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# Mathematics of Computation

A journal devoted to advances in numerical analysis,  
the application of computational methods, mathematical tables,  
high-speed calculators and other aids to computation



Formerly: Mathematical Tables and other Aids to Computation

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