Abscissas, Coefficients, and Error Term for the Generalized Gauss-Laguerre Quadrature Formula Using the Zero Ordinate

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1. Introduction. Several tables of abscissas and coefficients for the generalized Gauss-Laguerre quadrature formula not using the zero ordinate have been issued recently [1–5]. Such formulas are all of odd degree of precision, and the error term is known [2]. Formulas using the zero ordinate are of even degree of precision; abscissas and coefficients have been tabulated only very briefly by Burnett [6] and copied by Kopal [7], and more elaborately but only for $s = 0$ by Krylov and Fedenko [8]; no formula for the error term has been published—it is derived in Section 3 below.

Table I in Section 2 gives abscissas and coefficients for $s = 0, -1/3, -1/2, -2/3$, with $n = 2(1)16$, thus complementing three of the tables of [2] and [3] by providing the abscissas and coefficients for the quadratures of even degree of precision. A less accurate table covering

$$s = -.99(.01) -.90, -.80, -.75, -.70(.1) -.3, -.25, -.2, -.1, 0(1)10$$

may be obtained by requesting reference [9]. All the tables include values of $H_j e^{a_j}$ for convenience in the integration of functions not explicitly containing the factor e^{-x}.

In selecting a formula for practical use in numerical integration one is usually concerned with two matters: (1) a minimum amount of calculational labor and (2) an error which is acceptably small. The amount of calculational labor often is controlled by the number of times that $f(x)$ must be evaluated, and the only significant difference between one evaluation and another is that $f(0)$ may be zero or more easily calculated than other values. If the labor of calculating $f(0)$ may be neglected, the number of evaluations of $f(x)$ for the formula using the zero ordinate and of degree of precision $2n - 2$ is $n - 1$, the same as the number of evaluations for the formula not using the zero ordinate and of lower degree of precision $2n - 3$; otherwise the number of evaluations is n and equals that not using the zero ordinate of degree of precision $2n - 1$. Anticipating the result of Section 3, we see that the error terms of the degrees of precision $2n - 3$, $2n - 2$ and $2n - 1$ are, respectively,

$$\frac{(n-1)!\,\Gamma(n+s)f^{(2n-2)}(\eta)}{(2n-2)!}, \qquad \frac{(n-1)!\,\Gamma(n+s+1)f^{(2n-1)}(\eta)}{(2n-1)!},$$

$$\frac{n!\,\Gamma(n+s+1)f^{(2n)}(\eta)}{(2n)!}.$$

Disregarding the unknown behavior of the higher derivatives, the ratio of the second coefficient to the first is $(n + s)/(2n - 1)$; the ratio of the third coefficient

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to the second is 1/2. Thus, for the values of n and s included in Table I, the first ratio is nearly 1/2; the error terms decrease in a uniform way; if the labor to evaluate $f(0)$ is negligible, one should select the formula with the zero ordinate, otherwise use the formula with the same number of ordinates not including $x = 0$. For the larger values of s, such as some of those in reference [9], the first ratio becomes larger than 1, and the selection of a most suitable formula cannot be stated so simply.

2. Numerical Values of Abscissas and Coefficients. The abscissas a_j and the coefficients H_j given in Table I following are derived from the requirement of maximum degree of precision ($E = 0$ for $f(x)$ a polynomial of as high degree as possible) of the quadrature formula

$$\int_0^\infty x^s e^{-x} f(x)\, dx = H_1 f(0) + \sum_{j=2}^{n} H_j f(a_j) + E. \tag{1}$$

Since the a_j and H_j, including H_1 but excluding $a_1 = 0$, constitute a set of $2n - 1$ parameters, the degree of precision is $2n - 2$.

Numerical values have been calculated by using the formulas given by Kopal [7]. Equivalent formulas may be found in Mineur [10] and in Krylov and Fedenko [8]. (Facile comparison of formulas and numerical values is impeded by differences of notation. E. g., Kopal's H_1 does not include the factor $1 + s$ which is included by Krylov and Fedenko and in formula (1) herein; Krylov and Fedenko begin formula (1) with $Af(0)$ and run the summation from 1 to n so that their n is one smaller than the present n which is the same as Kopal's.) Krylov and Fedenko [8] have observed that the $a_j \neq 0$ of (1) are the same as the a_j of the formula not using the zero ordinate, but for a value of s increased by unity:

$$\int_0^\infty x^{s+1} e^{-x} f(x)\, dx = \sum_{j=2}^{n} K_j f(a_j)$$

and that $K_j = H_j a_j$ $(j = 2, \cdots, n)$.

The abscissas of (1) are roots of the polynomial

$$\Lambda_n^s(x) \equiv L_n^s(x) - (n + s) L_{n-1}^s(x) = 0, \tag{2}$$

where $L_n^s(x)$ represents the generalized Laguerre (or Sonine) polynomial. As given in [7],

$$H_j = \frac{\Gamma(n)\Gamma(n + s)}{(n + s)\{L_{n-1}^s(a_j)\}^2},$$

but the present notation incorporates the factor $1 + s$ into H_1.

For each combination of n and s, the accuracy of calculated a's and H's has been checked by the relations

$$\sum_{j=1}^{n} a_j = (n - 1)\cdot(n + s), \qquad \prod_{j=2}^{n} a_j = \frac{\Gamma(n + s + 1)}{\Gamma(s + 2)} \quad \text{and} \quad \sum_{j=1}^{n} H_j = \Gamma(s + 1).$$

In no case do the calculated and theoretical values differ in more than the last two digits. Since there are obvious cases (e.g., H_1 at $s = 0$, $n = 11$) in which the calculated values are incorrect in the last two digits, it is expected that only 14 digits of H are reliable. The a's are probably good to 15 digits.

TABLE I

S= 0.0

N	A	H	H X EXP(A)
14	0.0000000000000000D 00	7.1428571428571780D-02	7.1428571428571780D-02
	2.6258839817190845D-01	3.3970317475984510-01	4.4171311889095840-01
	8.8355030738771940D-01	3.3126121545698910-01	8.0147801519558620-01
	1.8690338151998080D 00	1.8076213791615820-01	1.1717056972563210 00
	3.2324186994511710 00	6.1495403075257720-02	1.5583473142508220 00
	4.9935756070742090D 00	1.3349836732775050-02	1.9686036476142370 00
	7.1806104941007960 00	1.8359717270997590-03	2.4119344648008800 00
	9.8328082510199980 00	1.5570326898229490-04	2.9015636085907660 00
	1.3005622405258620 01	7.7706542594632080-06	3.4572248694914410 00
	1.6779613641175660 01	2.1214031113373790-07	4.1107097614493970 00
	2.1277911755447570 01	2.8245945258516800-09	4.9185350993055930 00
	2.6705034098927450 01	1.5136712597238150-11	5.9962607932197310 00
	3.3452784965767680 01	2.2633023880736130-14	7.6401607553488840 00
	4.2524447566018220 01	3.7621928601771740-18	1.1055387735634260 01
15	0.0000000000000000D 00	6.6666666666666980-02	6.6666666666666980-02
	2.4503301509309880-01	3.2254095454441010-01	4.1209880316148500-01
	8.2408222004786560-01	3.2766970316747500-01	7.4701583807958860-01
	1.7418724007644670 00	1.9101972226977750-01	1.0903446102372420 00
	3.0091324590179290 00	7.1376361170824540-02	1.4467850943168900 00
	4.6416340665906600 00	1.7566132220463590-02	1.8218482907534200 00
	6.6613414966271700 00	2.8435634411518510-03	2.2225256204449620 00
	9.0983027037705280 00	2.9733203395971190-04	2.6581798445749040 00
	1.1993703433185720 01	1.9427381835351090-05	3.1420529152803450 00
	1.5404988031051520 01	7.5370521229970470-07	3.6940503819580250 00
	1.9414992828165710 01	1.6080663816091860-08	4.3463853090108190 00
	2.4149757952025670 01	1.6758955850891070-10	5.1564828152922790 00
	2.9818105119673750 01	7.0062051865194440-13	6.2419655349268620 00
	3.6819624257000880 01	8.0784678316756170-16	7.9047591240438770 00
	4.6177430016985000 01	1.0030402504955040-19	1.1374214282415580 01
16	0.0000000000000000D 00	6.2500000000000280-02	6.2500000000000280-02
	2.2968050542513540-01	3.0695869696599220-01	3.8621480580026980-01
	7.7214491037541170-01	3.2320216799611000-01	6.9953997812913050-01
	1.6310530990674460 00	1.9958462154190950-01	1.0197282136774920 00
	2.8151445900122550 00	8.0893567224677880-02	1.3505658013884020 00
	4.3371660773375610 00	2.2177638721508450-02	1.6963744798664830 00
	6.2146427645592390 00	4.1248203421073980-03	2.0624816680865910 00
	8.4711639813467160 00	5.1429144949694240-04	2.4557778408381930 00
	1.1138331965750800 01	4.1968202365334960-05	2.8856023147955300 00
	1.4258910021624240 01	2.1599994856730120-06	3.3652674747776990 00
	1.7892053438169480 01	6.6420015562263090-08	3.9148853784243580 00
	2.2122620174833620 01	1.1269621551339800-09	4.5671071339213780 00
	2.7079311499047550 01	9.3413033794928710-12	5.3802571678634850 00
	3.2974973552397410 01	3.0925807054179290-14	6.4739612665929540 00
	4.0216583711496550 01	2.7909692203439550-17	8.1555896295227460 00
	4.9846221708556570 01	2.6267530271263820-21	1.1677725790105310 01

S=-0.5

N	A	H	H X EXP(A)
14	0.0000000000000000D 00	4.2357728801506110-01	4.2357728801506110-01
	1.7961313656338840-01	7.0903460819951040-01	8.4854021130261480-01
	7.2082025894669730-01	4.1474096150202010-01	8.5275679971894050-01
	1.6308996559505540 00	1.6834616216810450-01	8.5999106920951320-01
	2.9225979212203510 00	4.6831867250683910-02	8.7058188079296810-01
	4.6151784764720220 00	8.7625596715851140-03	8.8506982366368760-01
	6.7361859722852350 00	1.0735392313762270-03	9.0428706092021620-01
	9.3243655946475390 00	8.2935160087683140-05	9.2952101366615920-01
	1.2434617774439650 01	3.8305618011906050-06	9.6282721524376200-01
	1.6146884936955450 01	9.7908008193991180-08	1.0076767623332930 00
	2.0583522072153260 01	1.2310063156463640-09	1.0704597326678250 00
	2.5947919119816550 01	6.2682352272460920-12	1.1646176735617740 00
	3.2629863503683360 01	8.9415449147862630-15	1.3255076117606930 00
	4.1627531579865930 01	1.4191558594102740-18	1.7007377637850670 00
15	0.0000000000000000D 00	4.0897117463523210-01	4.0897117463523210-01
	1.6741569967619430-01	6.9283787814334200-01	8.1910454839525110-01
	6.7157808860476350-01	4.2028538889842280-01	8.2263455873532850-01
	1.5183556670790830 00	1.8153768443115910-01	8.2866744855519880-01
	2.7179575122108100 00	5.5279311184045620-02	8.3744553697025990-01
	4.2856496876496940 00	1.1691007090543860-02	8.4934852052259750-01
	6.2429304553669540 00	1.6815850122803950-03	8.6494616127520700-01
	8.6194060789968300 00	1.5981840709044790-04	8.8508971700249540-01
	1.1455822731119150 01	9.6461469952952160-06	9.1107572127386690-01
	1.4809157304054100 01	3.4984582598999630-07	9.4495688890067748D-01
	1.8761704209826460 01	7.0406246552013920-09	9.9018376540860810-01
	2.3438802001796920 01	6.9682603226168790-11	1.0530930306525070 00
	2.9048224956026040 01	2.7802145428992160-13	1.1469965674766160 00
	3.5987656531811210 01	3.0690008892106540-16	1.3068858707396920 00
	4.5275339075781780 01	3.6489217055344940-20	1.6787826484002240 00
16	0.0000000000000000D 00	3.9577855609861140-01	3.9577855609861140-01
	1.5677065056447880-01	6.7754531578821820-01	7.9254325749664910-01
	6.2865390052235920-01	4.2426557773375370-01	7.9553394752461680-01
	1.4204515939960790 00	1.9343589632174100-01	8.0062907820911560-01
	2.5404721362130710 00	6.3694461462584280-02	8.0800621296033870-01
	4.0010342625897690 00	1.4965599828095410-02	8.1793959174694340-01
	5.8192767121752300 00	2.4673666146019130-03	8.3083244681203960-01
	8.0184141901108190 00	2.7904180790318490-04	8.4727094457709970-01
	1.0629691277554610 01	2.0997205515172530-05	8.6811600907258750-01
	1.3695501736560810 01	1.0087425117939160-06	8.9466645684683210-01
	1.7274594908211580 01	2.9223976689743430-08	9.2896758151766280-01
	2.1451343460517180 01	4.7049840064207440-10	9.7444705136694330-01
	2.6353803549347710 01	3.7207470429039460-12	1.0373889170925280 00
	3.2193838475018270 01	1.1799112997507800-14	1.1309821777486780 00
	3.9376612753227360 01	1.0221218015854350-17	1.2898769634341480 00
	4.8939540393390650 01	9.2379367889284210-22	1.6586216348623790 00

TABLE I (continued)

S=-1/3

N	A	H	H X EXP(A)
2	0.0000000000000000D 00	8.1247076365584020D-01	8.1247076365584020D-01
	1.6666666666666670D 00	5.4164717577056010D-01	2.8677455829824210D 00
3	0.0000000000000000D 00	6.0935307274188040D-01	6.0935307274188040D-01
	1.0336735048112150D 00	7.0407223365662630D-01	1.9794107142620170D 00
	4.2996598285221180D 00	4.0692633027894090D-02	2.9980186453358670D 00
4	0.0000000000000000D 00	4.9856160497062960D-01	4.9856160497062960D-01
	7.5427531616805340D-01	7.4344041431847900D-01	1.5806065372885900D 00
	2.9691486965685580D 00	1.1000382383605530D-01	2.1423611135773180D 00
	7.2765759872633900D 00	2.1120963012376950D-02	3.0541510651619880D 00
5	0.0000000000000000D 00	4.2733851854625410D-01	4.2733851854625410D-01
	5.9488311662037330D-01	7.3955909842505410D-01	1.3406868179067550D 00
	2.2945373791794590D 00	1.7645255959386210D-01	1.7503821863144030D 00
	5.3360190193713370D 00	1.0677641072000380D-02	2.2175781481067290D 00
	1.0441227151495500D 01	9.0121789231199600D-05	3.0860086680678540D 00
6	0.0000000000000000D 00	3.7706339871728320D-01	3.7706339871728320D-01
	4.9145763660772590D-01	7.1970508473236060D-01	1.1764999435293760D 00
	1.8767642380823950D 00	2.3101096616518670D-01	1.5090406851292110D 00
	4.2734123312278030D 00	2.5549454515705550D-02	1.8335849121934280D 00
	7.9650375423223410D 00	7.8563984475303080D-04	2.2614935526028460D 00
	1.3726661585093070D 01	3.3954511131263730D-06	3.1067825941572640D 00
7	0.0000000000000000D 00	3.3935705884555520D-01	3.3935705884555520D-01
	4.1881027022623220D-01	6.9430090779546360D-01	1.0554428506486170D 00
	1.5902573254038720D 00	2.7327541095582960D-01	1.3404188836080840D 00
	3.5813953423344940D 00	4.4413906440010040D-02	1.5955088153585600D 00
	6.5394117960968900D 00	2.7225229173370800D-03	1.8836577732933710D 00
	1.0772776604109710D 01	4.8015230576565990D-05	2.2905324704886190D 00
	1.7097348661828800D 01	1.1724163008974330D-07	3.1215193189576320D 00
8	0.0000000000000000D 00	3.0984774938072430D-01	3.0984774938072430D-01
	3.6494021273143360D-01	6.6763780047487850D-01	9.6168410576927400D-01
	1.3807727409175680D 00	3.0514832800073960D-01	1.2138722232142840D 00
	3.0892520789697140D 00	6.5015669203768030D-02	1.4277861609998090D 00
	5.5782588061849130D 00	6.2315979180497970D-03	1.6489460310685340D 00
	9.0101736825321690D 00	2.3422559593996870D-04	1.9173573556703810D 00
	1.3711333314848400D 01	2.5650591855715210D-06	2.3112865314863490D 00
	2.0531935830482470D 01	3.7931152489998060D-09	3.1325821748508940D 00
9	0.0000000000000000D 00	2.8601330712066850D-01	2.8601330712066850D-01
	3.2338288777975770D-01	6.4153650183759830D-01	8.8647149546077500D-01
	1.2206197099544380D 00	3.2879568728296800D-01	1.1143830991922080D 00
	2.7193024077660770D 00	8.5754263703052380D-02	1.3008695637881470D 00
	4.8766702434788880D 00	1.1304047544550630D-02	1.4830129803225420D 00
	7.7910411511600510D 00	6.9682960754672920D-04	1.6855155957987740D 00
	1.1635417945708010D 01	1.7178569739259580D-05	1.9417119828845510D 00
	1.6750412507491190D 01	1.2364363264345090D-07	2.3269308515877000D 00
	2.4016486479994910D 01	1.1664669229230290D-10	3.1412317695729580D 00

S=-2/3

N	A	H	H X EXP(A)
2	0.0000000000000000D 00	2.0092039010308120D 00	2.0092039010308120D 00
	1.3333333333333330D 00	6.6973463367693660D-01	2.5407507777375930D 00
3	0.0000000000000000D 00	1.7221747723121260D 00	1.7221747723121260D 00
	8.0580810168138650D-01	9.1682611753468180D-01	2.0523195800890150D 00
	3.8608585649852800D 00	3.9937644860942910D-02	1.8972825874904310D 00
4	0.0000000000000000D 00	1.5499572950809140D 00	1.5499572950809140D 00
	5.8046512131708260D-01	1.0168270098237180D 00	1.8169370166024140D 00
	2.6321997616344630D 00	1.1034091679285670D-01	1.5342156899106880D 00
	6.7873351170484540D 00	1.8133130102621610D-03	1.6075887774387610D 00
5	0.0000000000000000D 00	1.4307298108439200D 00	1.4307298108439200D 00
	4.5424410475084620D-01	1.0575771306007290D 00	1.6656653792296020D 00
	2.0192169205841790D 00	1.8127691565851480D-01	1.3654546150368220D 00
	4.9379857204000750D 00	9.2843200985859010D-03	1.2950604942362910D 00
	9.9218865875982300D 00	7.0357506001027960D-05	1.4332799692294267D 00
6	0.0000000000000000D 00	1.3413091976661760D 00	1.3413091976661760D 00
	3.7330742627062240D-01	1.0712722537007210D 00	1.5560559622529360D 00
	1.6435057353058020D 00	2.4320514206800050D-01	1.2581668108959050D 00
	3.9353349642709180D 00	2.2530576005442800D-02	1.1530988904472270D 00
	7.5274254517543930D 00	6.1890335388525410D-04	1.1501173971321590D 00
	1.3187093089064930D 01	2.4619135294870330D-06	1.3132716101447570D 00
7	0.0000000000000000D 00	1.2707139767363790D 00	1.2707139767363790D 00
	3.1692792573424330D-01	1.0715508566727350D 00	1.4711360472890100D 00
	1.3877180968233320D 00	2.9473393402123890D-01	1.1804952534469130D 00
	3.2867731719785390D 00	3.9769616382428200D-02	1.0640912053377620D 00
	6.1582619626772470D 00	2.1650029204313690D-03	1.0231932702261760D 00
	1.0307180703084080D 01	3.5067962077419810D-05	1.0501763447452460D 00
	1.6543138139702560D 01	8.0012465250523810D-08	1.2239150045153540D 00
8	0.0000000000000000D 00	1.2129542505210890D 00	1.2129542505210890D 00
	2.7537861189435200D-01	1.0648546890182930D 00	1.4024447441250860D 00
	1.2017146398701870D 00	3.3679997686495590D-01	1.1201342840189280D 00
	2.8278507256237700D 00	5.9148089528762370D-02	1.0001397470140050D 00
	5.2396485605770250D 00	5.0075118645028490D-03	9.4443580596834380D-01
	8.5973972869012540D 00	1.7225278885792340D-04	9.3318653242700970D-01
	1.3224770231774970D 01	1.7616621262455860D-06	9.7581469440750100D-01
	1.9966573277232920D 01	2.4591666903089610D-09	1.1538797836746490D 00
9	0.0000000000000000D 00	1.1644360805002450D 00	1.1644360805002450D 00
	2.4347866507469960D-01	1.0544753153425470D 00	1.3451721223366180D 00
	1.0601184237456460D 00	3.7104008739241540D-01	1.0719861786318410D 00
	2.4842245865329380D 00	7.9272942579614900D-02	9.5062670224155870D-01
	4.5716056142426790D 00	9.1857851517069850D-03	8.8825850619351020D-01
	7.4191504335641770D 00	5.1638043784168920D-04	8.6112436029617870D-01
	1.1198303590638250D 01	1.1862604655033500D-05	8.6604710244324930D-01
	1.6247476071517110D 01	8.0626396646322730D-08	9.1762757011786690D-01
	2.3442309281351160D 01	7.2331475090285250D-11	1.0969652900165180D 00

TABLE I (continued)

S=-1/3

N	A	H	H X EXP(A)
10	0.0000000000000000D 00	2.6628825145717460-01	2.6628825145717460-01
	2.9034150314484850-01	6.1674767994827400-01	8.2452008078528220-01
	1.0940654935135440D 00	3.4608731992994470-01	1.0335519112691560 00
	2.4302030112169140 00	1.0565008969581630-01	1.2003105623714380 00
	4.3383425428253450 00	1.7719012405973860-02	1.3569309213895560 00
	6.8833662508365750 00	1.5589400175461800-03	1.5213788871530380 00
	1.0170837375643470 01	6.5528000378033140-05	1.7122432547168260 00
	1.4381973875632790 01	1.1124744350870590-06	1.9602069859942550 00
	1.9869519905842420 01	5.4934177630538660-09	2.3391899028784530 00
	2.7541350041344090 01	3.4435519498396710-12	3.1482048693683760 00
11	0.0000000000000000D 00	2.4964523574110090-01	2.4964523574110090-01
	2.6343737341713630-01	5.9353684803625050-01	7.7242635549245640-01
	9.9146878422410540-01	3.5850669759739540-01	9.6624374496964190-01
	2.1976464242399860 00	1.2417256578941960-01	1.1180246278674340 00
	3.9105908131242920 00	2.5168087643438540-02	1.2566033950884090 00
	6.1757212863088180 00	2.9035394423570280-03	1.3963986053230420 00
	9.0632454606710180 00	1.7952909398411620-04	1.5497167634540380 00
	1.2684237639586660 01	5.3706824175093920-06	1.7327020379080830 00
	1.7226696612032450 01	6.5171570123949230-08	1.9747742892169360 00
	2.3054083027989720 01	2.2836917925640760-10	2.3490839385541140 00
	3.1099539245072460 01	9.8287484643773210-14	3.1539623499827710 00
12	0.0000000000000000D 00	2.3537979369875260-01	2.3537979369875260-01
	2.4110347571696370-01	5.7194530414107670-01	7.2788774537491040-01
	9.0657580632418200-01	3.6719450003607070-01	9.0911126174169250-01
	2.0063042582563580 00	1.4108245026412600-01	1.0490588746609260 00
	3.5616998654692570 00	3.3334992136930670-02	1.1741591174500400 00
	5.6059631446377870 00	4.7665266458489350-03	1.2967040101796030 00
	8.1886126533245240 00	3.9609508918851300-04	1.4258332818419880 00
	1.1383882071361180 01	1.7880441903442320-05	1.5715760947192980 00
	1.5307307278383770 01	3.9344887303232910-07	1.7489097120251295D 00
	2.0152735078159390 01	3.5146583896166390-09	1.9865740561672590 00
	2.6293333981203850 01	8.9816682650336970-12	2.3572566488400600 00
	3.4685815720496050 01	2.7268727424920440-15	3.1588080674775590 00
13	0.0000000000000000D 00	2.2299138350408190-01	2.2299138350408190-01
	2.2226505584284530-01	5.5191221994592420-01	6.8928447558509080-01
	8.3514568281183520-01	3.7301848082375250-01	8.5986349308052510-01
	1.8459895139203440 00	1.5631783859867350-01	9.9017418821654050-01
	3.2712574914332800 00	4.1935977804427560-02	1.1047801211219940 00
	5.1360389001193390 00	7.1427590721392050-03	1.2145678658684410 00
	7.4767900933235360 00	7.5110594689996120-04	1.3268745994589410 00
	1.0346129479512280 01	4.6527824157608870-05	1.4487040876654360 00
	1.3821514143433540 01	1.5794832567030540-06	1.5889943411688470 00
	1.8022256798806540 01	2.6245975743226610-08	1.7620953384285200 00
	2.3147403213888010 01	1.7678542062652910-10	1.9963465443708890 00
	2.9579068953669810 01	3.3701237814544230-13	2.3641350338698650 00
	3.8296140673238620 01	7.3839692936402810-17	3.1629509318406090 00

S=-2/3

N	A	H	H X EXP(A)
10	0.0000000000000000D 00	1.1228490776252370 00	1.1228490776252370 00
	2.1821191079729910-01	1.0421929738513690 00	1.2963323819690250 00
	9.4861052570115010-01	3.9889243442803860-01	1.0299878843072570 00
	2.2165424566476000D 00	9.9225861956918160-02	9.1045196602215770-01
	4.0605431095640980 00	1.4567061855689370-02	8.4497416448071090-01
	6.5444491063797050 00	1.1648761745174270-03	8.1002401858647510-01
	9.7727692030669450 00	4.5516648707356660-05	7.9878521213548240-01
	1.3925472497027110 01	7.2874019444028570-07	8.1344573701609170-01
	1.9353394821015710 01	3.4250318205140210-09	8.7043668750806250-01
	2.6960036369804370 01	2.0529616025496260-12	1.0494525934407900 00
11	0.0000000000000000D 00	1.0866281396373270 00	1.0866281396373270 00
	1.9770198348194150-01	1.0290101029182090 00	1.2539508645597310 00
	8.5846574848105260-01	4.2159443583486360-01	9.9476800372431400-01
	2.0017738520888530 00	1.1845060282730700-01	8.7679207006762040-01
	3.6554505395507420 00	2.0937385462942110-02	8.0996163204135980-01
	5.8641166968759130 00	2.1887852171972010-03	7.7082655830985810-01
	8.6972071949872590 00	1.2550707381461690-04	7.5130676073126780-01
	1.2264966038017210 01	3.5348043690191330-06	7.4984664755571490-01
	1.6754337571260210 01	4.0794808916216540-08	7.7076370976001050-01
	2.2527075312956430 01	1.3686175124778050-10	8.3112547593265620-01
	3.0512238395633900 01	5.6568616341540090-14	1.0089562196133480 00
12	0.0000000000000000D 00	1.0546684884715240 00	1.0546684884715240 00
	1.8071989339364390-01	1.0155064175112430 00	1.2166574645484720 00
	7.8405139699415380-01	4.4014495605787980-01	9.6406191067211170-01
	1.8254527993491700 00	1.3663877166393330-01	8.4792614578974520-01
	3.3257206620605270 00	2.8062843999595960-02	7.8068136702056120-01
	5.3174274255861920 00	3.6261561027440170-03	7.3922370476073040-01
	7.8494969536715520 00	2.7882252823371410-04	7.1502474793599750-01
	1.0995559362626150 01	1.1828995248850660-05	7.0511282506058260-01
	1.4870462684121880 01	2.4725826382651770-07	7.1008559872401160-01
	1.9667130446799230 01	2.1138944742111080-09	7.3520682739583450-01
	2.5757171569943620 01	5.1956182561303490-12	7.9769104458540710-01
	3.4093473466286750 01	1.5201675546118500-15	9.7386521629383620-01
13	0.0000000000000000D 00	1.0261639347290520 00	1.0261639347290520 00
	1.6642678548923690-01	1.0020218675758570 00	1.1834650421392890 00
	7.2156317113212560-01	4.5533753430211900-01	9.3692398095048990-01
	1.6779981507805330 00	1.5364270792558920-01	8.2272991808052400-01
	3.0517109043047590 00	3.5723069859357730-02	7.5559657208036950-01
	4.8672856714201000 00	5.4848802573925240-03	7.1285706750578610-01
	7.1607058530269390 00	5.3257492271591670-04	6.8586043648235250-01
	9.9841261779128020 00	3.0952366746709860-05	6.7103437438041800-01
	1.3414509625917020 01	9.9682660637526690-07	6.6752437563888650-01
	1.7570577142525860 01	1.5839537911012830-08	6.7693536673085140-01
	2.2650544998702150 01	1.0260428311190300-10	7.0497253129503680-01
	2.9035082923576500 01	1.8881024496836360-13	7.6877960080433400-01
	3.7699468596111940 01	3.9981759841533250-17	9.4304645777403860-01

TABLE I (continued)

S=-1/3

N	A	H	H X EXP(A)
14	0.0000000000000000D 00	2.1211375504046678D-01	2.1211375504046678D-01
	2.0616020298030740D-01	5.3333357502690100D-01	6.5544033419704320D-01
	7.7419757174390580D-01	3.7663776941476560D-01	8.1687123347387460D-01
	1.7096484346709870D 00	1.6991893886402760D-01	9.3914503841842350D-01
	3.0254542156251460D 00	5.0735187791159020D-02	1.0453153871624550D 00
	4.7410827619334240D 00	9.9972573875475190D-03	1.1452675491424940D 00
	6.8842691936418570D 00	1.2748045264719510D-03	1.2452132108454010D 00
	9.4939417268667680D 00	1.0169983803210180D-04	1.3504756710973740D 00
	1.2625187898089160D 01	4.8238052842822880D-06	1.4670307128753530D 00
	1.6358158669472700D 01	1.2610648311473080D-07	1.6032286249989740D 00
	2.0815469612775310D 01	1.6168802663083570D-09	1.7730519138104390D 00
	2.6200888957675050D 01	8.3776736740711750D-12	2.0045871176127690D 00
	3.2904879765429470D 01	1.2143481048123130D-14	2.3700139875928450D 00
	4.1927327655762540D 01	1.9578430718086830D-18	3.1665395770805840D 00
15	0.0000000000000000D 00	2.0247222072044670D-01	2.0247222072044670D-01
	1.9223363965745770D-01	5.1609101318507640D-01	6.2547838423202200D-01
	7.2157336799439190D-01	3.7855445080140500D-01	7.7893943051128670D-01
	1.5922323692143830D 00	1.8198069858347310D-01	8.9438201963475240D-01
	2.8145769270763310D 00	5.9546413062498370D-02	9.9359828687335390D-01
	4.4040428375206930D 00	1.3275944736868680D-02	1.0857177507948430D 00
	6.3822863262434990D 00	1.9892425588257180D-03	1.1761953972946810D 00
	8.7790634024270190D 00	1.9537887393133580D-04	1.2693354060677860D 00
	1.1635268982030520D 01	1.2117836901572760D-05	1.3694879561092020D 00
	1.5008037734292260D 01	4.4974030962449030D-07	1.4820736526842510D 00
	1.8979846505217220D 01	9.2332884484688890D-09	1.6150984775189380D 00
	2.3676269469705710D 01	9.3006058425485220D-11	1.7823148361426120D 00
	2.9305432032503220D 01	3.7701870282120090D-13	2.0116402486959340D 00
	3.6265655953704990D 01	4.2238488661248820D-16	2.3751040136867010D 00
	4.5576813785745620D 01	5.0963245156870430D-20	3.1696828088381670D 00
16	0.0000000000000000D 00	1.9385638154085330D-01	1.9385638154085330D-01
	1.8007106433759810D-01	5.0006587295860370D-01	5.9873009265634040D-01
	6.7567126890846770D-01	3.7915285494327000D-01	7.4516875080598630D-01
	1.4900275730645930D 00	1.9262387948777990D-01	8.5471411929104440D-01
	2.6315763300248090D 00	6.8228477663404090D-02	9.4807950878723160D-01
	4.1127788560245460D 00	1.6915253401897930D-02	1.0337979595718840D 00
	5.9509067732893120D 00	2.9075243052541500D-03	1.1167843717884620D 00
	8.1692997060163390D 00	3.4005214167231360D-04	1.2006800585910360D 00
	1.0799323941674780D 01	2.6309972883774900D-05	1.2888643065873490D 00
	1.3883498029670130D 01	1.2944165825842460D-06	1.3851601413228940D 00
	1.7480708068232560D 01	3.8262034145168760D-08	1.4946623974702940D 00
	2.1875489851194880D 01	6.2712050173112290D-10	1.6251619058739360D 00
	2.6596115954173500D 01	5.0392916747342480D-12	1.7902591210233340D 00
	3.2454778916805510D 01	1.6215735377308870D-14	2.0177531942463880D 00
	3.9657249950949550D 01	1.4242025638821940D-17	2.3795595864295550D 00
	4.9242503715633410D 01	1.3051151165415830D-21	3.1724622187496490D 00

S=-2/3

N	A	H	H X EXP(A)
14	0.0000000000000000D 00	1.0005098363608250D 00	1.0005098363608250D 00
	1.5423040536426020D-01	9.8875579687647930D-01	1.1536404006234080D 00
	6.6833636990480200D-01	4.6780001247287970D-01	9.1267262962798530D-01
	1.5527953507535980D 00	1.6941526643863720D-01	8.0042933267978440D-01
	2.8201662706591950D 00	4.3726193244622460D-02	7.3370979879658220D-01
	4.4895038134599740D 00	7.7495173434186960D-03	6.9030558517629860D-01
	6.5881579734734980D 00	9.1072098490558000D-04	6.6158552944516320D-01
	9.1546873891038980D 00	6.8055337672336800D-05	6.4371319250366610D-01
	1.2243802419620620D 01	3.0583229846674090D-06	6.3518345466078930D-01
	1.5935234361954280D 01	7.6379366039638010D-08	6.3615127298890650D-01
	2.0351075198115430D 01	9.4125435173318750D-10	6.4873399498196720D-01
	2.5694334010338170D 01	4.7082476562984540D-12	6.7883898551182490D-01
	3.2354119145871510D 01	6.6073869480203360D-15	7.4343867994819310D-01
	4.1326890624714070D 01	1.0320514353560640D-18	9.1567613347728130D-01
15	0.0000000000000000D 00	9.7724216574778310D-01	9.7724216574778310D-01
	1.4370063521327800D-01	9.7582295403376470D-01	1.1266250999759610D 00
	6.2244759284891710D-01	4.7802990655804900D-01	8.9080084984232230D-01
	1.4451224657644370D 00	1.8397001809645240D-01	7.8046919038072340D-01
	2.6217847746409960D 00	5.1913137397368700D-02	7.1433830155275380D-01
	4.1675275591719940D 00	1.0388726690051930D-02	6.7065094825829950D-01
	6.1036877598773650D 00	1.4320748383963930D-03	6.4086068198471600D-01
	8.4597190404277530D 00	1.3155201066461910D-04	6.2102171755413040D-01
	1.1276216429103670D 01	7.7204420006748810D-06	6.0931340489262290D-01
	1.4609996930622900D 01	2.7344879951614840D-07	6.0522529207088310D-01
	1.8543170669617850D 01	5.3918142996267460D-09	6.0944093973283890D-01
	2.3200838737024390D 01	5.2412271050969440D-11	6.2435158322242760D-01
	2.8790422241087730D 01	2.0575279655005910D-13	6.5594474591257480D-01
	3.5708963470696920D 01	2.2372524558568630D-16	7.2097595885908430D-01
	4.4973068369568460D 01	2.6205344587969500D-20	8.9113869501102050D-01
16	0.0000000000000000D 00	9.5599777084022260D-01	9.5599777084022260D-01
	1.3451753379819630D-01	9.6328622799800150D-01	1.1019847231417950 00
	5.8247302171699060D-01	4.8642317397781320D-01	8.7092162942208700D-01
	1.3515141230956900D 00	1.9735557589711630D-01	7.6243798852858330D-01
	2.4498291923374420D 00	6.0156458885479500D-02	6.9699484073800070D-01
	3.8895918587982650D 00	1.3361727149306300D-02	6.5326724066634430D-01
	5.6878052559213330D 00	2.1095122463947920D-03	6.2282277862953970D-01
	7.8675568261205630D 00	2.3042584212145690D-04	6.0168294170363440D-01
	1.0459967456299790D 01	1.6849000556555840D-05	5.8786863801629350D-01
	1.3507304412348130D 01	7.9009622871518570D-07	5.8053413428114140D-01
	1.7068178756181500D 01	2.2417385194293600D-08	5.7969666815410070D-01
	2.1226798623099570D 01	3.5437599654333960D-10	5.8633498484728400D-01
	2.6111002311097780D 01	2.7570559973269140D-12	6.0298958038466410D-01
	3.1932322222607520D 01	8.6138115319472180D-15	6.3566265998761560D-01
	3.9095313721464950D 01	7.3580236167236960D-18	7.0087439616423360D-01
	4.8635824685112550D 01	6.5573934829868950D-22	8.6896348716623170D-01

TABLE I (continued)

S= 0.0

N	A	H	H X EXP(A)
2	0.000000000000000D 00	5.000000000000000D-01	5.000000000000000D-01
	2.000000000000000D 00	5.000000000000000D-01	3.694528049465324D 00
3	0.000000000000000D 00	3.333333333333336D-01	3.333333333333336D-01
	1.267949192431123D 00	6.220084679281462D-01	2.210342807746914D 00
	4.732050807568877D 00	4.465819873852047D-02	5.069962602869399D 00
4	0.000000000000000D 00	2.500000000000002D-01	2.500000000000002D-01
	9.358222752408878D-01	6.290526808677528D-01	1.603649477140665D 00
	3.305407289332279D 00	1.183563854551006D-01	3.226352682838583D 00
	7.758770483143633D 00	2.590933671469500D-03	6.068026981613486D 00
5	0.000000000000000D 00	2.000000000000002D-01	2.000000000000002D-01
	7.432919279814314D-01	6.012046901038596D-01	1.264241209835609D 00
	2.571635007646278D 00	1.857323340768451D-01	2.430717063851168D 00
	5.731178751689099D 00	1.294284962045381D-02	3.990701202751512D 00
	1.095389431268319D 01	1.201261988423295D-04	6.868368539084384D 00
6	0.000000000000000D 00	1.666666666666670D-01	1.666666666666670D-01
	6.170308532782703D-01	5.640148108872614D-01	1.045354524616280D 00
	2.112965958785240 00	2.377135666068174D-01	1.966542897573834D 00
	4.610833151017532D 00	3.056192121447185D-02	3.073548327966233D 00
	8.399066971204841D 00	1.038197820781175D-03	4.612629282018880D 00
	1.426010306592083D 01	4.836804002523279D-06	7.544710189245802D 00
7	0.000000000000000D 00	1.428571428571431D-01	1.428571428571431D-01
	5.276681217111288D-01	5.261832780981877D-01	8.918676179602462D-01
	1.796299809643409D 00	2.749599929804610D-01	1.657267460105989D 00
	3.876641520476912D 00	5.236602241177979D-02	2.527281439675224D 00
	6.918816566704721D 00	3.565468188288595D-03	3.605125785483134D 00
	1.123461042908311D 01	6.791893428616242D-05	5.141859180201521D 00
	1.764596355238071D 01	1.765298550164625D-07	8.135097515376769D 00
8	0.000000000000000D 00	1.250000000000003D-01	1.250000000000003D-01
	4.610242198049948D-01	4.906696582640980D-01	7.780533075785770D-01
	1.563586189654263D 00	3.004047431789662D-01	1.434708395686778D 00
	3.352050502536734D 00	7.551273945331229D-02	2.156737363815450D 00
	5.916297249020539D 00	8.079903167838808D-03	2.997929423835922D 00
	9.420699383021589D 00	3.291114204974204D-04	4.061633296986200D 00
	1.419416554800746D 01	3.838525675154048D-06	5.605472452095731D 00
	2.109217690795441D 01	5.989613528510852D-09	8.661933465824037D 00
9	0.000000000000000D 00	1.111111111111114D-01	1.111111111111114D-01
	4.093835732031851D-01	4.583294340259185D-01	6.901933577927619D-01
	1.384963184803140D 00	3.169653645295573D-01	1.266174833818485D 00
	2.956254556168862D 00	9.809577127794184D-02	1.885972374788269D 00
	5.181943101040071D 00	1.450061939152049D-02	2.581522102616627D 00
	8.161709688145817D 00	9.719344294037857D-04	3.405825926052979D 00
	1.207005512683715D 01	2.557089703153508D-05	4.463795615470539D 00
	1.724973552614899D 01	1.941454815188554D-07	6.019941080402073D 00
	2.458595524365278D 01	1.920361757923611D-10	9.139611669055636D 00

S=-0.5

N	A	H	H X EXP(A)
2	0.000000000000000D 00	1.181635900603677D 00	1.181635900603677D 00
	1.500000000000000D 00	5.908179503018386D-01	2.647862350427287D 00
3	0.000000000000000D 00	9.453087204829425D-01	9.453087204829425D-01
	9.188611699158103D-01	7.872386463044827D-01	1.973161993502857D 00
	4.081138830841900D 00	3.990648411809183D-02	2.362977251071975D 00
4	0.000000000000000D 00	8.102646175568078D-01	8.102646175568078D-01
	6.663259077023707D-01	8.512145052202561D-01	1.657374606567279D 00
	2.800775054150257D 00	1.290311656382542D-01	1.794369203450369D 00
	7.032899038147372D 00	1.943562490199040D-03	2.202661459220646D 00
5	0.000000000000000D 00	7.202352156060519D-01	7.202352156060519D-01
	5.235260767382690D-01	8.653031180051120D-01	1.460604905490185D 00
	2.156648763269094D 00	1.769490547887532D-01	1.529216253189101D 00
	5.137387546176711D 00	9.887248551073905D-03	1.683505402957341D 00
	1.018243761381592D 01	7.921355452652889D-05	2.094007161953367D 00
6	0.000000000000000D 00	6.547592869145927D-01	6.547592869145927D-01
	4.313988071478514D-01	8.587195047122512D-01	1.321926838881923D 00
	1.759753698423696D 00	2.344557503354173D-01	1.362423762133341D 00
	4.104465362828315D 00	2.382279088982309D-02	1.443907249456781D 00
	7.746703779542557D 00	6.936389326466919D-04	1.605033737702062D 00
	1.345767835205758D 01	2.879120787428517D-06	2.013053572344778D 00
7	0.000000000000000D 00	6.043931879211627D-01	6.043931879211627D-01
	3.669498773983707D-01	8.432325937970879D-01	1.217059167406643D 00
	1.488534292312452D 00	2.806466413740472D-01	1.243432110573528D 00
	3.434037968424071D 00	4.172555059233992D-02	1.293518926631685D 00
	6.349067925683379D 00	2.414919985438777D-03	1.381236069675692D 00
	1.054046985448340D 01	4.086072080541419D-05	1.545161646703497D 00
	1.682097007782838D 01	9.651463700146278D-08	1.949160791279821D 00
8	0.000000000000000D 00	5.641003087264190D-01	5.641003087264190D-01
	3.193636339206300D-01	8.240573749977984D-01	1.134042306893209D 00
	1.290758622959153D 00	3.169778315918721D-01	1.152386700567001D 00
	2.958374458696650D 00	6.156306774509232D-02	1.186054899528422D 00
	5.409031597244433D 00	5.556137685825564D-03	1.241325207054377D 00
	8.804079578056775D 00	2.000088824650C04D-04	1.332332010218091D 00
	1.346853574325148D 01	2.118231095422137D-06	1.497214732033815D 00
	2.024991636587088D 01	3.044951608507C35D-09	1.896737941655210D 00
9	0.000000000000000D 00	5.309179376248646D-01	5.309179376248646D-01
	2.826336481165991D-01	8.036529389408251D-01	1.066141309147196D 00
	1.139873801581614D 00	3.452965953401950D-01	1.079526227816999D 00
	2.601528434060029D 00	8.184006829951271D-02	1.103554706156333D 00
	4.724114537527790D 00	1.013469991525510D-02	1.141478588248913D 00
	7.605256299231614D 00	5.972865733955C72D-04	1.199785465335550D 00
	1.141718207654583D 01	1.422457828004266D-05	1.292583400425758D 00
	1.649941079765581D 01	9.954157963261613D-08	1.457496741174207D 00
	2.373000399593471D 01	9.161157861597277D-11	1.852508279979031D 00

TABLE I (continued)

S= 0.0

N	A	H	H X EXP(A)
10	0.0000000000000000D 00	1.0000000000000003D-01	1.0000000000000003D-01
	3.6817845294174150D-01	4.2922371539702210D-01	6.2027114746529970D-01
	1.2433579621404820D 00	3.2700364001017710D-01	1.1337990505734780D 00
	2.6460338413842070D 00	1.1903947664566650D-01	1.6782200488461150D 00
	4.6168825146350600D 00	2.2477401060444530D-02	2.2742211519797690D 00
	7.2217865393965700D 00	2.1570338086426040D-03	2.9528337474004930D 00
	1.0567320807741860D 01	9.6985034191440110D-05	3.7673315142837790D 00
	1.4835914515260930D 01	1.7390374796668590D-06	4.8246381871563830D 00
	2.0382181985449250D 01	8.9958943133831520D-09	6.3960744459099720D 00
	2.8118343381049890D 01	5.8834527822319710D-12	9.5779737623976560D 00
11	0.0000000000000000D 00	9.0909090909091200D-02	9.0909090909091200D-02
	3.3452867632475250D-01	4.0312404907393340D-01	5.6327783969847070D-01
	1.1282533558766370D 00	3.3230509721298370D-01	1.0269072254023970D 00
	2.3958699247473060D 00	1.3784017545904490D-01	1.5131741201516130D 00
	4.1668409879287670D 00	3.1565611700168700D-02	2.0363398387024780D 00
	6.4873530313808110D 00	3.9834527405419730D-03	2.6162623581105800D 00
	9.4283548133356060D 00	2.6405940858044050D-04	3.2838560789412400D 00
	1.3101723580367800D 01	8.3567994512385090D-06	4.0930423133526770D 00
	1.7696487566846220D 01	1.0630784256341420D-07	5.1529058082610080D 00
	2.3577787088360150D 01	3.8819101191621760D-10	6.7413721660356870D 00
	3.1682800974831940D 01	1.7363690478467590D-13	9.9840712170504270D 00
12	0.0000000000000000D 00	8.3333333333333710D-02	8.3333333333333710D-02
	3.0652670213005210D-01	3.7971460570790350D-01	5.1591737932268020D-01
	1.0327973987797250D 00	3.3418502469584950D-01	9.3869649413674820D-01
	2.1896119419683880D 00	1.5434561494218940D-01	1.3785758915681490D 00
	3.7990476060549640D 00	4.1332599160959610D-02	1.8458573199346300D 00
	5.8949141715027450D 00	6.4817899620344860D-03	2.3540861217510780D 00
	8.5272920080702700D 00	5.7870428983312760D-04	2.9228906391942940D 00
	1.1771006565490690D 01	2.7682630496463740D-05	3.5833660925972442D 00
	1.5742260287026190D 01	6.3930828245739160D-07	4.3902283131625700D 00
	2.0635805668641130D 01	5.9533264825605600D-09	5.4547673407212900D 00
	2.6826349949370700D 01	1.5788712323653030D-11	7.0612642539242990D 00
	3.5274390709965130D 01	4.9661325042784840D-15	1.0363144097249260D 01
13	0.0000000000000000D 00	7.6923076923077320D-02	7.6923076923077320D-02
	2.8285834823991410D-01	3.5867340065049720D-01	4.7592990032668450D-01
	9.5232604136461340D-01	3.3360369197473470D-01	8.6461107267886090D-01
	2.0164921385776810D 00	1.6860282510908350D-01	1.2665322592359680D 00
	3.4923540697780250D 00	5.1407330371462500D-02	1.6894103577568240D 00
	5.4054910200157220D 00	9.6257562620420870D-03	2.1429388703161150D 00
	7.7928139404124150D 00	1.0896728063443950D-03	2.6404150875813290D 00
	1.0707388688989090D 01	7.1645460864559920D-05	3.2014631247238670D 00
	1.4227152363789980D 01	2.5558235600533830D-06	3.8574884467665420D 00
	1.8471996634233000D 01	4.4307971519618420D-08	4.6641023904193640D 00
	2.3641783752400890D 01	3.0975471974104230D-10	5.7347456213630760D 00
	3.0120058626106340D 01	6.1078979203140505D-13	7.3599171740342730D 00
	3.8889284376095320D 01	1.3828047002006350D-16	1.0719202039904230D 01

S=-0.5

N	A	H	H X EXP(A)
10	0.0000000000000000D 00	5.0297488827618830D-01	5.0297488827618830D-01
	2.5353255497441900D-01	7.8321797722606240D-01	1.0092306627044770D 00
	1.0208442777203900D 00	3.6726540261399510D-01	1.0193587501965270D 00
	2.3230960770224660D 00	1.0162077381810440D-01	1.0372663874005480D 00
	4.1993506006572930D 00	1.5977733555446000D-02	1.0648047211031030D 00
	6.7139743166150280D 00	1.3417550428143660D-03	1.1053892419342910D 00
	9.9720091595393480D 00	5.4418395526323350D-05	1.1655590595256610D 00
	1.4154053671278040D 01	8.9764862944462720D-07	1.2593132652736630D 00
	1.9611902191655950D 01	4.3261020197271180D-09	1.4237637486742500D 00
	2.7251236523027060D 01	2.6525941889970420D-12	1.8143975921857050D 00
11	0.0000000000000000D 00	4.7902370312017920D-01	4.7902370312017920D-01
	2.2987298051865620D-01	7.6333814722700600D-01	9.6061537506315970D-01
	9.2448154698665730D-01	3.8424064813399640D-01	9.6859204188048980D-01
	2.0994104627087980D 00	1.2035929331782480D-01	9.8229516942261870D-01
	3.7828808737072900D 00	2.2828131674868840D-02	1.0031240853275820D 00
	6.0199180277014610D 00	2.5099640834528250D-03	1.0329628149850620D 00
	8.8803475979967080D 00	1.4956797734620160D-04	1.0752874567770900D 00
	1.2474832404836200D 01	4.3437697961133520D-06	1.1366262383617780D 00
	1.6990847293542550D 01	5.1424603611864400D-08	1.2308416255594790D 00
	2.2791002894948940D 01	1.7637222484099950D-10	1.3945593045494320D 00
	3.0806405917052720D 01	7.4407301402721280D-14	1.7810153024030910D 00
12	0.0000000000000000D 00	4.5819658559321490D-01	4.5819658559321490D-01
	2.1025741848317980D-01	7.4428764975513180D-01	9.1844773879594950D-01
	8.4483941642521240D-01	3.9728979715604630D-01	9.2473332482763190D-01
	1.9155655747361730D 00	1.3778057885817500D-01	9.3563737937437060D-01
	3.4435372681407950D 00	3.0414168008968360D-02	9.5188651515015690D-01
	5.4616447726944180D 00	4.1391357499212880D-03	9.7470199944143120D-01
	8.0191126887823450D 00	3.3112339828374920D-04	1.0061118427593780D 00
	1.1189875716176440D 01	1.4498591836004570D-05	1.0496079234149230D 00
	1.5089128220021150D 01	3.1106786582915450D-07	1.1116812283153600D 00
	1.9910257642768470D 01	2.7192593008057870D-09	1.2060509363168490D 00
	2.6025655381891420D 01	6.8166218960611070D-12	1.3688886635545700D 00
	3.4390125899880690D 01	2.0320769241273530D-15	1.7513867283362060D 00
13	0.0000000000000000D 00	4.3986872216948670D-01	4.3986872216948670D-01
	1.9372964450476220D-01	7.2617797855178350D-01	8.8141165782411690D-01
	7.7789358225852890D-01	4.0724227335624900D-01	8.8651837850370420D-01
	1.7616740620956190D 00	1.5377799035161810D-01	8.9532251317480060D-01
	3.1612879981441600D 00	3.8486197930817910D-02	9.0831177105523690D-01
	5.0015749999647210D 00	6.2314174402512850D-03	9.2628209314999280D-01
	7.3187575856579170D 00	6.3016727749097000D-04	9.5049947600437990D-01
	1.0165224525114880D 01	3.7831945914681170D-05	9.8301376377517530D-01
	1.3618187712267260D 01	1.2514064999382250D-06	1.0273115795491330D 00
	1.7796665644267490D 01	2.0340765006036980D-08	1.0898355443888010D 00
	2.2899291018196020D 01	1.3439276835412450D-10	1.1841633861730150D 00
	2.9307458587209690D 01	2.5176299754930970D-13	1.3460445804441170D 00
	3.7998254641218930D 01	5.4238470280768640D-17	1.7248039774620640D 00

For $s = 0$, comparison with the 8-place tables of [8] discloses that 17 of 240 entries therein are incorrect in the 8th digit, presumably due to rounding, and 4 cases (in their notation: A_2 of $n = 6$, A_3 of $n = 12$, A_{12} of $n = 14$, X_{15} of $n = 15$) of misprints or errors. Among the 24 possible comparisons with $s = 1$ of [1], there are 5 cases in which, rounding their a's to 16 figures, our last digits are in error by 1 unit; there are 4 cases in which their weights and our $a_j \cdot H_j$ differ by one when rounded to 14 digits. Since reference [5] is known to the author only via Mathematical Reviews, comparison with it cannot be made.

3. The Error Term. Since published formulas do not include that for the error term, this term is derived below, some of the essential ideas being found in [6].

If $f(x)$ is a polynomial of degree $2n - 2$ or less, it can be written

$$f(x) = \Lambda_n^s(x) \cdot \sum_{r=0}^{n-2} A_r \Lambda_r^s(x) + \sum_{k=1}^{n} B_k \frac{\Lambda_n^s(x)}{x - a_k} \tag{3}$$

where $\Lambda_0^s(x) = 1$ and the B's are to be evaluated from

$$f(a_j) = B_j \left[\frac{d}{dx} \Lambda_n^s(x) \right]_{x=a_j} \qquad j = 1, \cdots, n \tag{4}$$

and the A's are to be determined from the requirements

$$\begin{aligned} f'(a_j) = \left[\frac{d}{dx} \Lambda_n^s(x) \right]_{x=a_j} \cdot \sum_{r=0}^{n-2} A_r \Lambda_r^s(a_j) + \sum_{k=1;k\neq j}^{n} (B_k + B_j) \\ \cdot \frac{\Lambda_n^s(x)}{(x - a_j)(x - a_k)} \Bigg|_{x=a_j} \qquad (j = 2, \cdots, n). \end{aligned} \tag{5}$$

Thus the left and right members of (3) are polynomials of degree $2n - 2$ or less coinciding at the n points $x = a_j$ and with equal derivatives at $n - 1$ points $x = a_j \neq 0$, hence are identical.

If $f(x)$ is not a polynomial of degree $2n - 2$ or less, then (3) will not be exact, but must contain a remainder term:

$$f(x) = \Lambda_n^s(x) \cdot \sum_{r=0}^{n-2} A_r \Lambda_r^s(x) + \sum_{k=1}^{n} B_k \frac{\Lambda_n^s(x)}{x - a_k} + R_{2n-1}(x) \tag{6}$$

and some form for $R_{2n-1}(x)$ is to be sought. We examine the possibility that

$$R_{2n-1}(x) = K(x) \cdot \frac{[\Lambda_n^s(x)]^2}{x}. \tag{7}$$

Let the right member of (3) be denoted by $\theta(x)$, and consider the function

$$\Phi(x) = f(x) - \theta(x) - R_{2n-1}(x).$$

With A's and B's determined by (4) and (5), and with R in the form (7), $\Phi(x)$ has zeros at $x = a_j$ and vanishing first derivative at $x = a_j \neq 0$. Then, selecting some arbitrary $x = \bar{x}$ different from all a_j, $K(\bar{x})$ can be determined so that $\Phi(\bar{x}) = 0$. Then $\Phi(x)$ has double zeros at $x = a_j$ $(j = 2, \cdots, n)$ and simple zeros at $x = 0$ and $x = \bar{x}$. Hence, by $(2n - 1)$-fold application of Rolle's Theorem,

$$\Phi^{(2n-1)}(\xi) = f^{(2n-1)}(\xi) - K(\bar{x}) \cdot (2n - 1)! = 0 \quad \text{for some } \xi.$$

This permits writing the remainder term in the form

$$R_{2n-1}(x) = \frac{f^{(2n-1)}(\xi)}{(2n-1)!}\frac{[\Lambda_n{}^s(x)]^2}{x}, \qquad 0 \leqq \xi \leqq \max(a_j, \bar{x}).$$

When (6) is substituted into the left member of (1), the terms of the first summation vanish because they are orthogonal with respect to the weight function; the terms of the second summation become the nonremainder terms of the right member; the integral of $R_{2n-1}(x)$ becomes E as follows:

$$\begin{aligned} E &= \int_0^\infty x^s e^{-x} \frac{f^{(2n-1)}(\xi)}{(2n-1)!}\frac{[\Lambda_n{}^s(x)]^2}{x}\,dx, \qquad 0 \leqq \xi \leqq \max(a_j, x), \\ &= \frac{f^{(2n-1)}(\eta)}{(2n-1)!}\int_0^\infty x^{s+1}e^{-x}\frac{[\Lambda_n{}^s(x)]^2}{x}\,dx, \qquad 0 \leqq \eta \leqq \infty, \\ &= \frac{f^{(2n-1)}(\eta)}{(2n-1)!}\int_0^\infty x^{s+1}e^{-x}[\Lambda_{n-1}^{s+1}(x)]^2\,dx, \qquad 0 \leqq \eta \leqq \infty, \\ &= \frac{f^{(2n-1)}(\eta)}{(2n-1)!}\cdot(n-1)!\,\Gamma(n+s+1), \qquad 0 \leqq \eta \leqq \infty. \end{aligned}$$

4. Acknowledgment. The author wishes to thank Mr. Hugh Hopper for his interest in the problem and for his careful programming of equations for machine computing of Table I.

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