REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

42[F].—A. Gloden, *Table des Solutions de la Congruence* $x^4 + 1 \equiv 0 \pmod{p}$ *pour* $800\,000 \leqq p \leqq 1\,000\,000$, published by the author, rue Jean Jaurès, 11, Luxembourg, 1959, 22p., 30 cm., mimeographed. Price 150 Belgian francs.

This volume represents the culmination of independent efforts of table-makers such as Cunningham, Hoppenot, Delfeld, and the author, extending over a period of several decades. The foreword contains an extensive list of references to earlier publications of this type, which combined with the tables under review give the two least positive solutions of the congruence $x^4 + 1 \equiv 0 \pmod{p}$ for all admissible primes less than one million. [See RMT **109,** *MTAC*, v. 11, 1957, p. 274 for a similar list of references.]

Professor Gloden has used such congruence tables in the construction of manuscript factor tables of integers $N^4 + 1$, which now extend to $N = 40\,000$, with certain omissions. The bibliography in the present set of tables also contains references to this work. Numerous references to these factor tables are also listed in RMT **2,** *MTAC*, v. 12, 1958, p. 63.

J. W. W.

43[G, X].—F. R. Gantmacher, *Applications of the Theory of Matrices*, translated by J. L. Brenner, Interscience Pub., New York, 1959, ix + 317 p., 24 cm. Price $9.00.

This is a remarkable book containing material, not easily available elsewhere, related to the analysis of matrices as opposed to the algebra of matrices. In this I use the word analysis to mean broadly that part of mathematics largely dependent upon inequalities (and limits) as opposed to algebra, which depends largely on equalities.

In particular, the material in this book is directed largely toward studies of stability of solution of linear differential equations (in Chapters IV and V) and of matrices with nonnegative elements (in Chapter III).

Several aspects of the book will be useful to numerical analysts. These include some parts of the chapter on matrices with nonnegative elements, the implications of the chapters on stability of solutions of differential equations to the stability of numerical methods of solving differential equations, and (as the author points out) a numerically feasible method of finding the roots of polynomials.

Many topics included in this text are not easily available elsewhere. For example, product integration is expounded; this is an amusing version of Euler's method applied to the solution of linear first-order homogeneous differential equations. Most of the other exposition is unique in one way or another, and on the whole the book is a valuable contribution to literature.

This is a translation, augmented to some extent by bibliographic and other notes, of the second part of a successful Russian book. It is interesting to note that another publisher has announced the impending publication of translations of both parts.

The printing is good, and the reviewer noticed no serious errors. There are four short appendices devoted to standard elementary theorems, a bibliography with seventy-two entries, and a useful index. The chapter headings follow:

Chapter I. Complex Symmetric, Antisymmetric and Orthogonal Matrices
Chapter II. Singular Bundles of Matrices
Chapter III. Matrices with Nonnegative Elements
Chapter IV. Applications of the Theory of Matrices to the Study of Systems of Linear Differential Equations
Chapter V. The Routh-Hurwitz Problem and Related Questions.

C. B. T.

44[K].—Joseph Berkson, "Tables for use in estimating the normal distribution by normit analysis," *Biometrika*, v. 44, 1957, p. 411–435.

In a quantal response assay a number of independent tests are made at each of a number of dose levels, and the result of each test is graded as "success" or "failure". If the probability of "success" at dose metameter value x is assumed to follow the "normal" law

$$P(x) = 1/\sqrt{2\pi} \int_{-\infty}^{(x-\mu)/\sigma} e^{-t^2/2}\, dt$$

the method of "normit analysis" is proposed by Berkson as a replacement for the familiar (iterative) method of "probit analysis" for estimating the parameters μ and σ.

Suppose that the dose metameter values used are $x_1, \cdots, x_k$, that n_i tests are made at level x_i, and that r_i of these tests result in success. Let

$$p_i = \begin{cases} 1/(2n_i) \text{ if } r_i = 0 \\ r_i/n_i \text{ if } 0 < r_i < n_i \\ 1 - 1/(2n_i) \text{ if } r_i = n_i, \end{cases}$$

$X_i = X(p_i)$, where $X(p)$ is defined by the relation

$$p = (1/\sqrt{2\pi}) \int_{-\infty}^{X(p)} e^{-u^2/2}\, du,$$

$$Z_i = (1/\sqrt{2\pi})\, e^{-X_i^2/2},$$

$$w_i = Z_i^2/p_i(1 - p_i).$$

The method consists of a weighted regression analysis, which is facilitated by tables which give for each p_i the corresponding values of w_i and w_iX_i.

In table 2 w_i and w_iX_i are given to $6D$ for $p = 0.001(0.001)0.500$. (For $p > \frac{1}{2}$, $w(p) = w(1 - p)$ and $wX(p) = -wX(1 - p)$.) For moderate n_i interpolation may be avoided by use of Table 1, which gives w_i and w_iX_i, also to $6D$, for all combinations of r_i and n_i for which $1 < n_i \leqq 50$ and $0 \leqq r_i \leqq n/2$. (For $r > n/2$, $w(r, n) = w(n - r, n)$ and $wX(r, n) = -wX(n - r, n)$.) It is stated that the entries in both tables are correct to within ± 1 in the final digit.

Paul Meier

University of Chicago
Chicago, Illinois

45[**K**].—I. D. J. Bross & E. L. Kasten, "Rapid analysis of 2 x 2 tables", Amer. Stat. Assn., *Jn.*, v. 52, 1957, p. 18–28.

Conventional statistical analysis of 2 x 2 tables such as

Sample	A	$\bar{A}$	
1	a	b	$N_1 = NP$
2	c	d	$N_2 = NQ$
	T		N

involves use of triple-entry tables for critical values of a. These tables are entered with N, N_1, and T, or some equivalent combination of three numbers. The body of the table then usually gives critical values for the observation a. The authors remark that the statistical test is relatively insensitive to variation in N and propose to reduce the complexity of the tabular entry to double entry by ignoring N and using only the parameters T and P. Charts I to IV inclusive present lower tail critical values for a at 5 %, 2.5 %, 1 % and .5 % levels of probability for $.1 < P < .9$ and $5 \leqq T \leqq 49$. Interchange of P and Q produces lower tail critical values for c (and by subtraction) upper tail critical values for a.

The authors claim that the approximation is good, provided P and Q are both at least .1 and T is not larger than $.2N$.

Leo Katz

Michigan State University
East Lansing, Michigan

46[**K**].—F. E. Clark, "Truncation to meet requirements on means," Amer. Stat. Assn., *Jn.*, v. 52, 1957, p. 527–536.

The problem under consideration is that of truncating a given distribution so that the resulting population will meet specified sampling requirements. This problem arises when one wishes to screen the output of some production process in order to reduce the risk (probability) of having lots rejected on the basis of a requirement that only those lots will be accepted for which the mean $\bar{X}$ of a random sample of n items shall, for example, exceed or be less than some value, say UAL (upper average level) or LAL (lower average level).

Methods are given for determining a single point of truncation A such that the mean $\bar{X}_A$ of a random sample from a normal population (μ, σ) screened or truncated at A will meet a specification requirement $\bar{X}_A \geqq$ LAL or $\bar{X}_A \leqq$ UAL with risk of rejection r.

Methods are also given for determining double points of truncation A and B such that a normal population (μ, σ) truncated at $X = A$ and at $X = B$ will meet the requirement LAL $\leqq \bar{X}_{AB} \leqq$ UAL with risk r.

As aids in carrying out the computations involved in the above methods, a table is included which lists values of the mean μ_{ab} and the standard deviation σ_{ab} of the standard normal population $(0, 1)$ truncated at a and at b $(a \leqq b)$. Entries are given to $4D$ for $a = -3.00(.25).50$ and for $b = 3.00(-.25)0$. A chart is included which contains curves of constant μ_{ab} and σ_{ab} for fixed degrees of truncation p, where p is the proportion of the complete population which is eliminated by truncation. In this chart, a extends from -3.0 to 0.5, b extends from -0.5 to $+3$, $p = .05$, .10(.10)1.0, $\mu_{ab} = -1.0(.1)1.0$, $\sigma_{ab} = 0(.1).9$. A second chart contains a set of five curves for selected values of n and r to be used in determining a and p as a function of h, where $h = (\mu - \text{LAL})/\sigma$. Values of a extend from -1.4 to 0.4, h extends from -1.0 to 0.4, and p from .10 to .65.

A. C. Cohen, Jr.

University of Georgia
Athens, Georgia

47[K].—W. H. Clatworthy, *Contributions on Partially Balanced Incomplete Block Designs with Two Associate Classes*, NBS Applied Mathematics Series, No. 47, U. S. Government Printing Office, Washington 25, D. C., 1956, iv + 70 p., 26 cm. Price $.45.

This publication contains six papers dealing with various aspects (enumeration, dualization, and tabulation) of partially balanced incomplete block designs with two associate classes, and with the construction of some new group divisible designs, triangular incomplete block designs, and Latin square type designs with two constraints. Approximately 75 new designs not contained in the monograph of Bose, Clatworthy, and Shrikhande [1] are given in the present paper. A number of theorems are proved in the six papers. Two of the theorems give bounds on the parameters v, p_{11}^1, and p_{12}^1 in terms of the parameters r, k, n_1, n_2, λ_1, and λ_2 of the partially balanced incomplete block design with two associate classes. The two theorems on the duals of partially balanced designs are useful in identifying certain partially balanced incomplete block designs.

W. T. Federer

Cornell University
Ithaca, New York

1. R. C. Bose, W. H. Clatworthy & S. S. Shrikhande, *Tables of Partially Balanced Designs with Two Associate Classes*, North Carolina Agricultural Experiment Station Technical Bulletin No. 107, 1954.

48[K].—W. J. Dixon, "Estimates of the mean and standard deviation of a normal population," *Ann. Math. Stat.*, v. 28, 1957, p. 806–809.

Four estimates of the mean in samples of N from a normal population are compared as to variance and efficiency. These are (a) median, (b) mid-range, (c) mean of the best two, (d) $\bar{X}_{)1,N(} = \sum_{i+2}^{N-1} [X_i/(N - 2)]$. The sample values are denoted $X_1 \leqq X_2 \leqq \cdots \leqq X_N$. The results for the median and mid-range are given primarily for comparison purposes, since results are well known. The mean of the best two is reported as the small sample equivalent of the mean of the 27th and 73rd percentiles.

The variance and efficiency are given to $3S$ for $N = 2(1)20$. The estimate (d) is compared to the best linear systematic statistics (BLSS) as developed in [1] and [2]. It is noted that the ratio Var (BLSS)/Var ($\bar{X}_{)1,N(}$) is never less than 0.990.

Two estimates of the standard deviation are given in Table II. One, the range, is well known. The quantity k which satisfies $E(kW) = \sigma$ is tabulated to $3D$ for $N = 2(1)20$. Denote the subranges $X_{N-i+1} - X_i$ by $W_{(i)}$ and $W_{(1)} = W$. The estimate $S' = k'(\sum W_{(i)})$, where the summation is over the subset of all $W_{(i)}$ which gives minimum variance, is the other estimate. Table II compares variances and efficiences of these two estimates to $3S$ for $N = 2(1)20$. Also a column gives the ratio of the variance of (BLSS) as given in [2] to the variance of S' to $3D$ for $N = 2(1)20$.

J. R. VATNSDAL

State College of Washington
Pullman, Washington

1. A. K. GUPTA, "Estimation of the mean and standard deviation of a normal population for a censored sample", *Biometrika*, v. 39, 1952, p. 260–273.

2. A. E. SARHAN & B. G. GREENBERG, "Estimation of location and scale parameters by order statistics from singly and doubly censored samples. Part I", *Ann. Math. Stat.* v. 27, 1956, p. 427–451.

49[K].—H. F. DODGE & H. G. ROMIG, *Sampling Inspection Tables*, Second Edition, John Wiley & Sons, Inc., New York, 1959, xi + 224 p., 29 cm. Price $8.00.

The first feature one notices about the second edition of the Dodge-Romig tables, as compared to the first edition, is its size. Measuring $8\frac{1}{2}''$ x $11''$, with a total of 224 pages, it stands in bold contrast to the pocket-sized $5\frac{1}{2}''$ x $8\frac{1}{2}''$, 106-page first edition,—a four-fold increase, in a sense indicative of the growth of statistical quality control since the first edition was published in 1944.

There are 60 pages of text covering an introduction, and four chapters which describe the principles, procedure for application, and the mathematics of the sampling plans. The titles of these chapters are, respectively: A Method of Sampling Inspection; Single Sampling and Double Sampling Inspection Tables; Using Double Sampling Inspection in a Manufacturing Plant; and Operating Characteristics of Sampling Plans. The remaining 158 pages include a table of contents, seven appendices, and an index. The last four appendices give the same four sets of tables as appear in the first edition; these are respectively entitled: Single Sampling Tables for Stated Values of Lot Tolerance Per Cent Defective (LTPD) with Consumer's Risk of 0.10; Double Sampling Tables for Stated Values of Lot Tolerance Per Cent Defective (LTPD) with Consumer's Risk of 0.10; Single Sampling Tables for Stated Values of Average Outgoing Quality Limit (AOQL); Double Sampling Tables for Stated Values of Average Outgoing Quality Limit (AOQL). The first three appendices are devoted to 120 pages of operating characteristic curves; these are, respectively, OC Curves for all Single Sampling Plans in Appendix 6; OC Curves for all Double Sampling Plans in Appendix 7; and OC Curves for Single Sampling Plans with $c = 0, 1, 2, 3$, and $n \leqq 500$ (based on binomial probabilities).

As can be seen from the above list of contents, the largest part of the increase in size is due to the inclusion of three sets of operating characteristic curves,—two sets for the AOQL plans, and one set for a separate inventory of single sampling plans. This separate inventory has a wide enough range in sample size and acceptance numbers to provide a useful independent reference of OC curves for those who prefer to derive their own single sampling plans (charts to derive such plans have been retained from the first edition). Although separate sets of OC curves are not given for the LTPD plans, the authors point out how these may be obtained or estimated from the OC curves which are given.

It is heartening to see this inclusion of OC curves and the authors' comment that it "has been urged over the years by a number of engineers", as well as the inclusion of a completely new chapter devoted to a discussion of the operating characteristic curves. This overt endorsement by this distinguished team in the field of quality control of the use of the OC curve to evaluate an acceptance sampling plan should impress upon quality control practitioners the importance of describing the assurance provided by a decision-making procedure in probability terms. Perhaps it will also emphasize the fact that while one may prefer to attach the label of LTPD plan, AOQL plan, or AQL plan to an acceptance sampling plan, depending on first considerations in deriving or classifying a plan, these sampling plans are one and the same as far as assurance in decision making is concerned, if they have the same operating characteristic curve.

The clear distinction made by the authors between OC curves giving the probability of lot acceptance based on lot quality as distinguished from process quality is most welcome, as this distinction is seldom clearly made. That there is a difference is often not realized; sometimes it is misunderstood and the importance of the difference exaggerated; at best it is ignored, since most often, but not always, the difference in OC curves is slight, as the authors point out. It is, however, unfortunate, in this reviewer's opinion, that the authors chose to attach the special labels of Type A and Type B to the corresponding OC curves, rather than simply to identify them as finite lot quality and infinite lot or process quality OC curves. The additional labels are not essential, and can do nothing more than add mystery to an already confused situation to the many who will not look beneath the labels. Unfortunately, also, an inaccuracy has crept into a statement with regard to these OC curves. On page 59 starting at the bottom of the first column the authors state, "When the sample sizes are a larger percentage of the lot size, the Type A OC curve will fall somewhat below the Type B curve shown on the chart, as can be seen in Fig. 4-1 where the Type A OC Curve for $N = \infty$ is identically the Type B OC curve." That the statement is inaccurate can indeed be seen from a careful examination of Fig. 4-1, since the finite lot (Type A) and infinite lot (Type B) OC curves intersect and cross. It would be better to remember that for the same sampling plan the OC curve for a finite lot is always more discriminating than the OC curve for an infinite lot.

Little need be said about the tables of sampling plans, which are well known as a result of the first edition. Derived on the principle of minimizing the total amount of inspection, sampling as well as screening of rejected lots, they are particularly suited to producers who are responsible for both the production and sampling inspection of their finished product. The sampling plans may also be used by a purchaser for acceptance inspection, but the choice of plan for this purpose should be based principally on the properties of the operating characteristic curve.

The format and typesetting, including tables and graphs, are considerably improved over the first edition. The division of each page of text into two columns also improves on readability. A minor typographical error which occurs in the second edition, but not in the first, appears on page 33, equation (2-1a), where C_N^M should be C_m^M.

The book can be highly recommended to those with modest or little mathematical background. The improvements in the second edition are sufficient to warrant its own place, along with other worthy texts, on the bookshelf of students and practitioners of quality control who are interested in a comprehensive account of sampling inspection as well as in the procedures and tables for its application.

HARRY M. ROSENBLATT

Federal Aviation Agency
Washington, District of Columbia

50[K].—C. W. DUNNETT & R. A. LAMM, "Some tables of the multivariate normal probability integral with correlation coefficients $\frac{1}{3}$," Lederle Laboratories, Pearl River, New York. Deposited in UMT File.

The probability integral of the multivariate normal distribution in n dimensions, having all correlation coefficients equal to ρ (where necessarily $-\dfrac{1}{n-1} < \rho < 1$), is given by

$$\int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_m} \left(\frac{1}{2\pi}\right)^{n/2} \frac{[1 + (n-1)\rho]^{-1/2}}{(1-\rho)\,\frac{(n-1)}{2}} \exp\left[\frac{1 + (n-2)\rho}{(1-\rho)[1 + (n-1)\rho]} \cdot \left\{\sum x_i^2 - \frac{2\rho}{1 + (n-2)\rho} \sum_{i \neq j}\sum x_i x_j\right\}\right] dx_1 \cdots dx_n$$

This function, which we shall denote by $F_{n,\rho}(x_1, \cdots, k_n)$, has been tabulated for $\rho = \frac{1}{2}$ and $x_1 = \cdots = x_n$ by Teichroew [1]. In the present paper, we present a table for the case $\rho = \frac{1}{3}$ and $x_1 = \cdots = x_n$. The need for this table arose in connection with a multiple-decision problem considered by one of the authors [2].

In computing the table, use was made of the fact that, for $\rho \geqq 0$, $F_{n,\rho}(x_1, \cdots, x_n)$ belongs to a class of multivariate normal probability integrals which can be written as single integrals (see Dunnett and Sobel [3]), a fact which greatly facilitates their numerical computation. In this case, we have

$$F_n,\ \rho(x_1, \cdots, x_n) \equiv \int_{-\infty}^{+\infty} \prod_{i=1}^{n} \left[F\left(\frac{x_i + \sqrt{\rho}y}{\sqrt{1-\rho}}\right)\right] f(y)\, dy$$

where

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \quad \text{and} \quad F(y) = \int_{-\infty}^{y} f(y)\, dy.$$

The attached table was computed by replacing the right-hand side of this identity by the series based on the roots of Hermite polynomials described by Salzer *et al.* [4]. Those tabular values marked with an asterisk have been checked by comparison with the values obtained by applying Simpson's rule. The values checked were found to be systematically less than the Simpson's rule values by an amount which varied between .0000000 and .0000013, depending on n. This indicates that the error in the tabular values may be no more than 1 or 2 units in 6th decimal place, but further checks are required in order to substantiate this.

The table gives $F_{n,1/3}(x, \cdots, x)$ to six decimal places, with x varying from 0 to $7.0/\sqrt{3}$ in steps of $0.1/\sqrt{3}$ for $n = 1\ (1)\ 10$, and from $1.5/\sqrt{3}$ to $2.1/\sqrt{3}$ in steps of $0.01/\sqrt{3}$ for $n = 1\ (1)\ 10, 13, 18$.

AUTHORS' ABSTRACT

1. D. Teichroew, "Probabilities associated with order statistics in samples from two normal populations with equal variance," Chemical Corps Engineering Agency, Army Chemical Center, Maryland, 1955.
2. C. W. Dunnett, "On selecting the largest of k normal population means," (to be published in *Jn.*, Roy. Stat. Soc, Series B, 1960).
3. C. W. Dunnett & M. Sobel, "Approximations to the probability integral and certain percentage points of a multivariate analogue of Student's t-distribution," *Biometrika*, v. 42, 1955, p. 258.
4. H. E. Salzer, R. Zucker & R. Capuano, "Table of the zeros and weight factors of the first twenty Hermite polynomials," *Jn. Res.*, Nat. Bur. Standards, v. 48, 1952, p. 111.

51[K].—E. C. Fieller, H. O. Hartley & E. S. Pearson, "Tests for rank correlation coefficients. I," *Biometrika*, v. 44, 1957, p. 470–481.

This paper is concerned with sampling determination of the approximate distribution for $z_S = \tanh^{-1} r_S$ and $z_K = \tanh^{-1} r_K$, where r_S is Spearman's rank correlation coefficient and r_K is Kendall's rank correlation coefficient, for the case of sample of size n from a bivariate normal distribution. It is concluded that z_S and z_K are approximately normally distributed if n is not too small, with var $(z_S) \doteq 1.060/(n - 3)$ and var $(r_K) \doteq 0.437/(n - 4)$. Eight tables are presented. Table 1 contains 4D values of three versions of var (r_S) for $\rho = 0.1(0.1)0.9$ and $n = 10, 30, 50$; one version is Kendall's approximate formula (adjusted), another is the observed value, and the third is a smoothed form of the observed value. Table 2 contains 3D values of var $(r_S)/[1 - (Er_S)^2]$ and 4D values of var $(r_K)/[1 - (Er_K)^2]$, also an average over ρ for each of these, for $\rho = 0.1(0.1)0.9$ and $n = 10, 30, 50$. Table 3 contains 3D approximate theoretical and observed values for Ez_S, while Table 4 contains these values for Ez_K, where $\rho = 0.1(0.1)0.9$ and $n = 10, 30, 50$; the second-order correction terms for the theoretical values are also stated to 3D. Table 5 contains 4D values of the observed variance of z_S and 3D values of its observed standard deviation, likewise for Table 6 with z_K, where $\rho = 0.1(0.1)0.9$ and $n = 10, 30, 50$. Table 7 contains values of χ^2 for goodness of fit tests of the normality of z_S and z_K for $n = 30, 50$. Table 8 contains 2D and 3D values of

$$(Ez_1 - Ez_2)/\sqrt{\text{var } (z_1) + \text{var } (z_2)}$$

for z_1 and z_2 representing the same correlation coefficient but with different ρ values ($\rho_2 = \rho_1 + 0.1$); this is for the product moment correlation coefficient, Spearman's coefficient and Kendall's coefficient with $\rho_1 = 0.1(0.1)0.8$ and $n = 10, 30, 50$.

J. E. Walsh

Operations Research Group
System Development Corporation
Santa Monica, California

52[K].—G. Horsnell, "Economical acceptance sampling schemes," Roy. Stat. Soc., *Jn.*, sec. A., v. 120, 1957, p. 148–201.

This paper is concerned with acceptance sampling plans designed to minimize the effective cost of accepted items produced under conditions of normal production. Effective cost per accepted item is defined to be the production cost per lot plus the average cost of inspection per lot when apportioned equally over the average number of items accepted per lot from production of normal quality. Single-sample plans are examined in detail. Double-sample plans are considered briefly.

An appendix contains thirty-one separate tables for single-sampling plans, ten of which are applicable when inspection is non-destructive and the remaining twenty-one are applicable when inspection is destructive. The table for non-destructive inspection displays c_m/c_s which is the ratio of manufacturing cost to inspection cost; n, the number of items to be sampled; k, the accepted number; $A(n, k)$, the probability of acceptance; and c/c_m, which is the ratio of effective cost to manufacturing cost. In the tables for destructive inspection, $A(n, k)$ is replaced by $A'(n, k)$, which is the expected number of accepted items per lot.

Plans for non-destructive inspection are given only for a nominal lot size of 10,000. Plans for destructive inspection are given for lot sizes of 10,000 and 20,000. For non-destructive inspection, the process average $p_0 = .01(.01).04$, the consumer's risk point $p_1 = .03(.01).07, .09$; at which the consumer's risks are .05 and .01. For destructive inspection $p_0 = .01$ and $.02$; $p_1 = .03(.01).06$, and consumer's risks are .05 and .01. The tables, however, do not include all possible combinations of the above listed parameter values.

A. C. Cohen, Jr.

University of Georgia
Athens, Georgia

53[K].—N. L. Johnson, "Optimal sampling for quota fulfillment," *Biometrika*, v. 44, 1957, p. 518–523.

This article contains two tables to assist with the problem of obtaining a preset quota m_i of individuals from each of k strata by selecting first a sample N of the whole population and then completing quotas by sampling from separate strata. Individual cost in the first case is c and in the second c_i. Table I gives for $m_i = m$ optimal values of N for $k = 2(1)10$; $mk \doteq 50, 100, 200, 500$; $d = c_i/c = 1.25, 1.5(.5)3.0$; $d' = c_i'/c = .9, .7, .25, 0$. Here c_i' is the worth of first sample individuals in excess of quota. The tabulated values of N are solutions of the equation $Pr(N_i < m) = (c - c_i')/(c_i - c_i)$.

Table 2 gives ratio of minimized cost to cost of choosing the whole sample by sampling restricted to each stratum. This quantity is

$$\frac{1}{d} + \left(1 - \frac{d'}{d}\right)\left(1 - \frac{1}{k}\right)^{N+1} \binom{N}{m} (k - 1)^{-m}$$

and is tabulated for $k = 2(1)5, 10$; $km \doteq 50, 100, 500$; $d = 1.5, 2.5, 3$; $d' = .5, .1, 0$.

W. J. Dixon

University of California
Los Angeles, California

54[K].—P. G. Moore, "The two-sample t-test based on range," *Biometrika*, v. 44, 1957, p. 482–489.

This paper provides a sample statistic for unequal sample sizes for a two-sample t-test based on observed sample ranges instead of sums of squares. The statistic used by the author is simply

$$u = \frac{|\bar{x}_1 - \bar{x}_2|}{w_1 + w_2},$$

where $\bar{x}_1$ and $\bar{x}_2$ are the sample means, and w_1 and w_2 are the sample ranges. Since unequal sample sizes are now permitted, the mean range $(w_1 + w_2)/2$ proposed by Lord [1] in his original paper is no longer used. Moore shows that there is very little loss in power resulting from the use of the simple sum $w_1 + w_2$, rather than a weighted sum of sample ranges, although for the estimation of σ separately, he gives a table of $f(n_1, n_2)$ and $d_{n_1} + fd_{n_2}$ to $3D$ for $n_1, n_2 = 2(1)20$ to estimate σ from

$$g = \frac{w_1 + fw_2}{d_{n_1} + fd_{n_2}},$$

which minimizes the coefficient of variation of the range estimates of population standard deviation.

The main use of this paper is, of course, the tables of percentage points (10%, 5%, 2%, and 1% points) to $3D$ for the statistic u, above. The tables of percentage points were computed by making use of Patnaik's chi-approximation for the distribution of the range, which resulted in sufficient accuracy. The limits for sample sizes are $n_1, n_2 = 2(1)20$, which fulfills the most usual needs in practice. With this work of Moore, therefore, the practicing statistician has available a very quick and suitably efficient procedure for testing the hypothesis of equal means for two normal populations of equal variance.

F. E. Grubbs

Ballistic Research Laboratory
Aberdeen Proving Ground, Maryland

1. E. Lord, "The use of range in place of standard deviation in the t-test", *Biometrika*, v. 34, 1947, p. 41–67.

55[K].—National Bureau of Standards, *Tables of the Bivariate Normal Distribution Function and Related Functions*, Applied Mathematics Series, No. 50, 1959, xliii + 258 p., 27cm. U. S. Government Printing Office, Washington, D. C. Price $3.25.

These tables, compiled and edited by the National Bureau of Standards, provide values for the probability content $L(h, k, r)$ of an infinite rectangle with vertex at the cut-off point (h, k) under a standardized and centered bivariate distribution with correlation coefficient r:

$$L(h, k, r) = \frac{1}{2\pi\sqrt{1 - r^2}} \int_h^\infty \int_k^\infty \exp\left[-\frac{1}{2}\left(x^2 + y^2 - 2rxy\right)/(1 - r^2) \right] dx\, dy$$

The range of tabulation is $h, k = 0(.1)4$, $r = 0(.05)0.95(.01)1$, the values of $L(h, k, r)$ being given to 6 decimal places. For negative correlations, the range of tabulation is $h, k = 0(.1)h_n, k_n$, $r = 0(.05)0.95(.01)1$, the values of $L(h, k, r)$ being given to 7 decimal places, where $L(h_n, k_n, -r) \leqq \frac{1}{2}.10^{-7}$ if h_n and k_n are both less than 4. The two tables of $L(h, k, r)$ for positive and negative r, respectively (Tables I and II in the text), may therefore be regarded as extensions of Karl Pearson's tables of the bivariate normal distribution in his celebrated *Tables for Statisticians and Biometricians*, Part II, since the range of parameters in the latter tables is $h, k = 0(.1)2.6$, $r = -1(.05)1$. In this connection, the authors of the present tables present a list of 31 errors in Pearson's tables, together with the corresponding correct values.

Table III gives the values to 7 decimal places, with the last place uncertain by 2 units, of the probability content, $V(h, \lambda h)$, of a certain right-angled triangle under a centered circular normal distribution with unit variance in any direction. The triangle has one vertex at the center of the distribution, with the angle at that vertex arc tan λ, while the lengths of the two bounding sides at the vertex are h and $h\sqrt{1 + \lambda^2}$. Formally,

$$V(h, \lambda h) = \frac{1}{2\pi} \int_0^h dx \int_0^{\lambda x} \exp\left[-\frac{1}{2} (x^2 + y^2) \right] dy .$$

The range of tabulated values for $V(h, \lambda h)$ is $h = 0(.01)4(.02)4.6(.1)5.6$ and ∞, and $\lambda = 0.1(.1)1$. Table IV gives the values of

$$V(\lambda h, h) = \frac{1}{2\pi} \int_0^{\lambda h} dx \int_0^{x/\lambda} \exp\left[-\frac{1}{2} (x^2 + y^2) \right] dy$$

for the parameters $h = 0(.01)4(.02)5.6$ and ∞, and $\lambda = 0.1(.1)1$, the degree of accuracy being the same as for the values of $V(h, \lambda h)$ in Table III.

Finally, a short table (Table V) for values of $y = \text{arc sin } r/2\pi$, $r = 0(.01)1$, correct to 8 decimal places is provided.

The two-parameter function V is related to the three-parameter function L by the formula

$$L(h, k, r) = V\left(h, \frac{k - rh}{\sqrt{1 - r^2}}\right) + V\left(k, \frac{h - rk}{\sqrt{1 - r^2}}\right) + F,$$

where

$$F = \tfrac{1}{4}[1 - \alpha(h) - \alpha(k)] + y$$

and

$$\alpha(x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} \exp\left(-\frac{1}{2} t^2\right) dt.$$

This relationship enables $L(h, k, r)$ to be computed to 6-decimal accuracy from the 7-decimal values of $V(h, \lambda h)$ in regions where interpolation in $L(h, k, r)$ is difficult.

The function V is also of considerable intrinsic interest, and finds applications in such fields as the probability content of polygons when the underlying distribution is bivariate normal, the distribution of range for normal samples of size 3, the non-central t-distribution for odd degrees of freedom, and one-dimensional heat flow problems. These applications are illustrated clearly and in detail in the section of the book headed Application of the Tables, due to Dr. D. B. Owen. The latter section also provides illustrations of the use of $L(h, k, r)$ in problems ranging from measurement errors, calibration systems, double-sample test procedures, and percentage changes in sample means to some interesting problems in selection (involving the correlation between aptitude test and job performance) and estimation of correlation. Finally, an introductory section discusses the mathematical properties of the L- and V-functions as well as methods of interpolation in the tables.

The authors are to be heartily commended for this most useful book, which will place many statisticians, both practising and those more theoretically inclined, in their debt. The visual appearance and general presentation of the material are excellent. Perhaps one very minor flaw is that since $L(h, k, r) = L(k, h, r)$, tables of $L(h, k, r)$ for $h \geqq k$ would have been sufficient. However, this is a flaw (if at all) from the point of view of economics, but hardly so from the point of view of the user of the tables! The cost of the book is remarkably low.

HAROLD RUBEN

Columbia University
New York, New York

56[K].—E. NIEVERGELT, "Die Rangkorrelation U," *Mitteilungsblatt für Math. Stat.*, v. 9, 1957, p. 196–232.

In contrast to Spearman's rank correlation R and Kendall's coefficient T, the author studies the van de Waerden coefficient U. Let p_i and q_i ($i = 1, 2, 3 \cdots n$) be the ranks of n observations on two variables x and y, and let ξ_i, η_i, ζ_i be the inverses of the normal probabilities: $F(\xi_i) = p_i/(n + 1)$; $F(\eta_i) = q_i/(n + 1)$; $F(\zeta_i) = i/(n + 1)$. Then, U is defined by $U = \sum_{i=1}^{n} \xi_i\eta_i / \sum_{i=1}^{n} \zeta_i^2$.

If x and y are independent the expectation of U is zero and its standard deviation is $\sigma_U = (n - 1)^{-1/2}$, as for Spearman's coefficient. The author calculates also the 4th and 6th moments of U and R, which differ. For n large, U is asymptotically normally distributed about mean zero with standard deviation σ_U. The distribution of U (to 4D) is tabulated completely for $n = 4$, and over the upper 5% tail for $n = 5, 6, 7$. For larger values of n the Gram-Charlier development is used. Tables testing independence based on 5%, 2.5%, 1%, and .5% probabilities are given to 3D for $n = 6(1)\ 30$. For $n > 30$ the normal probability function can be used.

In the case of dependence the correlation between R and U decreases slowly with n increasing. If x and y are normally distributed with zero mean, unit standard deviation and correlation ρ a generalization U^* of U to the continuous case leads to $U^* = \rho$. A consistent estimate for ρ is given. The U test is more powerful than the R test.

E. J. GUMBEL

Columbia University
New York, New York

57[K].—D. B. OWEN & D. T. MONK, *Tables of the Normal Probability Integral*, Sandia Corporation Technical Memorandum 64-57-51, 1957, 58 p., 22 x 28 cm. Available from the Office of Technical Services, Dept. of Commerce, Washington 25, D. C., (Physics (TID-4500, 13th Edn.), Price $.40.

The following forms of the normal probability integral

$$G(h) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{h} e^{-t^2/2}\, dt, \quad h \geqq 0$$

$$G(-h) = 1 - G(h)$$

are given for $h = 0(.001)\ 4(.01)\ 7$, to 8D. For those having frequent use of $G(h)$ these tables eliminate the simple yet troublesome computation necessary when using the more comprehensive tables prepared at the New York Mathematical Tables Project [1] for

$$F(h) = \frac{1}{\sqrt{2\pi}} \int_{-h}^{h} {}^{-t^2/2}\, dt$$

The present tables are more comprehensive than the Pearson and Hartley tables [2] for $G(h)$ up to $h = 7$. These tables have been checked by the authors against other tables. (The reviewer could undertake no systematic checking, but such occasional checks as were made revealed no errors.)

Computations (made on a CRC-102A digital computer) were facilitated by using the following continued fractions:

$$R_1(x) = \cfrac{x}{1 - \cfrac{x^2}{3 + \cfrac{2\,x^2}{5 - \cfrac{3\,x^2}{7 + \cdots}}}}, \quad h \leqq 2.5;$$

$$R_2(x) = \cfrac{1}{x + \cfrac{1}{x + \cfrac{2}{x + \cfrac{3}{x + \cdots}}}}, \quad h \geqq 2.5.$$

Then $G(h)$ was computed from

$$G(h) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} e^{-h^2/2} R_1(h), \quad \text{or} \quad 1 - \frac{1}{\sqrt{2\pi}} e^{-h^2/2} R_2(h).$$

For large values of h, $G(-h)$ can be obtained from the formula

$$G(-h) = \frac{1}{\sqrt{2\pi}} e^{-h^2/2} M(h), \qquad h \geqq 0,$$

where

$$M(h) = e^{-h^2/2} \int_{h}^{\infty} e^{-t^2/2}\, dt.$$

$M(h)$, Mill's ratio [3], provides constants a, an integer, and b $(0.1 < b < 1)$ such that

$$G(-h) = b.10^{-a}$$

For large $h = 50(1)\,150(5)\,500$, $M(h)$ and b are tabulated to $8S$; a, of course, exactly.

The authors checked $-\log_{10} G(-h)$ given in [2] for large h, and found one discrepancy, namely, $-\log_{10} G(-500)$ should read 54289.90830.

S. B. Littauer

Columbia University
New York, New York

1. *Tables of Normal Probability Functions*, National Bureau of Standards Applied Math. Series, No. 23, U. S. Government Printing Office, Washington, D. C., 1953.

2. E. S. Pearson, H. O. Hartley, *Biometrika Tables for Statisticians*, v. 1, Cambridge University Press, London, 1954, p. 104–111.
3. R. D. Gordon, "Values of Mill's ratio of area to bounding ordinate of the normal probability integral for large values of the argument," *Ann. Math. Stat.*, v. 12, 1941, p. 364–366.

58[K].—A. E. Sarhan & B. G. Greenberg, "Tables for best linear estimates by order statistics of parameters of single exponential distributions from singly and doubly censored samples," Amer. Stat. Assn., *Jn.*, v. 52, 1957, p. 58–87.

Tables are provided for the exact coefficients of the best linear systematic statistics for estimating the scale parameter of a one-parameter single exponential distribution and the scale and location parameters of a two-parameter single exponential distribution. All possible combinations of samples of size n with the r_1 lowest and r_2 highest values censored are considered for $n \leqq 10$. Exact coefficients for the best linear systematic statistic for estimating the mean (equal to the location parameter plus the scale parameter) are also given for the two parameter case. Other tables give the variances, exact or to $7D$, of the estimates obtained and the efficiency relative to the best linear estimate to $4D$ based on the complete sample. These extensive tables are of immediate practical importance in many fields, such as life testing and biological experimentation.

C. A. Bennett

Hanford Laboratories Operation
General Electric Company
Richland, Washington

59[K].—Y. S. Sathe & A. R. Kamat, "Approximations to the distributions of some measures of dispersion based on successive differences," *Biometrika*, v. 44, 1957, p. 349–359.

Let $x_1, \cdots, x_n$ be a random sample from a normal population with variance σ^2 and let

$$\delta^2 = \frac{1}{n-1}\sum_{i=1}^{n-1}(x_i - x_{i+1})^2, \qquad d = \frac{1}{n-1}\sum_{i=1}^{n-1}|x_i - x_{i+1}|,$$

$$\delta_2^2 = \frac{1}{n-2}\sum_{i=1}^{n-2}(x_i - 2x_{i+1} + x_{i+2})^2, \qquad d_2 = \frac{1}{n-2}\sum_{i=1}^{n-2}|x_i - 2x_{i+1} + x_{i+2}|.$$

The problem is to develop approximations to the distributions of these four types of statistics. Let u be any one of these statistics. The method followed is to assume that u is approximately distributed as $(\chi_\nu^2/c)^\alpha$, where χ_ν^2 has a chi-square distribution with ν degrees of freedom; that is, taking $\lambda = 1/\alpha$, that cu^λ is approximately distributed as χ^2 with ν degrees of freedom. The constants c, α (or λ), and ν are then determined by equating the first three moments of u to those of $(\chi_\nu^2/c)^\alpha$. The results show that a fixed value can be used for α (or λ) if $n \geqq 5$. This allows two independent measures of variability u_1 and u_2, based on the same type of statistic, to be compared by use of the F test when $n \geqq 5$ for both statistics. The basic results of the paper are given in Table 1. There, for each of δ^2/σ^2, d/σ, δ_2^2/σ^2, and d_2/σ, fixed values are stated for λ, while $3D$ values for ν and $4D$ values for $\log_{10} c$ are given for $n = 5(1)\,20, 25, 30, 40, 50$. Table 2 deals with an example. Table 3 lists the results of some approximations to δ^2/σ^2 by $(\chi_\nu^2/c)^\alpha$ for $n = 5, 10, 20, 30, 50$. Table 4 lists for comparison purposes, the upper and lower 1% and 5% points for four approximations to δ^2/σ^2 when $n = 15, 20$. Table 5 is important; it contains $2D$ values of upper and lower 0.5%, 1.0%, 2.5%, and 5% points for the approximate distribution developed for δ^2/σ^2. Table 6 lists the results of some approximations to d/σ by $(\chi_\nu^2/c)^\alpha$ for $n = 5, 10, 20, 30, 50$. Finally, Table 7 furnishes $4D$ values of the β_1, β_2 differences that result from using a fixed λ for the $(\chi_\nu^2/c)^\alpha$ approximation to the distribution of δ_2^2/σ^2, and from using a fixed λ for the $(\chi_\nu^2/c)^\alpha$ approximation to the distribution of d_2/σ, for $n = 5, 7, 10(5)30, 40, 50$.

J. E. Walsh

Operations Research Group
System Development Corporation
Santa Monica, California

60[K].—C. C. Sekar, S. P. Agarwala & P. N. Chakraborty, "On the power function of a test of significance for the difference between two proportions," *Sankhya*, v. 15, 1955, p. 381–390.

The authors determine the power function of the following statistical test: a sample of size n is drawn from each of two binomial distributions with unspecified probabilities of success p_1 and p_2, respectively. The null hypothesis is $H_0 : p_1 = p_2 = p$. For the two-sided test (alternative hypothesis: $p_1 < p_2$ or $p_1 > p_2$) at significance level α, the critical region is determined by the following conditions:

1) For a given total number r of successes in the two samples, the conditional probability of rejection under H_0 is $\leqq \alpha$.

2) If the partition $(a, r - a)$ of r successes is contained in the critical region and $0 < a < r - a$, then the partition $(a - 1, r - a + 1)$ is contained in the critical region.

3) If the partition $(a, r - a)$ is contained in the critical region, the partition $(r - a, a)$ is contained in the critical region.

A similar definition is used for the one-sided test of H_0 against the alternative $p_1 > p_2$. The critical region is determined using the exact conditional probabilities for these partitions given by S. Swaroop, [1].

The power function for the two-sided test is given to $5D$ for p_1 and $p_2 = .1(.1).9$; $n = 5(5)20(10)50, 100, 200$, and for $a = .05$. For the one-sided test the power function to $5D$ is given for the same levels of p_1, p_2 and n, and for $\alpha = .025$.

The critical region used by the authors is the one defined for the exact test by E. S. Pearson, [2]. However, for small sample sizes the power differs considerably from Patnaik's determinations, which are based on an approximately derived critical region and which use a normal distribution approximation of the probabilities.

Examples of the use of the tables are included.

M. L. Epling

National Bureau of Standards
Washington, District of Columbia

1. Satya Swaroop, "Tables of the exact values of probabilities for testing the significance of differences between proportions based on pairs of small samples," *Sankhya*, v. 4, 1938, p. 73–84.

2. E. S. Pearson, "The choice of statistical tests illustrated on the interpretation of data classed in a 2 x 2 table," *Biometrika*, v. 34, 1947, p. 139–167.

3. P. B. Patnaik, "The power function of the test between two proportions in a 2 x 2 table," *Biometrika*, v. 35, 1948, p. 157–175.

61[K].—B. SHERMAN, "Percentiles of the ω_n statistic," *Ann. Math. Stat.*, v. 28, 1957, p. 259–261.

The statistic

$$\omega_n = \frac{1}{2} \sum_{k=1}^{n+1} \left| L_k - \frac{1}{n+1} \right|, \qquad 0 \leqq w_n \leqq \frac{n}{n+1},$$

is one of several which have been suggested in connection with the null hypothesis that x_i, $(i = 1, \cdots, n)$ is a random sample from the uniform distribution and the L_k are the lengths of the $n + 1$ subintervals of the unit interval defined by the ordered sample. In most cases of interest, the x_i are the probability transforms of observations on a random variable with a continuous distribution function. Based on the distribution function derived by the author [1], the 99th, 95th, and 90th percentiles of ω_n to $5D$ for $n = 1(1)20$ have been computed, and are given in Table I. Values of two standardized forms of this statistic (based on the exact and asymptotic mean and variance, respectively) which are asymptotically normal are given in Table II to $5S$ for the same percentiles as in Table I and for $n = 5(5)15(1)20$. The author points out that the rate of convergence to the limiting values is slow.

C. A. BENNETT

Hanford Laboratories Operation
General Electric Company
Richland, Washington

1. B. SHERMAN, "A random variable related to the spacing of sample values," *Ann. Math. Stat.*, v. 21, 1950 p. 339–361.

62[K, X, Z].—E. D. CASHWELL & C. J. EVERETT, *A Practical Manual on the Monte Carlo Method for Random Walk Problems*, Pergamon, 1959, 152 p., 23 cm. Price $6.00.

This is volume I of the publisher's series "International Tracts in Computer Science and Technology and Their Application". It is devoted to a direct and elementary attack on the Monte Carlo principle (that is, the principle of using simulation for calculation and recording the sample statistics obtained from the simulation) in random walk problems, such as mean free path and scatter problems. Many examples are given in the text, and an appendix is added listing twenty more or less typical problems in which the Monte Carlo method was used at the Los Alamos Scientific Laboratory.

Computational details are given and in many cases flow charts are included. No full machine codes are given, but most of the calculations were done on the MANIAC I computer at Los Alamos, and coding from the descriptions given and the flow charts is probably easier than any attempt to translate a MANIAC I code to a code suitable for another machine. A disappointingly short chapter on statistical considerations is included; the reader should be warned that this is not a suitable exposition of the theory or even the practice of the statistical handling of the statistics gathered in his simulation. However, it also is treated from a definitely computational point of view, including flow charts, and is interesting from this point of view.

An interesting small chapter titled "Remarks on Computation" is also included.

There is a section on scaling, a section on debugging, a section on special routines, a section on a Monte Carlo device for determining the square root of r, and a section on a Monte Carlo device for the cosine of an equi-distributed angle. The random number routine is the familiar routine of selecting the middle digits of the square of a quasi-random number; it is frowned on by many random-number specialists. The logarithm routine presented depends on the power series expansion of the log, with restriction of the size of the arguments to assure fast convergence. An exponential routine is given as a quadratic approximation with scaling of the argument. A cosine routine is given through the use of a trigonometric identity and a truncated power series for the sine of a related angle. There is no detailed discussion of the accuracy of any of these routines.

The exposition in this book is far from perfect, and the editors have included the following statement: "It is realized that many workers in this fast moving field cannot devote the necessary time to producing a finished monograph. Because of their concern for speedy publication, the Editors will not expect the contributions to be of a polished literary standard if the originality of the ideas they contain warrant immediate and wide dissemination." The reviewer feels that the present book in its present form is more than justified on the basis of this philosophy, and he recommends the book as a most valuable contribution to numerical analysis.

The chapter headings follow:

C. B. T.

63[L].—CENTRE NATIONAL D'ÉTUDES DES TÉLÉCOMMUNICATIONS, *Tables numériques des fonctions associées de Legendre. Fonctions associées de première espece,* P_n^m (cos θ), deuxième fascicule, Éditions de la Revue Optique, Paris, 1959, xii + 640 p., 31 cm. Price 5600 F.

The first volume of these Tables was reviewed in *MTAC*, v. 7, p. 178. The present second volume was designed to extend the range of tabulation from $\theta = 90°$ to $\theta = 180°$. In the process of constructing these tables, however, it was found desirable to increase the number of decimals and to add second and fourth central differences, thus facilitating interpolation. For this reason, the range up to $\theta = 90°$, already covered in the first volume, is included (in an improved form) in the volume under review. Perhaps because of the increase in size consequent upon increased numbers of decimals and added differences, tabulation has been restricted to $m =$

0, 1, 2 (whereas vol. 1 has $m = 0(1)5$). Otherwise, the range and intervals of this volume match those of the first.

Since P_n^m (cos θ) has a singularity at $\theta = 180°$ (except when n is an integer), an auxiliary function T_n^m (cos θ) is introduced by the relation

$$P_n^m(\cos\theta) = (\csc^m\theta)T_n^m(\cos\theta) + (-1)^m\left(A_n \log_{10}\left(\cot\frac{\theta}{2}\right) P_n^m(\cos(180° - \theta)\right)$$

where $A_n = (2\sin n\pi)/(\pi \log_{10} e)$. The function T_n^m (cos θ) is tabulated for $135° \leqq \theta \leqq 180°$. The use of these auxiliary functions is facilitated by the provision of tables of A_n, csc θ, csc^2 θ, and $\log_{10}$ cot $(\theta/2)$.

The introductory material contains formulas, an account of the tables, hints for interpolation, and level curves of P_n (cos θ), P_n^1 (cos θ), and P_n^2 (cos θ).

A. Erdélyi

California Institute of Technology
Pasadena, California

64[L].—Louis Robin, *Fonctions sphérique de Legendre et fonctions sphéroidales*, tome 3, Gauthier-Villars, Paris, 1959, viii + 289 p., 24 cm. Price 5500 F.

The first two volumes of this work were reviewed in *MTAC*, v. 13, p. 325f. The present, final, volume contains chapters VII to X.

Chapter VII is devoted to the addition theorems of Legendre functions. Both Legendre functions of the first and second kind are included, and two cases are distinguished according as the composite argument lies in the complex plane cut from $-\infty$ to $+1$ or else on the cut between -1 and 1. Addition theorems are also developed for the associated Legendre functions of the first kind.

Chapter VIII is devoted to zeros of Legendre functions. First the zeros of $P_n^m(\mu)$ as functions of μ, for fixed real m and n are discussed, then the zeros of $P^m_{-\frac{1}{2}+ip}(\mu)$ when m is an integer and p a fixed real number, and then the zeros of $Q_n^m(\mu)$. This chapter contains also a discussion of zeros of Legendre functions considered as functions of n, m and μ being fixed. (These zeros are of importance in certain boundary-value problems.)

In Chapter IX, applications of Legendre functions are given to partial differential equation problems relating to surfaces of revolution other than spheres. Prolate and oblate spheroidal harmonics, toroidal harmonics, and conal harmonics are discussed.

Chapter X contains the discussion of some functions related to Bessel functions, namely, Gegenbauer polynomials and functions, and spheroidal wave functions.

Appendix I summarizes relevant information on "spherical Bessel functions", and Appendix II lists numerical tables of Legendre functions and tables connected with these functions.

The third volume maintains the high standards set by the first two volumes, and the author must be congratulated upon the completion of this valuable work.

A. Erdélyi

California Institute Technology
Pasadena, California

65[L. M.].—R. G. Selfridge & J. E. Maxfield, *A Table of the Incomplete Elliptic Integral of the Third Kind*, Dover Publications, Inc., 1959. xiv + 805 p., 22 cm. Price $7.50.

The advent of fast automatic computers has made a considerable difference to the art of making and publishing tables. It has speeded up the computing processes fantastically, without, except in relatively minor ways, modifying the labor and care needed during publication processes. It has also increased the care needed in *planning* calculations; the plan has to be quite precise and exact in all details for the machine to produce proper results, whereas in desk computation, the plan can be built up and modified as the work proceeds.

The glamour of fast computation has led quite a number of people to enter the table-making field; people who appear to imagine that the whole problem is simplified by automatic computers, who perhaps do not even realize the need to seek expert advice. It is with some reluctance, but with the feeling that it is an urgent duty that needs to be performed on general grounds, that I suggest that the table now reviewed presents one of the most deplorable examples of inadequate planning and poor execution that I have met.

The tables give entries that purport to be 6-decimal values of

$$\Pi(\phi, \alpha^2, k) = \int_0^\phi \frac{d\theta}{(1 - \alpha^2 \sin^2\theta)\sqrt{1 - k^2 \sin^2\theta}}$$

for

$$k = \sin\theta, \qquad \theta = .1(.1)1.5, \qquad \phi = 0(.01)1.57,$$

$$\alpha^2 = -1(.05) - .1(.02) - .02, .05(.05).5(.02).8(.01).99.$$

Also given are two lines for $\phi = 1.5707963$, representing a direct and a check calculation.

One example of bad planning is illustrated by the arguments θ, and the heading, erroneously labelled α. These end arbitrarily in 4 or 5 zeros or 4 or 5 nines. The latter is obviously wrong and quite intolerable in print. The present authors are not unique in exhibiting this fault, which is quite inexcusable. It is a matter of, at most, a few minutes to modify a program, on any automatic machine, to round-off at the appropriate figure and to suppress printing thereafter, giving better-looking and more convincing argument values. The authors have, in any case given special treatment to the argument θ which has one *more* decimal than function values; why not treat it properly?

Both bad planning and poor execution are exhibited by the check up. The foreword states that "the greatest difficulty was encountered not in constructing the table, but in obtaining satisfactory checking". This has not, in fact, been achieved! The method described could have been—but is not—satisfactory for $\alpha^2 < 0$ and for $0 < k^2 < \alpha^2$, but the method as described, of integrating through a singularity when $\alpha^2 > 0$, is absurdly inadequate. It is not surprising then, that the final two lines, both for $\phi = 1.5707963$ as mentioned before, should often be in disagreement. It is surprising, however, that the authors accept this as a legitimate problem to hand on to their readers. The discrepancies indicate errors; the authors' duty is to find and remove these. This states the obvious, but it is equally obviously necessary to do so. Let it be said again, that to make a program to deal with this properly is a job to be done once and for all. It may take a little effort, time, and money, but this is not to be compared with effort, time, and money wasted if it is *not* done before publishing a book of inadequate tables.

I believe, for my part, the electronic computation is so fast and easy that a discrepancy of more than a single unit (known or unknown) is not tolerable in a published table (of which the computing effort and cost are now often only a small part of the whole effort and cost); however, I am willing to concede that discrepancies up to perhaps two units *might*, on rare occasions, be justified, provided this is clearly stated. It is quite intolerable to have the following discrepancies; to pick out some of the worst:

Page	α^2	k^2	$\Pi(\alpha^2, k)$
661	0.87	0.86869...	8.654098 and 8.654259
733	0.93	0.92844...	15.309882 and 15.310251
745	0.94	0.92844...	16.857725 and 16.857859
793	0.98	0.97111...	45.498015 and 45.498457
805	0.99	0.92844...	49.243943 and 49.244046
805	0.99	0.97111...	70.018520 and 70.018897

The discrepancies are, in fact, highly systematic throughout; they are all of the same sign, except for the violent cases mostly listed above near the singularity $k = \alpha$ mentioned in the introduction. They indicate clearly that at least one set of the check values is erroneous because of an inadequate method, and not merely because of rounding; severe doubt is cast on *both* discrepant values. Dr. J. W. Wrench, Jr. has computed anew the value on p. 733 for $\alpha^2 = .93$, $k^2 = \sin^2 1.3$, and finds 15.3098662, which is not even between the two values quoted from the book; neither published value is correct and one errs by more than the discrepancy between them. I repeat again, it is the duty of the table compiler to remove *all* these doubts.

The poor execution is also exhibited by the fact that in the heading, α and k appear in place of the correct α^2 and k^2, while the 10-decimal values of k^2 given (which are simply $\sin^2 \lambda$ for $\lambda = .1(.1)1.5$) have end figure errors running up to 11 units. Again, the argument is given as θ in the tables; this corresponds to ϕ in the introduction. It is only fair to add that the heading errors in α^2, k^2, ϕ have been announced as errata. Another awkward point for the user is that absence of values for $\alpha = 0$, and for $k = 0$, makes the tables harder to interpolate.

From all this, it is evident that the authors are lacking in experience of table-making so that their remark "With the argument as outlined, no attempt has been made to proof or check the printed sheets in any way other than by comparison of the resultant complete integrals" causes less surprise than might otherwise be the case. If there were no other faults occurring other than those mentioned above, they would have been exceptionally and unduly lucky. Photographic processes seem as far from infallibility as printing from letter-press; the possible faults are different, but nevertheless exist just the same. In fact, a rather superficial examination of the table reveals unsightly irregularities in spacing of lines on pages 51, 623, 700 and some lesser ones elsewhere. It would have been easy to reprint the pages before reproduction. More serious are several digits that are not fully legible:

p. 446	$\theta = .25$	$k = .03946 \cdots$	3rd digit 3
p. 507	$\theta = .75$	$k = .15164 \cdots$	1st digit 8
p. 579	Bottom right corner.	Very faint.	

All these imperfections occur in at least two copies of the tables; such imperfections are common in tables printed from typescript and should be expected and sought out. The real surprise is, however, as mentioned above, that, after the numerical comparison of check values mentioned had been made, its lack of success seems simply to have been ignored.

It is hoped that possible users may, with the exercise of necessary—but undue—caution, obtain adequate results, maybe $5\frac{1}{2}$ correct figures, if they need them. The publication of this book will undoubtedly make it much more difficult to publish a good and proper version; this is a major criticism of such a book. The only consolation I can offer the authors is that I have seen several tables that are even worse.

As I have said, I have expressed myself so freely with some reluctance, from a sense of duty; it is no part of my desire to discourage the enthusiasm of table-makers, but they must realize the magnitude and duties of the task so taken on, and seek competent advice before proceeding with the work, and potential users must be adequately warned.

J. C. P. Miller

The University Mathematical Laboratory
Cambridge, England; and
The Mathematics Research Center, U. S. Army
University of Wisconsin, Madison, Wisconsin.

66[S].—D. R. Hartree, *The Calculation of Atomic Structures*, John Wiley and Sons, Inc., New York, 1957, xiii + 181 p., 23 cm. Price $5.00.

This book by the late D. R. Hartree is the fruit of a lifetime of experience in the calculation of the outer, electronic structure of atoms. It is concerned with methods for the calculation of atomic structures rather than with the results of such calculations for particular atoms. Emphasis is deliberately placed on means of obtaining "best" approximations which can be both represented and applied simply. The student who wants an introduction to the essential methods of approximation and computation of shell structures may read the first hundred pages. The mathematician will find in this book the physical background for the author's well-known text on numerical analysis.

In the Introduction are outlined the seven main steps in the development of atomic theory up to the point at which quantitative calculations are possible. The atomic units are introduced and the point change approximation of the electron justified. Then, properties of the Schroedinger equation are summarized to prepare the reader for the main problem of the book, the numerical solution of the self-consistent field equations with and without exchange. The variation principle is carefully introduced, and the total energy of closed shell configurations discussed. Also, configurations comprising incomplete groups are treated. In the later part, the main ideas and methods are extended to more complicated or more complete cases, and, in general, are described only very briefly. The text concludes with a chapter on "Better Approximations".

Many tables, relating to Slater coefficients, mean radii, screening numbers and reduced radial wave functions, are found in the text and in Appendix 2. For the bibliography up to 1947, the author refers to *Reports on Progress in Physics*, v. 11, 1946–47, p. 141–143, and completes the list to October 1956 in Appendices 1 and 3.

Numerical procedures, most of them recommended by the author's experience in hand and machine calculations, are described in detail, giving step-by-step instructions and numerical examples. The reviewer would have liked to have seen some comments on the numerical stability of methods using finite differences approximations to differential equations. Numerical stability is obvious in the Numerov and Fox-Goodwin process, but this is not so in Hartree's method of paragraph 4.6 (page 72), although the application is correct. Familiarity with numerical stability prevents the physicist from blindly refining the methods given in the text and will save costly "numerical experimentation".

In general, the book would gain by stating briefly the mathematical reasons why certain procedures are recommended (it would be mostly in the light of numerical stability!), as, for example, for the separation of integration of the radial wave equation into an outward and inward integration (§5.2). The physical reasons are stated adequately.

It seems that many equations were renumbered before the manuscript went to the printer. Cross references to equations are quite often unreliable. Otherwise, the number of misprints for a book of this kind is rather low.

Erwin Bareiss

Argonne National Laboratory
Lemont, Illinois

67[W, X].—Robert O. Ferguson & Lauren F. Sargent, *Linear Programming*, McGraw-Hill Book Co., New York 1958, xiv + 342 p., 24 cm. Price $10.00.

With increasing frequency the professional mathematician, especially if he is working in applied mathematics, finds himself approached by friends or colleagues who lack advanced mathematical training but want to know more about the latest techniques, such as linear programming. In the course of the past year, several books on linear programming have appeared to which such inquirers might be referred.

This book should probably be regarded as the best of the group. It is addressed primarily to "people engaged in management activities at all levels in the firm and students of management. . . ." Its major virtues include a simple expository style without condescension, a wealth of illustrative examples, and a somewhat broader coverage of the subject than other works currently available. As a result of these qualities, it should prove suitable for individual study by management personnel with substantial practical experience in industry. It should, however, have its greatest value as a textbook for classroom instruction (on the job or off) of groups in which some or all participants lack the mathematical prerequisites which would permit use of a more advanced text, such as the well-known volume by Gass (from the same publishing house, interestingly enough).

The three sections into which the book is divided are entitled Introduction, Methods, and Application, while two technical appendices treat the mathematical foundations of the Simplex Algorithm, and its relationships to the modi method, respectively. The second section presents the transportation method, the modi method, the simplex method, all of which are exact, and two approximation methods: the index method and the authors' own ratio-analysis method. Applications illustrated in detail in the third section include a product-mix problem, a production smoothing problem, and a problem in optimal assignment of orders to plants, where production costs as well as distribution costs affect the decision. Emphasis in this section is on obtaining and utilizing a maximal amount of information of value to management from the linear programming solution. Other applications, it may be added, are illustrated in presenting the computational methods in the second section. Further, two technical appendices treat the mathematical foundations of the Simplex Algorithm and its relationships to the modi method, respectively.

The professional mathematician should probably be warned that, while he may safely recommend this book to non-mathematicians, he should not attempt to read it himself, unless he is willing to take the risk of apoplexy. As evidence for this conclusion, one may cite a number of quite apoplectic reviews in which this book and others like it have been severely castigated (by mathematicians) for a low level of scholarship, lack of rigor, and similar mortal sins. To some, this attitude may seem unfair. Texts on business statistics, to take an analogy at random, are not usually criticized for omitting discussion of Borel sets, Stieltjes integrals, and the Cramer-Rao inequality. One might expect that there would be a place for a comparable treatment of linear programming without reference to theorems on matrix inversion or convex polyhedral cones.

For good or for ill, linear programming is being dished up for the common folk, and this book represents probably the most workmanlike presentation currently available. As might be expected, the book is weakest where it is most technical. The attempt, on page 50, to explain degeneracy explains nothing. Experience indicates that a better treatment of this concept is possible without resorting to advanced mathematics. Similarly, on page 5 and again on page 77, the difference between linear programming and the solution of simultaneous equations is explained in terms of the non-optimizing character of the latter, but the authors do not go on to explain, as they easily might have done, the reasons for this difference. Also, one of the examples (p. 119 ff.) used to illustrate the simplex method contains one redundant equation (because it is inherently a transportation problem) but the text makes no mention of this fact. As is all too often the case in technical works, the index is far from complete. For example, under "degeneracy" there is no reference to page 50, which has the only non-technical discussion of the topic (such as it is).

All these omissions could easily be remedied, and the many good qualities of the book warrant the hope that a future edition will see such improvements made.

Thomas A. Goldman

Corporation for Economic and Industrial Research
Arlington, Virginia

68[X].—Kurt Arbenz, *Integralgleichungen für einige Randwertprobleme für Gebiete mit Ecken*. Promotionsarbeit. Eidgenössische Technische Hochschule, Zürich, 1958, 43 p.

This paper is devoted to the problem of finding a procedure, suitable in numerical application, for the conformal representation of a simply connected plane domain over the unity circle, in the more difficult case of a boundary with corners. A useful tool which has been employed by Todd [1] is the integral equation of Lichtenstein and Gerschgorin, but it cannot be directly applied in this case. Modifications have been proposed by Stiefel and by Birkhoff, Young and Zarantonello [2].

The author generalizes theorems given by Radon, on the potential theory for domains bounded by smooth arcs. He makes extensive use of methods of functional analysis, with particular reference to the book of Riesz and Nagy [3].

The last seven pages contain numerical examples: the conformal representation of a square on the unit circle, obtained in a rather simple way with good accuracy; and the displacements in a square plate with built-in boundary. It appears that no use has been made of electronic computers, and it would be of interest to start numerical experiments on computers with the procedure here suggested.

ENZO L. APARO

University of Rome
Rome, Italy

1. NBS APPLIED MATHEMATICS SERIES, No. 42, *Experiments in the Computation of Conformal Maps*, U. S. Government Printing Office, Washington, D. C., 1955.
2. *Proceedings of Symposia in Applied Mathematics*, v. 4, 1953, p. 117–140.
3. F. RIESZ & B. V. SZ. NAGY, *Leçons d'analyse fonctionelle*, Third Edition, Gauthier-Villars, Paris, 1955.