Gauss Weights and Ordinates for $\int_0^1 f(x)x^2\,dx$

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1. Introduction. In the physics of many-particle systems, starting from the Fermi gas model, one frequently encounters integrals with a quadratic weight factor. The most efficient numerical procedure for their evaluation is Gauss quadrature [1]:

$$\int_0^1 f(x)x^2\,dx = \sum_i^N H_i f(a_i). \tag{1.1}$$

As is well known, the N-point formula is exact for polynomials $f(x)$ of degree $(2N - 1)$ or less. In the course of some recent work we have evaluated the weights and ordinates for equation (1.1) correct to 15 decimals, up to $N = 20$. This range is adequate for physical applications. Besides, it is only when using a relatively low order formula that there will be a significant gain in accuracy when using these results as opposed to ordinary Gauss quadrature.

2. Procedure. Since the theory of Gaussian quadrature is well known, we state here only the special results needed for this case. The ordinates a_i of equation (1.1) are the zeros of a polynomial, say $P_n(x)$, which is a member of a set of orthogonal polynomials over the interval (0, 1). It is convenient to take

$$\int_0^1 P_n(x)P_m(x)x^2\,dx = \delta_{nm}/(2n + 3). \tag{2.1}$$

Then

$$P_n(x) = \frac{1}{n!}\frac{\partial^n}{\partial x^n}\{x^{n+2}(x - 1)^n\}. \tag{2.2}$$

With the particular normalisation chosen here, one finds the recurrence formulae

$$\begin{aligned} P_{n+1}(x) &= \frac{2n + 3}{(n + 1)(n + 3)}\left[(2n + 4)x - \frac{(n^2 + 3n + 3)}{(n + 1)}\right]P_n(x) \\ &\quad - \frac{n(n + 2)^2}{(n + 3)(n + 1)^2}P_{n-1}(x), \end{aligned} \tag{2.3}$$

$$\begin{aligned} (x^2 - x)P_n'(x) &= \frac{n(n + 1)(n + 3)}{2(n + 2)(2n + 3)}P_{n+1}(x) \\ &\quad + \frac{n(n + 3)}{2(n + 2)(n + 1)}P_n(x) - \frac{n(n + 2)(n + 3)}{2(2n + 3)(n + 1)}P_{n-1}(x). \end{aligned} \tag{2.4}$$

At first a FORTRAN program was written which found the zeros a_i by Bairstow's method of quadratic factors, having constructed the coefficients of the polynomial $P_n(x)$. It was found advisable, however, to improve the zeros by Newton's method, using the recurrence relations to evaluate the polynomial. The weights H_i were

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TABLE 1

a_i	H_i
$N = 1$	
.75000 00000 00000	.33333 33333 33333
$N = 2$	
.45584 81559 88775	.10078 58820 79825
.87748 51773 44559	.23254 74512 53508
$N = 3$	
.29499 77901 11502	.02995 07030 08581
.65299 62339 61648	.14624 62692 59866
.92700 59759 26850	.15713 63610 64887
$N = 4$	
.20414 85821 03227	.01035 22407 49918
.48295 27048 95632	.06863 38871 72923
.76139 92624 48137	.14345 87897 99214
.95149 94505 53003	.11088 84156 11278
$N = 5$	
.14894 57870 52984	.00411 38252 03099
.36566 65273 69113	.03205 56007 22962
.61011 36129 34481	.08920 01612 21590
.82651 96792 28304	.12619 89618 99911
.96542 10600 81785	.08176 47842 85771
$N = 6$	
.11319 43838 22438	.00183 10758 06869
.28431 88726 88286	.01572 02971 84945
.49096 35868 35248	.05128 95711 29616
.69756 30819 77109	.09457 71867 48541
.86843 60583 42014	.10737 64997 36781
.97409 54449 06333	.06253 87027 26581
$N = 7$	
.08881 68334 36997	.00089 26880 33689
.22648 27534 08562	.00816 29256 32305
.39997 84867 21007	.02942 22112 89529
.58599 78554 02940	.06314 63787 08891
.75944 58739 51942	.09173 38032 79795
.89691 09708 51951	.09069 88246 12686
.97986 72262 26599	.04927 65017 76438
$N = 8$	
.07149 10350 40093	.00046 85177 84035
.18422 82964 16716	.00447 45217 13014
.33044 77281 75639	.01724 68637 80235
.49440 29218 15511	.04081 44263 88544
.65834 80085 22798	.06844 71834 21653
.80452 48315 11260	.08528 47691 71939
.91709 93825 14349	.07681 80932 67223
.98390 22404 48079	.03977 89578 06691
$N = 9$	
.05875 41357 86726	.00026 12623 46519
.15256 31579 86763	.00257 43986 45608
.27673 14282 59665	.01040 61165 79352
.42030 59436 48837	.02634 32809 02550
.57054 63546 12187	.04854 06278 64507
.71410 89116 60780	.06928 43956 89805
.83824 04832 35925	.07767 35691 60556
.93191 54704 06232	.06548 95370 33397
.98683 41144 02886	.03276 01451 11040
$N = 10$	
.04912 57073 59477	.00015 32400 36697
.12828 99254 25592	.00154 55231 94737
.23465 23204 51892	.00646 98890 68559
.36039 05113 45290	.01719 75750 46553
.49619 28735 85126	.03385 45650 16814
.63199 21495 50662	.05288 37887 66964
.75771 88038 67476	.06745 22193 81438
.86404 97659 49771	.07006 95077 08666
.94310 18494 66342	.05627 29364 02808
.98903 15475 43826	.02743 40887 10097
$N = 11$	
.04167 48562 13352	.00009 38004 86010
.10930 89006 45608	.00096 30205 75543
.20121 56414 09085	.00414 00042 63219
.31167 57601 37043	.01141 45909 18098
.43376 01608 45432	.02358 96635 38807
.55980 16001 41138	.03929 02177 98436
.68188 20480 93766	.05463 22223 21045
.79233 16584 02940	.06418 35593 49756
.88421 13733 70605	.06297 95430 00367
.95175 03079 05714	.04874 58610 30664
.99072 10261 68651	.02330 08500 51388
$N = 12$	
.03579 34647 07328	.00005 95503 59808
.09420 49264 55456	.00061 99533 98545
.17428 14857 85213	.00272 08421 44104
.27175 35470 46927	.00771 63670 23886
.38137 61365 41544	.01654 73436 44609
.49724 26395 50968	.02890 80518 38727
.61310 79900 02310	.04276 47635 70189
.72272 65422 00685	.05457 39411 64232
.82018 92109 47577	.06025 06074 87605
.90024 23744 57386	.05658 41847 49474
.95857 21895 45705	.04255 66153 67406
.99204 79542 97362	.02003 11125 84749
$N = 13$	
.03107 11057 71411	.00003 90169 93264
.08200 03544 62793	.00041 07208 91791
.15230 89560 52768	.00183 23911 39239
.23875 02336 91723	.00531 35693 38833

TABLE 1—*Continued*

$N = 13$—*Cont'd*

a_i			H_i		
.33729	06864	39611	.01172	82595	93102
.44332	53773	70545	.02125	49967	14757
.55189	73610	94290	.03294	04516	78433
.65793	04873	53925	.04462	46007	55615
.75646	71004	12872	.05334	00216	05872
.84290	00258	03491	.05611	15371	30827
.91318	82198	42415	.05090	98307	12695
.96404	66372	99830	.03742	67081	68844
.99310	91686	90041	.01740	12286	10062

$N = 14$

a_i			H_i		
.02722	28792	58817	.00002	62761	61669
.07200	42612	53248	.00027	90993	40947
.13417	50992	68403	.00126	17838	95157
.21122	57615	65439	.00372	48756	75283
.30000	90828	30094	.00841	29767	37136
.39689	26032	72403	.01569	43322	85073
.49791	08602	50022	.02521	61835	36839
.59892	86213	02846	.03573	90102	92515
.69581	04718	58344	.04525	34322	54461
.78459	02429	31104	.05138	99240	39472
.86163	34740	40526	.05202	58654	10702
.92378	63775	25146	.04591	47721	14671
.96850	57401	81342	.03313	95954	69961
.99397	11911	28933	.01525	52061	19448

$N = 15$

a_i			H_i		
.02404	61256	64976	.00001	81287	85570
.06371	81782	64481	.00019	39872	63064
.11905	04712	84686	.00088	65736	35234
.18807	45227	24990	.00265	56629	85774
.26830	20269	47477	.00611	10501	10206
.35683	53982	31215	.01166	85824	84777
.45047	56444	06449	.01929	34165	85275
.54583	87642	45523	.02832	65531	74241
.63947	83064	12942	.03747	38757	92101
.72801	00340	70514	.04499	15558	20616
.80823	42937	22346	.04904	12678	94409
.87725	17137	84502	.04813	20576	68530
.93256	81341	61001	.04153	13076	51604
.97218	54380	81740	.02952	76507	56798
.99468	09479	97156	.01348	16627	25134

$N = 16$

a_i			H_i		
.02139	35625	01303	.00001	27784	10570
.05677	57142	87754	.00013	75781	03976
.10631	40254	51196	.00063	44599	99569
.16844	41169	73538	.00192	35319	26067
.24117	47420	50859	.00449	48488	51480
.32217	00825	85439	.00874	75454	95612
.40882	76146	83497	.01480	37766	57752
.49836	24438	46812	.02235	71856	49530
.58789	70358	83140	.03061	20379	17838
.67455	37899	75870	.03834	80469	51186
.75554	75682	55489	.04411	26576	75899
.82827	52405	16998	.04650	62393	97363
.89039	93907	41229	.04449	65734	72543
.93992	35622	83362	.03768	58246	81919
.97525	70323	80565	.02646	08425	17340
.99527	23128	77065	.01199	94056	24692

$N = 17$

a_i			H_i		
.01915	61061	52587	.00000	91808	04200
.05090	29834	03455	.00009	93565	25335
.09549	45766	85671	.00046	16740	58777
.15167	29142	90049	.00141	38837	58956
.21782	68969	18779	.00334	65172	91820
.29205	49986	14914	.00661	64197	60362
.37222	24614	53186	.01141	36422	38637
.45602	33219	09524	.01763	87893	31050
.54104	69466	46981	.02482	92561	11976
.62484	74583	59182	.03216	35701	12379
.70501	41337	01977	.03855	52419	47102
.77924	07748	07849	.04282	54039	27935
.84539	20717	34114	.04392	26187	72593
.90156	50631	69459	.04114	39758	47068
.94614	39851	72815	.03430	78897	95672
.97784	72946	88170	.02383	77620	18703
.99577	02345	13509	.01074	81510	30769

$N = 18$

a_i			H_i		
.01725	15924	05713	.00000	67100	86125
.04589	21044	82683	.00007	29370	08664
.08623	01358	89207	.00034	10912	03688
.13724	32792	99945	.00105	35502	83957
.19761	48501	63583	.00252	07216	38436
.26578	26918	64987	.00505	02487	88355
.33998	18727	74684	.00885	22782	25095
.41829	09473	29082	.01394	38976	53648
.49868	18972	49726	.02007	86416	96370
.57907	27241	29682	.02672	34243	68960
.65738	14021	97992	.03309	59762	02072
.73157	98172	13398	.03826	27946	72349
.79974	63040	76370	.04128	36658	09958
.86011	54291	64831	.04137	71822	87520
.91112	37408	83503	.03807	51342	53448
.95145	03546	95775	.03133	39076	71730
.98005	16489	52657	.02157	87726	79194
.99619	34177	52498	.00968	23988	03766

$N = 19$

a_i			H_i		
.01561	71934	48298	.00000	49807	02332
.04158	30975	76593	.00005	43426	43132
.07823	93182	70860	.00025	55308	58891
.12474	68370	16173	.00079	50443	25449
.18001	96429	36763	.00191	97884	72692
.24276	34808	26702	.00388	97848	12553
.31150	82090	85905	.00691	08110	77932
.38464	27551	95453	.01106	14406	29037
.46045	30372	07912	.01623	18153	21996
.53716	22014	47980	.02209	05631	49824
.61297	22972	30668	.02809	16079	36103
.68610	64325	81782	.03352	50680	70610
.75485	04309	04831	.03760	75005	20348
.81759	30163	30908	.03959	82482	14881
.87286	35901	38440	.03892	23862	22262
.91936	67236	33339	.03527	81171	99476
.95601	26049	09157	.02870	81497	78704
.98194	30216	62099	.01962	08563	40232
.99655	61095	96138	.00876	72970	56879

TABLE 1—*Continued*

a_i			H_i			a_i			H_i		
$N = 20$						$N = 20$					
.01420	42111	59358	.00000	37492	20993	.57185	49590	66743	.02368	22904	14246
.03785	12878	49502	.00004	10391	02087	.64323	91298	97547	.02900	24047	45589
.07130	09850	81249	.00019	38883	09618	.71154	72878	18728	.03355	64314	53475
.11385	86970	85453	.00060	70457	04267	.77532	35806	15695	.03669	74204	72018
.16462	10853	68438	.00147	74444	41971	.83320	87465	18991	.03784	73903	76922
.22250	74079	64513	.00302	24891	70907	.88396	90923	44513	.03658	79165	53227
.28628	43889	84712	.00543	20908	06123	.92652	28082	14715	.03273	46443	17573
.35459	29542	22632	.00881	35560	78243	.95996	30995	38093	.02638	25644	29126
.42597	73418	17173	.01314	09192	56662	.98357	79118	66012	.01791	37523	25739
.49891	61845	17151	.01822	05034	18269	.99686	93162	59256	.00797	57927	36277

then found from the relation [2]

$$H_i = -(2n + 4)/(n + 1)(n + 3)P_{n+1}(a_i)P_n{'}(a_i). \tag{2.5}$$

The weights and ordinates were checked by applying them to the integrals of x^r, $r = 0, 1, \cdots, 2n$.

The initial calculations were carried out on the IBM 7040 at McMaster University. The results presented in Table 1 were computed in double precision on the CDC 1604 at Cornell Computing Center. The latter machine has a 48 bit word length in single precision. The numbers in Table 1 are rounded correct to 15 decimals. The checks mentioned above were satisfied in every case to better than 20 figures, $r \leqq (2n - 1)$.

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1. Z. KOPAL, *Numerical Analysis*, 2nd ed., Wiley, New York, 1961; Chapter VII. MR **25** #733.
2. Z. KOPAL, ibid., p. 393.