Polynomial Approximations to Integral Transforms

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1. Introduction. The symmetric Jacobi polynomials $P_n^{(\alpha,\alpha)}(x)$, orthogonal on the interval $-1 \leqq x \leqq 1$, are widely used for approximating functions, but the integral which defines the coefficients for the expansion of a function $g(x)$ in these polynomials usually is quite difficult to evaluate. The problem is simplified if $g(x)$ is an integral transform of the Fourier or Laplace type, since the kernel of the transform generates a series of the above polynomials. The coefficients in such cases are found to be Hankel transforms, which are widely tabulated.

Examples include Chebyshev polynomial expansions of $1/(x + a)^k$, $\psi(x + a)$, $\log \Gamma(x + a)$, $Ci(x)$ and $Si(x)$.

2. Formulas When $g(x)$ is a Laplace or Fourier Transform. The symmetric Jacobi polynomials [1, v. 2, p. 168] may be defined by

$$(1) \qquad P_n^{(\alpha,\alpha)}(x) = \binom{n+\alpha}{n} {}_2F_1[-n, n + 2\alpha + 1; \alpha + 1; \tfrac{1}{2} - \tfrac{1}{2}x].$$

A function $g(x)$ satisfying certain conditions has the expansion

$$(2) \qquad g(x) = \sum_{n=0}^{\infty} A_n P_n^{(\alpha,\alpha)}(x), \qquad -1 \leqq x \leqq 1,$$

where

$$(3) \quad A_n = \frac{(2n + 2\alpha + 1)n!\Gamma(n + 2\alpha + 1)}{2^{2\alpha+1}[\Gamma(n + \alpha + 1)]^2} \int_{-1}^{1} g(x)(1 - x^2)^\alpha P_n^{(\alpha,\alpha)}(x)\, dx.$$

Suppose now that $g(x)$ is the Laplace transform of some $f(t)$,

$$(4) \qquad g(x) = \mathfrak{L}\{f(t)\} = \int_0^\infty e^{-xt} f(t)\, dt = \sum_{n=0}^{\infty} A_n P_n^{(\alpha,\alpha)}(x).$$

To determine the A_n's replace the kernel of the Laplace transform by its Neumann series [1, v. 2: p. 98, No. (1); p. 175, No. (16); p. 174, No. (6); and the duplication formula for the gamma function].

$$(5) \qquad e^{-xt} = \sum_{n=0}^{\infty} (-)^n \Omega_n \frac{I_{n+\alpha+1/2}(t)}{t^{\alpha+1/2}} P_n^{(\alpha,\alpha)}(x),$$

$$(6) \qquad \Omega_n = \frac{2^{1/2-\alpha}\pi^{1/2}(n + \alpha + \frac{1}{2})\Gamma(n + 2\alpha + 1)}{\Gamma(n + \alpha + 1)}.$$

Then (4) yields

$$(7) \qquad A_n = e^{(n-\alpha-1)[\pi i/2]}\Omega_n \mathfrak{H}\left\{\frac{f(t)}{t^{\alpha+1}}\right\}_{\substack{y=i \\ \nu=n+\alpha+1/2}},$$

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$$\mathcal{H}\{F(t)\} = \int_0^\infty F(t) J_\nu(yt)(yt)^{1/2}\, dt. \tag{8}$$

$\mathcal{H}\{F(t)\}$ denotes the Hankel transform of $F(t)$ [2].

When $\alpha = -\frac{1}{2}$, (7) furnishes the coefficients for the Chebyshev expansion

$$g(x) = \int_0^\infty e^{-xt} f(t)\, dt = \sum_{n=0}^\infty C_n T_n(x), \qquad -1 \leqq x \leqq 1, \tag{9}$$

where

$$C_n = \epsilon_n e^{(n-1/2)[\pi i/2]} \mathcal{H}\left\{\frac{f(t)}{t^{1/2}}\right\}_{\substack{y=i \\ \nu=n}}, \qquad \epsilon_n = \begin{matrix} 1, n = 0, \\ 2, n > 0. \end{matrix} \tag{10}$$

If we replace t by it in (5), we find that the same method is applicable when $g(x)$ is a Fourier transform of $f(t)$. We omit details, but the key results for the sine and cosine transforms are as follows.

$$\begin{matrix} g_1(x) \\ g_2(x) \end{matrix} = \int_0^\infty f(t) \begin{matrix} \sin \\ \cos \end{matrix} (xt)\, dt = \sum_{n=0}^\infty \begin{matrix} S_n \\ C_n \end{matrix} P_n^{(\alpha,\alpha)}(x), \qquad -1 \leqq x \leqq 1 \tag{11}$$

where

$$S_n = \begin{cases} 0, & n \text{ even}, \\ e^{(n-1)[\pi i/2]} \Omega_n \mathcal{H}\left\{\dfrac{f(t)}{t^{\alpha+1}}\right\}_{\substack{y=1 \\ \nu=n+\alpha+1/2}}, & n \text{ odd}, \end{cases} \tag{12}$$

and

$$C_n = \begin{cases} 0, & n \text{ odd}, \\ e^{n\pi i/2} \Omega_n \mathcal{H}\left\{\dfrac{f(t)}{t^{\alpha+1}}\right\}_{\substack{y=1 \\ \nu=n+\alpha+1/2}}, & n \text{ even}. \end{cases} \tag{13}$$

3. The Chebyshev Expansion for $1/(y + a)^k$. Let $g(x) = \left[\dfrac{x+1}{2} + a\right]^{-k}$. Then

$$\mathcal{L}^{-1}\{g(x)\} = \frac{2^k}{(k-1)!} e^{-(2a+1)t} t^{k-1} = f(t). \tag{14}$$

Use (10) and let $y = \dfrac{x+1}{2}$. Then $T_n(2y - 1) = T_n^*(y)$, $0 \leqq y \leqq 1$, is the shifted Chebyshev polynomial [3] and

$$\frac{1}{(y+a)^k} = \left\{\sum_{n=0}^\infty \frac{\epsilon_n(-)^n(k+n-1)!}{(k-1)!} P_{k-1}^{-n} \cdot \left[\frac{2a+1}{2\sqrt{a^2+a}}\right] T_n^*(y)\right\} \Big/ (a^2 + a)^{k/2} \qquad 0 \leqq y \leqq 1, \qquad a > 0, \tag{15}$$

where $P_\nu^\mu(x)$ is the Legendre function [1, v. 1. p. 120]. For $k = 1$, (15) agrees with a result of Luke [4].

TABLE 1

Coefficients for the Series

$$\psi(x+a) = \sum_{n=0}^{\infty} C_n T_n(x), \qquad \ln \Gamma(x+a) = \sum_{n=0}^{\infty} S_n T_n(x).$$

n	$a = 2$		$a = 3$		$a = 4$		$a = 5$	
	C_n	S_n	C_n	S_n	C_n	S_n	C_n	S_n
0	0.30459199	0.17002422	0.88194225	0.79383494	1.23549564	1.86343494	1.49369453	3.23372482
1	.72037978	.36686678	.41097870	.90276517	.28965835	1.24591092	.22406724	1.49994422
2	−.12454959	.17315258	−.04164582	.10135581	−.02083054	.07191856	−.01249938	.05578533
3	.02776946	−.01962889	.00555546	−.00680364	.00198412	−.00343655	.00092592	−.00207042
4	−.00677624	.00325570	−.00082401	.00067831	−.00021127	.00024503	−.00007686	.00011489
5	.00172388	−.00063281	.00012898	−.00008032	.00002385	−.00002085	.00000678	−.00000762
6	−.00044818	.00013383	−.00002083	.00001046	−.00000279	.00000196	−.00000062	.00000056
7	.00011794	−.00002978	.00000343	−.00000145	.00000033	−.00000020	.00000006	−.00000004
8	−.00003125	.00000685	−.00000057	.00000021	−.00000004	.00000002	−.00000001	—
9	.00000832	−.00000161	.00000010	−.00000003	—	—	—	—
10	−.00000222	.00000039	−.00000002	—	—	—	—	—
11	.00000059	−.00000009	—	—	—	—	—	—
12	−.00000016	.00000002	—	—	—	—	—	—
13	.00000004	−.00000001	—	—	—	—	—	—
14	−.00000001	—	—	—	—	—	—	—
15	—	—	—	—	—	—	—	—

4. The Psi and Log Gamma Functions. These examples show how a property of the Laplace transform may be used to advantage when applying (4) and (8). We know that

$$\mathfrak{L}\{e^{-at}f(t)\} = g(x+a). \tag{16}$$

If $g(x)$ cannot be expanded in symmetric Jacobi polynomials, a in (16) can often be chosen so that $g(x+a)$ has a convergent expansion. Let

$$g(x) = \psi^{(m)}(x) = D^{m+1}\log\Gamma(x). \tag{17}$$

Since $\psi^{(m)}(x)$ has poles at zero and the negative integers, we cannot expand the function over $-1 \leqq x \leqq 1$. However, if

$$g(x) = \psi^{(m)}(x+a), \tag{18}$$

then

$$f(t) = \mathfrak{L}^{-1}\{g(x)\} = (-)^{m+1}e^{-at}t^m[1-e^{-t}]^{-1}, \tag{19}$$

and if $\text{Re}(a) > 1$, (7), and in particular (10), may be used since (18) is analytic for $|x| \leqq 1$. Substituting (19) in (10) and expanding $(1-e^{-t})^{-1}$ by the binomial theorem, we have

$$C_n = -\epsilon_n \sum_{k=0}^{\infty} \frac{d^m}{dx^m}\left[\frac{(\sqrt{x^2-1}-x)^n}{\sqrt{x^2-1}}\right]\Bigg|_{x=k+a}. \tag{20}$$

Setting m equal to zero, we get

$$C_n = -\epsilon_n \sum_{k=0}^{\infty} \frac{[\sqrt{(k+a)^2-1}-(k+a)]^n}{\sqrt{(k+a)^2-1}}, \qquad n \geqq 1. \tag{21}$$

TABLE 2

Coefficients for the Series

$$Ci(x) = \int_{\infty}^{x} \frac{\cos t}{t}\,dt = \log(x) + \sum_{n=0}^{\infty} A_{2n}T_{2n}\left(\frac{x}{a}\right), \qquad 0 < x \leqq a$$

$$Si(x) = \int_{0}^{x} \frac{\sin t}{t}\,dt = \sum_{n=0}^{\infty} B_{2n+1}T_{2n+1}\left(\frac{x}{a}\right), \qquad -a \leqq x \leqq a$$

n	$a = 2$		$a = 5$	
	A_{2n}	B_{2n+1}	A_{2n}	B_{2n+1}
0	0.13529 62627	1.69809 09708	−0.96313 15550	2.08578 21107
1	−.42327 51922	−.09558 49521	−1.13103 16550	−.67042 59749
2	.01822 27219	.00295 78196	.34661 70891	.15186 68742
3	−.00041 57650	−.00005 14215	−.05698 43620	−.01861 43512
4	.00000 56716	.00000 05642	.00537 47844	.00138 96747
5	−.00000 00511	−.00000 00042	−.00032 52237	−.00006 95137
6	.00000 00003	—	.00001 36729	.00000 24908
7	—	—	−.00000 04226	−.00000 00671
8	—	—	.00000 00100	.00000 00014
9	—	—	−.00000 00002	—

If $n = 0$, (21) diverges, and for $n = 1$ the series is slowly convergent, but since $T_n(1) = 1$, $T_n(-1) = (-)^n$, we may solve for C_0 and C_1 in terms of higher computable coefficients, i.e.,

$$\tag{22} \begin{cases} C_0 = \dfrac{\psi(a+1) + \psi(a-1)}{2} - \displaystyle\sum_{k=1}^{\infty} C_{2k}, \\ C_1 = \dfrac{\psi(a+1) - \psi(a-1)}{2} - \displaystyle\sum_{k=1}^{\infty} C_{2k+1}. \end{cases}$$

Integration of the series defined by (21) yields a Chebyshev expansion for $\ln \Gamma(x + a)$ because [3]

$$\tag{23} \int T_n(x)\, dx = \frac{1}{2}\left[\frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1}\right] + C.$$

In Table 1 are listed coefficients for the Chebyshev expansions of $\psi(x + a)$ and $\log \Gamma(x + a)$, $a = 2(1)5$, $n = 0(1)15$ to 8D.

5. The Sine and Cosine Integrals. For examples of (11)–(13) let

$$\tag{24} \begin{matrix} g_1(x) \\ g_2(x) \end{matrix} = \begin{matrix} (1 - \cos ax)/x \\ \sin ax/x \end{matrix} = \int_0^{\infty} f(t) \begin{matrix} \sin xt \\ \cos xt \end{matrix}\, dt,$$

$$\tag{25} f(t) = \begin{cases} 1, & 0 < x < a, \\ 0, & a < x < \infty. \end{cases}$$

Using [2, v. 2, p. 333, No. (1)] to evaluate (12) and (13) for $\alpha = -\frac{1}{2}$, we find that

$$\tag{26} S_n = \begin{cases} 0, & n \text{ even}, \\ 4e^{(n-1)[\pi i/2]} \displaystyle\sum_{k=0}^{\infty} J_{n+2k+1}(a), & n \text{ odd}, \end{cases}$$

$$\tag{27} C_n = \begin{cases} 0, & n \text{ odd}, \\ 2\epsilon_n e^{n\pi i/2} \displaystyle\sum_{k=0}^{\infty} J_{n+2k+1}(a), & n \text{ even}. \end{cases}$$

Let $a = 2$ and 5 in (26) and (27), and use [1, v. 2, p. 145, No. (6)] and (23) to obtain the expansion whose coefficients are listed in Table 2.

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