Expansion of Dawson's Function in a Series of Chebyshev Polynomials

By David G. Hummer

Dawson's function

$$F(x) = e^{-x^2} \int_0^x e^{t^2}\, dt = \int_0^\infty e^{-t^2} \sin 2xt\, dt \tag{1}$$

is of importance, for instance, in the calculation of profiles of absorption lines [1], [2]. Extensive tables of $F(x)$ are given by Miller & Gordon [3], Rosser [4], and Lomander & Rittsten [5]; the last of these is the most satisfactory. Terrill & Sweeny [6] tabulate $e^{x^2}F(x)$. For use in machine computing in some astrophysical problems in which severe cancellation occurs, we have obtained a Chebyshev expansion of $F(x)$ capable of very high accuracy in the interval $-k \leqq x \leqq k$, where k is sufficiently large so that, for $x > k$, $F(x)$ may be obtained from the asymptotic series

$$F(x) \sim \frac{1}{2x} + \frac{1}{2^2x^3} + \frac{1\cdot 3}{2^3x^5} + \frac{1\cdot 3\cdot 5}{2^4x^7} + \cdots.$$

Since $F(x)$ is an odd function, we write

$$F(kx) = \sum_{n=0}^{\infty} a_n(k) T_{2n+1}(x), \qquad -1 \leqq x \leqq 1, \tag{2}$$

where

$$T_m(x) = \cos(m \cos^{-1} x).$$

From the orthogonality of the $T_m(x)$ we have

$$a_n(k) = \frac{2}{\pi} \int_0^\pi F(k \cos\theta) \cos(2n+1)\theta\, d\theta. \tag{3}$$

Integrating by parts and using the differential equation

$$F'(x) = 1 - 2xF(x),$$

we have

$$\begin{aligned} a_n(k) &= \frac{2}{\pi} \frac{k}{2n+1} \int_0^\pi [1 - 2k\cos\theta F(k\cos\theta)] \sin\theta \sin(2n+1)\theta\, d\theta \\ &= \frac{1}{\pi} \frac{k^2}{2n+1} \int_0^\pi F(k\cos\theta)[\cos(2n+3)\theta - \cos(2n-1)\theta]\, d\theta \end{aligned}$$

or

$$a_n(k) = \frac{k^2}{2(2n+1)} [a_{n+1}(k) - a_{n-1}(k)]. \tag{4}$$

The coefficients a_n may be obtained by the well-known method (see for example

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[7], p. 88–90) of setting

$$\tilde{a}_N = 1, \qquad \tilde{a}_{N+1} = 0$$

and obtaining $\tilde{a}_{N-1}, \cdots, \tilde{a}_0$ recursively from (4). Then

$$F^*(kx) = c\sum_{n=0}^{N} \tilde{a}_n(k)T_{2n+1}(x)$$

and c is obtained from the condition $\frac{d}{dx}F(0) = 1$,

$$c = k/\sum_{n=0}^{N}(-1)^n(2n+1)\tilde{a}_n(k).$$

The coefficients $a_n^*(k) = c\tilde{a}_n$ have been evaluated with $N = 35$ using double-precision arithmetic on the University of London Mercury Computer. In Table 1 we give $a_0^*, \cdots, a_{33}^*$ for $k = 5.0$. The values of $F(x)$ obtained by summing thirty terms in the series using the summation algorithm of Clenshaw [8] agree with the twenty-place value of Lomander and Rittsten to within two places in the 14th place. By including the terms corresponding to $n = 30, \cdots 33$, the error should be reduced to a few units in the 15th place.

The coefficients $a_n(k)$ may also be evaluated analytically. Substituting the second form of $F(x)$ given in (1) into (3) and interchanging the order of integration, we have

$$a_n(k) = \frac{2}{\pi}\int_0^\infty e^{-t^2}\int_0^\pi \sin(2k\cos\theta)\cos(2n+1)\theta\, d\theta.$$

Using some standard results from the theory of Bessel functions, we transform

TABLE 1

n	a_n^* (5)	n	a_n^* (5)
0	.19999999 99972224	17	−.00000278 76379719
1	−.18400000 00029998	18	.00000085 66873627
2	.15583999 99965025	19	−.00000025 18433784
3	−.12166400 00043988	20	.00000007 09360221
4	.08770815 99940391	21	−.00000001 91732257
5	−.05851412 48086907	22	.00000000 49801256
6	.03621573 01623914	23	−.00000000 12447734
7	−.02084976 54398036	24	.00000000 02997777
8	.01119601 16346270	25	−.00000000 00696450
9	−.00562318 96167109	26	.00000000 00156262
10	.00264876 34172265	27	−.00000000 00033897
11	−.00117326 70757704	28	.00000000 00007116
12	.00048995 19978088	29	−.00000000 00001447
13	−.00019336 30801528	30	.00000000 00000285
14	.00007228 77446788	31	−.00000000 00000055
15	−.00002565 55124979	32	.00000000 00000010
16	.00000866 20736841	33	−.00000000 00000002

this to

$$
\begin{aligned}
a_n(k) &= (-1)^n 2 \int_0^\infty e^{-t^2} J_{2n+1}(2kt)\, dt \\
&= (-1)^n \sqrt{\pi} e^{-k^2/2} I_{n+(1/2)}(k^2/2) \qquad (5) \\
&= \sum_{r=0}^{n} \frac{(n+r)!}{r!(n-r)!} k^{-2r-1}[(-1)^{r+n} - e^{-k^2}], \qquad n = 0, 1, 2 \cdots.
\end{aligned}
$$

This expression may easily be seen to be consistent with (4).

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First One Hundred Zeros of $J_0(x)$ Accurate to 19 Significant Figures

By Henry Gerber

1. Introduction. Some physical investigations require a knowledge of accurate values of the zeros of the Bessel function $J_0(x)$. The most extensive values previously published are those of the British Association for the Advancement of Science [1], which consist of 10 decimal places. More accurate values have now been computed, and are presented in Table 1. The minimum accuracy of the tabulated zeros is 19 significant figures.

2. Method of Computation. Two methods were used to compute the roots. The first twelve roots were computed by the method of "false position." The values of

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