Evaluation of the Integral $I_n(b) = \frac{2}{\pi} \int_0^\infty \left(\frac{\sin x}{x}\right)^n \cos(bx)\, dx$

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This integral appears in a number of diverse problems (see, e.g., [1], [2], and [6]). The present authors required it in connection with certain operations involving "white" noise. For example, the intermodulation distortion generated by taking the nth power of a narrow-band, high-frequency white noise is proportional to $I_n(b)$: $b = 0$ corresponds to the distortion level at the mid-band frequency, $b = 1$ to the distortion level at the edges of the band, $b = 2$ to the distortion level at frequencies spaced by twice the semi-bandwidth from the center frequency, and so on.

In closed form, $I_n(b)$ is given [3] by

$$I_n(b) = \frac{n}{2^{n-1}} \sum_{0 \leqq r < (b+n)/2} \frac{(-1)^r (b+n-2r)^{n-1}}{r!\,(n-r)!}, \qquad 0 \leqq b < n$$

(where r takes integral values),

$$= 0, \qquad n \leqq b < \infty.$$

(The lower limit of r is not correct in [3].) The special case of this formula for $b = 0$ is given in [4], [5], and [7]. This expression for $I_n(b)$ is useful for small n, but becomes prohibitively cumbersome, even for $b = 0$, as n increases beyond the range covered in [2]. One is thus led to seek a limiting form for large n.

We consider first the case $b = 0$. For the smaller n's it is practicable to express $I_n(0)$ in rational form, and this is done up to $n = 12$ in [4].

The following Table 1 extends Grimsey's table up to $n = 16$. In [2], a ten-place table is given for $1 \leqq n \leqq 30$. The entry for $n = 30$ is in error, and should read 0.25104 85150.

Goddard [5] takes to task the author of [1] for his lack of rigour, and attempts to supply a deficiency in that reference by deriving what he claims to be an asymptotic expansion for $I_n(0)$, applicable for large n. In fact, while Goddard's method is of considerable interest it involves steps of which the region of applicability needs careful scrutiny. (He has, in addition, made an analytical error resulting in an incorrect coefficient for his final term.)

Goddard's approach is based on a relation which, after correcting some misprints in the version in [5], we can give as

$$\log\left(\frac{\sin x}{x}\right) = -\sum_{K=1}^{\infty} \frac{B_K}{2K(2K)!} (2x)^{2K},$$

where the B_K's are the Bernoulli numbers.

Need for caution is immediately apparent, since the left-hand side becomes imaginary for x in the ranges $(2m-1)\pi$ to $2m\pi$ ($m = 1, 2, 3,$ etc.), while the right-hand side is always a sum of real terms. It can, in fact, be readily shown that the right-hand side diverges for $x \geqq \pi$. Goddard then writes

$$(1) \qquad \left(\frac{\sin x}{x}\right)^n = \exp\left[-n \sum_{K=1}^{\infty} \frac{B_K}{2K(2K)!} (2x)^{2K}\right]$$

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$$= \left[1 + \sum_{m=2}^{\infty} a_{2m} x^{2m}\right] \exp\left(-\frac{nx^2}{6}\right), \tag{1.1}$$

where the a's are functions of n to be determined, and proceeds to integrate expression (1.1) term by term, using the relation

$$\frac{2}{\pi}\int_0^{\infty} \exp\left(-\frac{nx^2}{6}\right) x^{2K}\, dx = \left(\frac{1}{2}\right)^{2K-1} \frac{(2K-1)!}{(K-1)!}\, 6^K n^{-K} \sqrt{\left(\frac{6}{\pi n}\right)}.$$

TABLE 1

n	$I_n(0)$
13	20646903199/54499737600
14	27085381/74131200
15	467168310097/1322526965760
16	2330931341/68108040000

The coefficients a_m are obtained by expanding the exponential expression (1) after removing the term $\exp(-nx^2/6)$, and collecting powers of x. Each term obtained by integrating the terms of (1.1) yields sums of inverse powers of n (together with the factor $\sqrt{(6/\pi n)}$), and these can be collected to yield a power series in $1/n$. Fortunately, there are only a finite number of contributions to each power of $1/n$.

Goddard obtained an expansion, in this way, up to $(1/n)^3$. However, the last term he gives is in error, since he failed to take account of all contributions. We have evaluated in closed form the coefficients of terms up to $(1/n)^6$. The labour increases rapidly with the order of the term, and after this point it only seemed worth computing the coefficients to a few significant figures. As far as it has been taken, the expansion is as follows:

$$\begin{aligned} I_n(0) \approx \sqrt{\left(\frac{6}{\pi n}\right)}\Bigg[1 &- \frac{3}{20}\frac{1}{n} - \frac{13}{1120}\frac{1}{n^2} + \frac{27}{3200}\frac{1}{n^3} + \frac{52791}{3942400}\frac{1}{n^4} \\ &+ \frac{482427}{66560000}\frac{1}{n^5} - \frac{124996631}{10035200000}\frac{1}{n^6} - 0.0386166\frac{1}{n^7} \\ &- 0.027677\frac{1}{n^8} + 0.12245\frac{1}{n^9} + 0.4523\frac{1}{n^{10}}\Bigg]. \end{aligned} \tag{2}$$

Presumably, this is a divergent series (though, since the coefficients are not known in analytical form, it is not easy to see how this could be proved), and Goddard claims it is asymptotic. In fact, after making the correction already mentioned to the last entry of the table of [2], formula (2) yields $I_n(0)$ values in agreement to the full ten places with those tabulated over the range $17 \leqq n \leqq 30$. At $n = 16$ there is a disagreement of one unit in the last place, while at $n = 15$ there is exact agreement. For $n = 14$, 13 and 12 the agreement is increasingly poor, the last two digits from formula (2) being 93, 56 and 02, respectively, whereas the corresponding exact values are 95, 45 and 52.

It should be emphasized that the discrepancy is not due to a failure of initial convergence. For $n = 12$, eleven terms of the series yield a sum $+0.39392\ 55601\ 869$, the final term being $+0.00000\ 00000\ 029$. This is to be compared with the true value

0.39392 55651 7: if the series were convergent, or were asymptotic in the sense that the error is less in magnitude than the last included term, it is clear that ten-place accuracy should have resulted.

The discrepancy has, in fact, a rather interesting source. The expression $((\sin x)/x)^n$, when plotted out, will clearly consist of a succession of peaks (positive for even n, and of alternating sign for odd n) centered round $x = 0$, $x = 4.4934$, etc. In the region $0 \leqq x \leqq \pi$, containing the first peak, the series in expression (1.1) converges so that (1.1) represents the required expression with increasing accuracy as more terms are taken. But in the region containing the second peak, the exponential in (1.1) becomes, for moderate n, vanishingly small to the order of accuracy in which we are interested, and the terms of the series are not such as to counterbalance the exponential. Consequently, (1.1) cannot include the contribution of the second and later peaks, and, in fact, it represents an approximation only to the contribution of the first peak to $I_n(0)$.

This can readily be put on a numerical basis. It is easy to show that a good approximation to the contribution of the second peak is

$$\sqrt{\left(\frac{6}{\pi n}\right)}\left\{\frac{2}{\sqrt{3}}(\cos x_0)^n\left[1-\left(\frac{1}{4}+\frac{1}{x_0^2}\right)\frac{1}{n}\right]\right\},$$

where x_0 is the second root of $\tan x = x$. For $n = 12$, this gives $+0.00000\ 00049\ 6$, and adding this to the contribution from expression (2) given above, we arrive at the value 0.39392 55651 5, which is in adequate agreement with the true value. The discrepancies at $n = 13$ and 14, already noted, are accounted for similarly. The correction is negligible for larger n's.

We have, accordingly, used formula (2), with the aid of an Hec 2M digital computer, to extend the table in [2] up to a value of n such that a small number of terms of the formula will suffice to compute further entries. Table 2 covers the range $30 \leqq n \leqq 100$. Thereafter, four terms of formula (2) give ten-place accuracy or better.

Returning to the general form $I_n(b)$, Goddard recommends using, for large n, an expansion derived in the same way as formula (2) above. Goddard's expansion, taken to two more terms than he gives, is

$$\begin{aligned}I_n(b) \approx \sqrt{\left(\frac{6}{\pi n}\right)}\exp\left(-\frac{3b^2}{2n}\right)\Bigg\{1 &- \frac{3}{20}\left(\frac{1}{n}\right)+\left(\frac{9b^2}{10}-\frac{13}{1120}\right)\left(\frac{1}{n}\right)^2\\ &-\left(\frac{9}{20}b^4+\frac{81}{280}b^2-\frac{27}{3200}\right)\left(\frac{1}{n}\right)^3\\ &+\left(\frac{603}{560}b^4-\frac{243}{1600}b^2+\frac{52791}{3942400}\right)\left(\frac{1}{n}\right)^4\Bigg\}.\end{aligned}$$

The leading term is given without derivation in [6].

Since in the fourth term of the series b appears to a higher power than $1/n$ (in later terms the largest excess of the power of b over that of $1/n$ increases without limit), the approximation is obviously useless for large n and b not small, but it may be useful for small b (say <1), where only limited accuracy is required.

For general b and large n we have not found a satisfactory computation procedure. However, for integral b, a suitable recurrence formula is readily derived.

TABLE 2

$$I_n(0) = \frac{2}{\pi} \int_0^\infty \left(\frac{\sin x}{x} \right)^n dx$$

n	$I_n(0)$	n	$I_n(0)$
30	0.25104 85150	66	0.16972 24991
31	0.24700 63826	67	0.16845 68919
32	0.24315 33907	68	0.16721 91807
33	0.23947 52362	69	0.16600 83556
34	0.23595 90855	70	0.16482 34572
35	0.23259 33864	71	0.16366 35732
36	0.22936 77077	72	0.16252 78358
37	0.22627 26033	73	0.16141 54185
38	0.22329 94960	74	0.16032 55342
39	0.22044 05769	75	0.15925 74323
40	0.21768 87196	76	0.15821 03967
41	0.21503 74049	77	0.15718 37440
42	0.21248 06563	78	0.15617 68214
43	0.21001 29831	79	0.15518 90047
44	0.20762 93306	80	0.15421 96975
45	0.20532 50370	81	0.15326 83288
46	0.20309 57944	82	0.15233 43521
47	0.20093 76156	83	0.15141 72438
48	0.19884 68037	84	0.15051 65022
49	0.19681 99255	85	0.14963 16462
50	0.19485 37880	86	0.14876 22145
51	0.19294 54167	87	0.14790 77639
52	0.19109 20377	88	0.14706 78691
53	0.18929 10594	89	0.14624 21216
54	0.18754 00585	90	0.14543 01285
55	0.18583 67656	91	0.14463 15123
56	0.18417 90530	92	0.14384 59096
57	0.18256 49233	93	0.14307 29708
58	0.18099 25000	94	0.14231 23593
59	0.17946 00172	95	0.14156 37508
60	0.17796 58125	96	0.14082 68331
61	0.17650 83184	97	0.14010 13048
62	0.17508 60560	98	0.13938 68757
63	0.17369 76283	99	0.13868 32655
64	0.17234 17150	100	0.13799 02041
65	0.17101 70663		

Integration by parts gives

$$I_n(b) = -\frac{2}{\pi}\frac{n}{b} \int_0^\infty \left(\frac{\sin x}{x} \right)^n \cot x \sin (bx)\, dx$$

$$+ \frac{2}{\pi}\frac{n}{b} \int_0^\infty \left(\frac{\sin x}{x} \right)^{n+1} \operatorname{cosec} x \sin (bx)\, dx.$$

The following results can be obtained straightforwardly.

b even:

$$\cot x \sin (bx) = \cos bx + 2 \cos (b - 2)x + 2 \cos (b - 4)x + \cdots + 2 \cos 2x + 1,$$

$$\operatorname{cosec} x \sin (bx) = 2 \cos (b - 1)x + 2 \cos (b - 3)x + 2 \cos (b - 5)x + \cdots + 2 \cos x.$$

b odd:

$$\cot x \sin (bx) = \cos bx + 2 \cos (b - 2)x + 2 \cos (b - 4)x + \cdots + 2 \cos x,$$

$$\operatorname{cosec} x \sin (bx) = 2 \cos (b - 1)x + 2 \cos (b - 3)x + 2 \cos (b - 5)x + \cdots + 2 \cos 2x + 1.$$

We consequently have the following relations.

b even:

$$I_n(b) = -\frac{n}{b}\left[I_n(b) + 2I_n(b-2) + 2I_n(b-4) + \cdots + 2I_n(2) + I_n(0)\right] + \frac{n}{b}\left[2I_{n+1}(b-1) + 2I_{n+1}(b-3) + \cdots + 2I_{n+1}(1)\right].$$

b odd:

$$I_n(b) = -\frac{n}{b}\left[I_n(b) + 2I_n(b-2) + 2I_n(b-4) + \cdots + 2I_n(1)\right] + \frac{n}{b}\left[2I_{n+1}(b-1) + 2I_{n+1}(b-3) + \cdots + 2I_{n+1}(2) + I_{n+1}(0)\right].$$

Using these relations, from a table of $I_n(0)$ there can be derived a table of $I_n(1)$, and then a table of $I_n(2)$, and so on (at each stage the largest n is one less than at the previous stage though, in view of the existence of the simple limiting form for $I_n(0)$ when n is large, this is no hardship).

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