of non-symmetric matrix for which SOR will always converge provided that a suitable value of ω is chosen.

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On Inverses of Finite Segments of the Generalized Hilbert Matrix

By Jean L. Lavoie

The purpose of this note is to show that two theorems given by Smith [1] on inverses of finite segments of the generalized Hilbert Matrix can be proved in a simple manner by using results from the theory of generalized hypergeometric series.

The usual notation for generalized hypergeometric functions will be used:

$$(1) \qquad {}_PF_Q(z) = {}_PF_Q\left(\begin{matrix} a_1 \cdots a_P \\ b_1 \cdots b_Q \end{matrix}\middle| z\right) = \sum_{K=0}^{\infty} \frac{\prod_{j=1}^{P} (a_j)_K}{\prod_{j=1}^{Q} (b_j)_K} \cdot \frac{z^K}{K!},$$

where

$$(\sigma)_\mu = \frac{\Gamma(\sigma + \mu)}{\Gamma(\sigma)}.$$

See Erdélyi [2], Chapters 2 and 4 for details.

Let H_n represent a finite segment of the generalized Hilbert matrix, i.e.,

$$(2) \qquad H_n = (h_{ij}), \qquad h_{ij} = (p + i + j - 1)^{-1}, \quad i, j = 1, 2, \cdots n.$$

Here n is the order of the segment and obviously

$$p \neq -1, -2, \cdots, -(2n - 1).$$

We shall assume that the above conditions on i, j, and p hold throughout this paper.

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It is well known that the inverses of the finite segments are given by:

$$S_n = (s_n^{ij})$$

$$(3)\quad s_n^{ij} = \frac{(-1)^{i+j}}{p+i+j-1} \cdot \frac{\Gamma(n+p+i)\Gamma(n+p+j)}{\Gamma(i)\Gamma(j)\Gamma(p+i)\Gamma(p+j)\Gamma(n-i+1)\Gamma(n-j+1)},$$

(see Smith [1] for references).

Smith has proved the following theorems. If s_n^{ij} is defined by (3), then

$$(4)\quad \text{I)}\qquad \sum_{i=1}^{n} s_n^{ij} = \sum_{i=1}^{n} s_n^{ji} = (-1)^{n+j} \frac{(p+j)_n}{\Gamma(j)\Gamma(n-j+1)},$$

$$(5)\quad \text{II)}\qquad \sum_{i,j=1}^{n} s_n^{ij} = n(p+n).$$

Now if we use (3) and (1), we easily obtain

$$(6)\quad \sum_{i=1}^{n} s_n^{ij} = \frac{(-1)^{j+1}\Gamma(n+p+1)\Gamma(n+p+j)}{\Gamma(j)\Gamma(n)\Gamma(p+1)\Gamma(p+j+1)\Gamma(n-j+1)} \cdot {}_3F_2\left(\begin{matrix}1-n, n+p+1, p+j\\ p+j+1, p+1\end{matrix}\,\middle|\, 1\right).$$

The ${}_3F_2(1)$ on the right of (6) is a terminating series but is not of Saalschützian type since the sum of the numerator parameters is equal to the sum of the denominator parameters.

To evaluate this particular ${}_3F_2(1)$, we start with the formula

$$(7)\quad {}_3F_2\left(\begin{matrix}a, b, c\\ e, f\end{matrix}\,\middle|\, 1\right) = \frac{\Gamma(e)\Gamma(f)\Gamma(s)}{\Gamma(a)\Gamma(s+b)\Gamma(s+c)}\, {}_3F_2\left(\begin{matrix}e-a, f-a, s\\ s+b, s+c\end{matrix}\,\middle|\, 1\right),$$

where $s = e + f - a - b - c$, (see Bailey [3], page 14, (3.2.1)).

If we substitute $1 - n, n + p + 1, p + j, p + j + 1, p + 1$ for a, b, c, e, f, then $s = 0$, the ${}_3F_2(1)$ on the right in (7) reduces to unity, and a simple limiting process shows that the ratio $\dfrac{\Gamma(0)}{\Gamma(1-n)}$ must be replaced by $(-1)^{n+1}\Gamma(n)$.

Hence we obtain

$$(8)\quad {}_3F_2\left(\begin{matrix}1-n, n+p+1, p+j\\ p+j+1, p+1\end{matrix}\,\middle|\, 1\right) = (-1)^{n+1}(p+j)\, \frac{\Gamma(n)\Gamma(p+1)}{\Gamma(n+p+1)}$$

for $1 \leqq j \leqq n$.

To prove Theorem I, we now only need to use (8) in (6).

To prove Theorem II, we sum from $j = 1$ to $j = n$ on both sides of (4) and we immediately obtain

$$\sum_{i,j=1}^{n} s_n^{ij} = (-1)^{n+1} \frac{\Gamma(n+p+1)}{\Gamma(n)\Gamma(p+1)}\, {}_2F_1\left(\begin{matrix}1-n, n+p+1\\ p+1\end{matrix}\,\middle|\, 1\right) = n(p+n)$$

by using Gauss's theorem, (see Erdélyi [2], page 104, (46)).

An interesting result can be obtained from the fact that

$$H_nS_n = S_nH_n = I$$

where I is the unit matrix.

In terms of the matrix elements, we have

$$\sum_{K=1}^{n} h_{iK}\, s_n^{Kj} = \delta_{ij} \qquad \delta_{ij} = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j. \end{cases} \tag{9}$$

Now from (2), (3), and (1), the last relation implies that

$$\begin{aligned} &{}_4F_3\left(\begin{matrix} 1-n, n+p+1, p+i, p+j \\ p+i+1, p+j+1, p+1 \end{matrix}\,\middle|\, 1\right) \\ &\qquad = (-1)^{j+1}\frac{(p+i)(p+j)\Gamma(j)\Gamma(n)\Gamma(n-j+1)}{(p+1)_n(p+j)_n}\,\delta_{ij} \end{aligned} \tag{10}$$

for $1 \leqq i, j \leqq n$.

This can be proved directly. Indeed the ${}_4F_3(1)$ is a terminating Saalschützian series, and hence, applying Whipple's transformation (Bailey [3], page 94), we obtain a terminating well-poised ${}_5F_4(1)$ which can be summed (Bailey [3], page 25, (4.3.3)) to yield (10).

However, it seems worth noting that if we use the contiguous function relation

$$(\alpha_1 - \alpha_2)F = \alpha_1 F(\alpha_1+) - \alpha_2 F(\alpha_2+)$$

in Rainville ([4], page 82, eq. (14)), with $\alpha_1 = p + i$, $\alpha_2 = p + j$, we obtain

$$\begin{aligned} &{}_4F_3\left(\begin{matrix} 1-n, n+p+1, p+i, p+j \\ p+i+1, p+j+1, p+1 \end{matrix}\,\middle|\, 1\right) \\ &\quad = \frac{1}{i-j}\left[(p+i)\,{}_3F_2\left(\begin{matrix} 1-n, n+p+1, p+j \\ p+j+1, p+1 \end{matrix}\,\middle|\, 1\right)\right. \\ &\qquad\qquad \left. - (p+j)\,{}_3F_2\left(\begin{matrix} 1-n, n+p+1, p+i \\ p+i+1, p+1 \end{matrix}\,\middle|\, 1\right)\right] = 0 \end{aligned}$$

for $i \neq j$ by using (8).

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