII, a knowledge of calculus is expected. The book attempts, according to the author, to fulfill a need for a book which "combines the discussion and methods of construction with a thorough presentation of the underlying theory", as well as to "make the presentation mathematically rigorous insofar as this could be done on a relatively elementary level".

The book has much to recommend it. The figures are good and the prose in most instances is lucid. The idea of presenting the notion of projective transformations from the "geometrical" point of view first seems to have merit, and useful descriptive terms such as shear and stretch assist in this endeavor. The entire approach to the subject is from the "determinant" point of view, which is desirable. Furthermore, the necessary properties of determinants and matrices are included (Chs. I and IV) for the benefit of the reader unfamiliar with them. In addition, the author does not hesitate to make the natural extension of using the word nomograph as a verb.

On the other hand, there are errors in the book. These are mathematical, pedagogical, grammatical or typographical.

Among the mathematical errors may be mentioned the definition of linearly related functions (p. 93), and some of its consequences. The statement of Theorem

7.1 ("A set of functions of a single variable are linearly related if and only if their wronskian vanishes") shows that the author intended to define what are usually called linearly dependent functions, since Theorem 7.1 would be true if the word "related" were replaced by "dependent," but is *not* true for the definition given, namely, "we shall say that four functions $f_1(x), f_2(x), f_3(x), f_4(x)$ are linearly related if there exist four *nonzero* [emphasis supplied by the reviewer] quantities c_1, c_2, c_3, c_4 independent of x such that $c_1f_1 + c_2f_2 + c_3f_3 + c_4f_4 = 0$". The word *nonzero* should, of course, be replaced by the phrase *not all zero*. Incidentally, "Wronskian" is usually capitalized.

Neither in Ch. VIII nor elsewhere was the reviewer able to discover the author's definition of the phrase "nonprojective transformation". Yet Ch. VIII, based largely on a paper by Gronwall (ref. 5 in the bibliography), has the phrase as its title. The reviewer was also unable to discover the phrase anywhere in Gronwall's paper, from which Theorem 8.1 (concerned with nonprojective transformation) is stated to have been taken.

The reviewer suggests that a worked example involving at least one scale with a curved support would have been appropriate for Ch. II. On p. 23, for equation (2.25) it would have been better to define the units in which the various physical quantities are measured. On p. 83, "antiparallel" should be defined. The reviewer doubts that a reader with merely a knowledge of the calculus would be able (as seems to be implied in the preface) to understand adequately the material of Chapters VII and VIII, even if these chapters were error-free. It is also suggested that too much may have been left as an exercise for the reader in several places.

In conclusion, the reviewer suggests that the unsophisticated reader follow the suggestion made in the preface that Chs. I, II, III, V, and VI form an elementary text. Ch. IV seems to be relatively elementary also. The sophisticated reader should go to the original papers of Kellogg and Gronwall, on which Chapters VII and VIII are respectively based.

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38[X].—ALEXANDER S. LEVENS, *Nomography*, 2nd edition, John Wiley & Sons, New York, 1959, vii + 296 p., 25 cm. Price $8.50.

Professor Levens' *Nomography* is judged by this reviewer to be an excellent elementary textbook. It reflects the author's mastery of pedagogic technique. This work should lead the student to more than the acquisition of a theoretical knowledge of nomography—it should enable him to become a skilled and experienced nomographer. To develop the requisite skill, the text abounds with realistic problems, taken in many cases from practical engineering or physical situations.

The geometric approach in this book is both simple and direct. Perhaps some will find the text overly simple in parts, with too many explicit steps. However, it is likely that most students will appreciate its clarity and ease of reading.

After a brief introduction and a careful discussion of functional scales, the author divides the study into various nomogram types, which are taken up serially. The geometry of each general type is thoroughly discussed and fully illustrated with a detailed application. The geometric approach in this text is considered to be a far better introduction to nomography than an analytical approach by the theory of determinants. Basic analytic theory is not neglected, however, and a chapter on determinants is included near the end.

The second edition is considered a definite improvement over the first; it has a more pleasing format and type arrangement and an over-all attractiveness that gives it more pedagogic appeal. (One minor defect: chapter numbers on the top of each page of the first edition were unaccountably omitted in the second edition.)

Besides numerous text revisions, important new material appears in the second edition. Most significant are: (a) the expansion of the chapter on determinants, (b) the addition of a chapter on projective transformations, and (c) the addition of a chapter indicating the relationship between concurrency and alignment nomograms (with applications to experimental data, including a description of the rectification of experimental curves).

An appendix supplies an assortment of nomographic solutions to different problems taken from various technical fields, and offers fine illustration of available techniques.

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39[X, Z].—Lyle R. Langdon, *Approximating Functions for Digital Computers.* Reprinted from *Industrial Mathematics* v. 6, 1955, p. 79–100.

This article is concerned with several methods for determining approximations to functions of one real variable. The methods mentioned include the use of Padé approximants, a modification of the Taylor expansion said to be due to Obrechkoff, the use of Chebyshev polynomials to economize truncated power series in the sense of Lanczos, the use of a rational interpolant which collocates $f(x)$ at five points, as well as some special devices based on a study of the particular function to be approximated.

The following approximations are given:

Function	Nature of approximation	Range of x	Stated upper bound for error
$\sin x$	Rational	$-\pi \leqq x \leqq \pi$	6×10^{-9}
$\cos x$	Rational	$-\pi \leqq x \leqq \pi$	1×10^{-9}
$\tan x$	Rational	$-\pi/4 \leqq x \leqq \pi/4$	7×10^{-8}
e^x	Rational	$-\pi \leqq x \leqq \pi$	1 unit in signifi digit
$\sqrt{x}$	Rational	$0.1 \leqq x \leqq 10.0$	Something in 5th significant digit
$\sin \dfrac{\pi x}{2}$	Polynomial	$-1 \leqq x \leqq 1$	4×10^{-9}
$\cos \dfrac{\pi x}{2}$	Polynomial	$-1 \leqq x \leqq 1$	7×10^{-9}

Function	Nature of approximation	Range of x	Stated upper bound for error
$\log_e x$	Polynomial in $y = \frac{x-a}{x+a}$, a being a suitably chosen constant	$0.10 \leqq x \leqq 100.0$ (for various choices of a)	3×10^{-9}
$\tan^{-1} y$	Polynomial	$-(\sqrt{2}-1) \leqq y \leqq (\sqrt{2}-1)$	4×10^{-9}

(used to represent $\tan^{-1} x$ on $0 \leqq x \leqq 1$ and $1 \leqq x \leqq \infty$ by expressing y as the ratio of two appropriately chosen linear functions of x)

Function	Nature of approximation	Range of x	Stated upper bound for error
$\sin^{-1} x$	Polynomial	$-\frac{1}{2} \leqq x \leqq \frac{1}{2}$	3×10^{-9}
$\sin^{-1} x$	$\frac{\pi}{2} - \sqrt{1-x}\cdot$polyn. in x	$0.966 \leqq x \leqq 1.0$	3×10^{-9}
$H(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$	Polynomial	$0 \leqq x \leqq 1$	6×10^{-10}
	Polynomial	$1 \leqq x \leqq 2$	2×10^{-10}
	Polynomial	$2 \leqq x \leqq 3$	8×10^{-9}
	Polynomial	$3 \leqq x \leqq 4$	7×10^{-9}
$\cosh x \cos x$	Polynomial	$-\frac{\pi}{2} \leqq x \leqq \frac{\pi}{2}$	1×10^{-10}
$\cosh x \sin x + \sinh x \cos x$	Polynomial	$-\frac{\pi}{2} \leqq x \leqq \frac{\pi}{2}$	2×10^{-10}
$\sinh x \sin x$	Polynomial	$-\frac{\pi}{2} \leqq x \leqq \frac{\pi}{2}$	1×10^{-8}
$\cosh x \sin x - \sinh x \cos x$	Polynomial	$-\frac{\pi}{2} \leqq x \leqq \frac{\pi}{2}$	3×10^{-9}

(Linear combinations of the last four functions give such functions as

$$e^{\pi x/2} \sin \pi x/2, \int e^x \sin x dx, \text{ etc.})$$

In addition, approximations of certain functions occurring in *Gas Tables* by J. H. Keenan & Joseph Kaye, Wiley & Sons, 1945, are given.

The paper has been reprinted in both an "uncorrected" and a "corrected" version. The following typographical errors were found to remain in the "corrected" version:

Page		
87	line 4 *fb*	Decimal point is missing from .00832 $86830x^5$, and x^7 is missing from 0.00019 $22123x^7$.
89	lines 15, 17, 18, 23, 25	Approximate value of $\sqrt{2} - 1$ should end in 4 instead of 3.
90	lines 8, 10	
93–4	Eqs. (15), (16), (17)	Prefix a_o to the given integral.
93	Eq. (15)	Summation should be $\sum_1^9 a_n(2x-3)^n$.
99	Range of h for Eq. (26)	Should read $395.74189\,76 < h \leqq 513.4$.

The reviewer compared Langdon's results with those of Cecil Hastings, Jr., *Approximations for Digital Computers*, where possible. Although both authors considered approximations for $\tan^{-1} x$, $\sin (\pi/2)x$, $\sin^{-1} x$, and $H(x)$, the only one of these in which the ranges are the same $(-1 \leqq x \leqq 1)$ is that for $\sin (\pi/2)x$. Here both authors use a five-term polynomial approximant; Langdon claims an error bound of 4×10^{-9}, compared to Hastings' claim of 5×10^{-9}. It is of interest to note that C. W. Clenshaw approximated the same function on the same range with a six-term polynomial, the stated error bound being 3×10^{-9} (*MTAC*, v. 1954, p. 143).

The reviewer made some spot checks of the approximations given and found no evidence that the error bounds or coefficients are incorrect, with the exception that the value of $H(2.5)$ was found to be 0.99959 30388, as compared with the value 0.99959 30480 given in the NBS *Tables of Probability Functions*, v. 1. Here the error in the value obtained from Langdon's approximation exceeds the stated bound of $8 \cdot 10^{-9}$.

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40[Z].—H. FREUDENTHAL, "Logique Mathématique Appliquée," *Collection de Logique Mathématique*, Série A, XIV, Gauthier-Villars, Paris, 1958, 57 p. Price $3.00.

This pamphlet is a prize-winning essay in a 1953 contest of the Institute for the Unity of Sciences on the general subject, Mathematical Logic as a Tool of Analysis—its uses and achievements in the sciences and philosophy. For the purposes of the present publication the most interesting portion is the second section entitled "Le calcul des propositions, les réseaux electriques et les machines à calculer." This gives a brief description of various electronic and electro-mechanical realizations of the propositional calculus, introduces the temporal problem, and suggests research in a temporally affected propositional calculus. Unfortunately, the span of five years between initial composition and publication means that no knowledge of the recent work on such temporal structures as the neuron model is demonstrated. The author's point that mathematical logicians should enter the field rather than surrender it to computer engineers retains its initial force.

Briefly, the introductory section comments on what the author considers as an unfortunate tendency to abstract, non-realizable research by logicians. (He concedes the right of number theorists to an ivory tower but thinks that logic must be primarily viewed as a part of applied mathematics.) The third section reviews certain clarifications introduced in philosophy and the foundations of science by the influence of mathematical logic. The fourth suggests an operational definition of implication in terms of an idea of "complete implication" which requires that all pairs (p, q) such that $p \rightarrow q$ be meaningful in a particular case before using the term "implication." The final portion suggests problems involving transitivity in modal forms.

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41[Z].—JOACHIM JEENEL, *Programming for Digital Computers*, McGraw-Hill Book Co., Inc., New York, 1959, viii + 517 p., 24 cm. Price $12.00.

This book is designed for the reader who wishes to become acquainted with stored-program calculators. It is not necessary for the reader to have had previous programming experience; however, a background in college mathematics is desirable. The principles and procedures presented apply mainly to the solution of scientific and technical problems. Some discussion of commercial data processing techniques is also given. The book covers recent developments in the field of programming. The approach is general, in that the procedures which are presented can apply to any stored-program calculator.

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