

# Truncation Errors in Two Chebyshev Series Approximations

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**1. Introduction.** Suppose a function  $f(x)$  is defined for  $-1 \leq x \leq 1$ , and is of bounded variation in this range. Then  $f(x)$  can be expanded in a convergent series of Chebyshev polynomials  $T_n(x)$  as

$$(1.1) \quad f(x) = \sum'_{n=0}^{\infty} a_n T_n(x).$$

$\sum'$  denotes a sum whose first term is halved, and  $T_n(x)$  denotes the Chebyshev polynomial of the first kind of degree  $n$ , defined by

$$(1.2) \quad T_n(x) = \cos n\theta, \quad \text{where } x = \cos \theta, \quad \text{for } n = 0, 1, 2, \dots.$$

The coefficients  $a_n$  are given by (see [1])

$$(1.3) \quad a_n = \frac{2}{\pi} \int_{-1}^1 \frac{f(x) T_n(x)}{\sqrt{1-x^2}} dx \quad \text{for } n = 0, 1, 2, \dots.$$

A useful polynomial approximation to  $f(x)$  can be found by truncating the infinite series in equation (1.1). Indeed, for many of the more commonly used functions, Clenshaw [3] has tabulated the coefficients  $a_n$ , as given by equation (1.3), to 20 decimal places. In each case we have a polynomial approximation  $P_N(x)$ , say, to  $f(x)$  where

$$(1.4) \quad P_N(x) = \sum'_{n=0}^N a_n T_n(x).$$

However, in attempting to find a suitable polynomial approximation to a general function  $f(x)$ , the integral occurring in equation (1.3) cannot be evaluated explicitly, and recourse has to be made to approximate methods for evaluating  $a_n$ . The most widely used method is the "curve-fitting" method described by Lanczos [1], and, in greater detail, by Clenshaw and Curtis [2]. There are two variations of the method which we shall call the "practical" and "classical" methods, respectively. Suppose we wish to approximate to  $f(x)$  by a polynomial of degree  $N$ . In the "practical" method, we construct a polynomial  $\Psi_N(x)$  by collocation with  $f(x)$  at the  $(N+1)$  points  $x_i = \cos(\pi i/N)$ ,  $i = 0(1)N$ , which are the zeros of the polynomial  $[T_{N+1}(x) - T_{N-1}(x)]$ . In the "classical" method, we construct a polynomial  $\Phi_N(x)$  by collocation with  $f(x)$  at the  $(N+1)$  points

$$x_i = \cos \frac{(2i+1)\pi}{2(N+1)}, \quad i = 0(1)N,$$

which are the zeros of the polynomial  $T_{N+1}(x)$ . Both  $\Psi_N(x)$  and  $\Phi_N(x)$  are "Lagrangian" interpolation polynomials to  $f(x)$ . In this paper we shall consider in some detail the truncation errors  $\psi_N(x) = f(x) - \Psi_N(x)$  and  $\phi_N(x) = f(x) -$

$\Phi_N(x)$ . First, we shall obtain estimates for  $\psi_N(x)$  and  $\phi_N(x)$  which may be used a priori to determine  $N$ , the degree of the required polynomial approximation to  $f(x)$ . Secondly, we shall attempt to compare the polynomial approximations  $\Phi_N(x)$  and  $\Psi_N(x)$  to a given  $f(x)$ . From the results of Sections 6 and 8, we conclude that, in general,  $\Psi_N(x)$  is to be preferred to  $\Phi_N(x)$  as an approximation to  $f(x)$  on the basis of minimising the maximum truncation error in  $-1 \leq x \leq 1$ . We shall consider only briefly the truncation error  $p_N(x) = f(x) - P_N(x)$ . In a recent paper [4], Clenshaw has considered the truncation error  $p_N(x)$  as compared with the maximum truncation error obtained using the polynomial of "best fit" of degree  $N$ , to  $f(x)$ . However, we have not attempted in this paper to compare  $\Phi_N(x)$  and  $\Psi_N(x)$  with the polynomial of "best fit."

In Sections 2 and 3, we discuss the computation of the polynomials  $\Psi_N(x)$  and  $\Phi_N(x)$ , respectively. Explicit forms for the truncation errors  $\phi_N(x)$  and  $\psi_N(x)$ , in terms of contour integrals, are derived in Section 4. Although these results are not new, the derivation does give explicit forms for the coefficients in the Chebyshev series expansions of  $\Phi_N(x)$  and  $\Psi_N(x)$ , also in terms of contour integrals. The evaluation of the contour integrals for the truncation error is discussed in Sections 5 and 7, where  $f(z)$  is considered to be a meromorphic function and an integral function, respectively. In Section 6, we use the results of Section 5 in order to make some comparison of the polynomial approximations  $\Phi_N(x)$  and  $\Psi_N(x)$ , and conclude that  $\Psi_N(x)$  is to be preferred to  $\Phi_N(x)$  in general. This conclusion is supported by the results of Section 8, where we have obtained asymptotic estimates for large  $N$  of the truncation error in the quadrature method proposed by Clenshaw and Curtis [2].

**2. The Polynomial  $\Psi_N(x)$ .** The computation of  $\Psi_N(x)$  has been discussed in some detail by Clenshaw and Curtis [2] and their results will be stated briefly here. It is shown that

$$(2.1) \quad \Psi_N(x) = \sum_{n=0}^N{}'' B_{n,N} T_n(x),$$

where  $\sum''$  denotes a sum whose first and last terms are halved. The coefficients  $B_{n,N}$  (which depend upon  $N$  as well as  $n$ ) are given by

$$(2.2) \quad B_{n,N} = \frac{2}{N} \sum_{i=0}^N{}'' f(x_i) T_n(x_i) = \frac{2}{N} \sum_{i=0}^N{}'' f(x_i) T_i(x_n),$$

where

$$(2.3) \quad x_i = \cos \frac{\pi i}{N} \quad \text{for } i = 0(1)N.$$

Since  $T_n(x_i) = T_i(x_n)$ , the coefficients  $B_{n,N}$  can be evaluated by the elegant method for summing a Chebyshev series described by Clenshaw [3]. The relation between the coefficients  $B_{n,N}$  and  $a_n$  is given by

$$(2.4) \quad B_{n,N} = a_n + \sum_{p=1}^{\infty} (a_{2pN-n} + a_{2pN+n}).$$

We note, in particular, that

$$B_{N-1,N} = a_{N-1} + a_{N+1} + a_{3N-1} + \dots,$$

so that, unless the coefficients  $a_n$  converge very rapidly,  $B_{N-1,N}$  will not be a good approximation to  $a_{N-1}$ . This immediately raises the question of comparing the truncation errors  $\psi_N(x) = f(x) - \Psi_N(x)$ , and  $p_N(x) = f(x) - P_N(x)$ . Throughout this paper we shall be interested in determining not only the truncation error for a given value of  $x$  in  $-1 \leq x \leq 1$ , but also the maximum modulus of the truncation error in this interval. If we define

$$(2.5) \quad p_N = \max_{-1 \leq x \leq 1} |p_N(x)|,$$

then equations (1.1) and (1.4) give immediately that

$$(2.6) \quad p_N \leq \sum_{n=N+1}^{\infty} |a_n|.$$

From equation (2.1),

$$(2.7) \quad \psi_N(x) = \sum_{n=0}^{N-1} (a_n - B_{n,N})T_n(x) + (a_N - \frac{1}{2}B_{N,N})T_N(x) + \sum_{n=N+1}^{\infty} a_n T_n(x),$$

and combining this with equation (2.4) we find

$$(2.8) \quad \psi_N \leq 2 \sum_{n=N+1}^{\infty} |a_n|,$$

where  $\psi_N$  is defined by

$$(2.9) \quad \psi_N = \max_{-1 \leq x \leq 1} |\psi_N(x)|.$$

Thus, if  $f(x)$  is such that the Chebyshev coefficients  $a_n$  are either of the same sign or of alternating sign, the truncation error of the polynomial  $\Psi_N(x)$  is less than or equal to twice that of the polynomial  $P_N(x)$ .

A simple estimate of  $\psi_N$  can be given when the coefficients  $a_n$  converge rapidly. From equation (2.4), we find

$$(2.10) \quad \psi_N(x) = a_{N+1}(T_{N+1}(x) - T_{N-1}(x)) + a_{N+2}(T_{N+2}(x) - T_{N-2}(x)) + \dots,$$

which gives

$$(2.11) \quad \psi_N = 2|a_{N+1}|, \quad \text{approximately.}$$

**3. The Polynomial  $\Phi_N(x)$ .** The calculation of this polynomial, found by collocation with  $f(x)$  at the zeros of  $T_{N+1}(x)$ , has not been discussed as extensively in the literature as that of  $\Psi_N(x)$ . We shall therefore consider the  $\Phi_N(x)$  polynomial in a little more detail than that of  $\Psi_N(x)$ . Now

$$(3.1) \quad \Phi_N(x) = \sum_{n=0}^N A_{n,N} T_n(x),$$

where the coefficients  $A_{n,N}$  are given by

$$(3.2) \quad A_{n,N} = \frac{2}{N+1} \sum_{i=0}^N f(x_i) T_n(x_i) \quad \text{for } n = 0(1)N, \quad \text{with}$$

$$x_i = \cos \frac{\pi(2i+1)}{2(N+1)} \quad \text{for } i = 0(1)N.$$

The derivation of the coefficients  $A_{n,N}$  is based on the orthogonality with respect to summation of the trigonometric functions, given by

$$(3.3) \quad \sum_{i=0}^N \cos(j\theta_i) \cos(k\theta_i) = \frac{N+1}{2} \delta_{j,k} \quad \text{for } j, k \leq N,$$

where  $\theta_i = \pi(2i + 1)/2(N + 1)$  and  $\delta_{j,k}$  is Kronecker's delta. To evaluate the coefficients  $A_{n,N}$  for a given  $f(x)$ , we first note that

$$T_n(x_i) = \cos \frac{n\pi(2i + 1)}{2(N + 1)} = T_{2i+1}(y_n)$$

say, where

$$y_n = \cos \frac{n\pi}{2(N + 1)}.$$

The coefficients  $A_{n,N}$  can then be evaluated in a similar way to the coefficients  $B_{n,N}$ , by summing a finite Chebyshev series of polynomials  $T_n(x)$  of odd degree only. Clenshaw [3] has shown in this case that we construct a sequence  $\{b_i\}$ , where  $b_i$  satisfies the recurrence relation

$$(3.4) \quad b_i - (4y_n^2 - 2)b_{i+1} + b_{i+2} = f(x_i),$$

with  $b_{N+1} = b_{N+2} = 0$ , to give in turn  $b_N, b_{N-1}$ , etc. Then

$$(3.5) \quad A_{n,N} = \frac{2}{N+1} \sum_{i=0}^N f(x_i) T_n(x_i) = \frac{2}{N+1} y_n (b_0 - b_1).$$

Thus the coefficients  $A_{n,N}$  may be computed as readily as the coefficients  $B_{n,N}$ .

Let us now consider the relation between the coefficients  $A_{n,N}$  and  $a_n$  analogous to equation (2.4). From equations (1.1) and (3.2), we have

$$(3.6) \quad A_{n,N} = \frac{2}{N+1} \sum_{i=0}^N f(x_i) T_n(x_i) = \frac{2}{N+1} \sum_{m=0}^{\infty} a_m \left( \sum_{i=0}^N T_m(x_i) T_n(x_i) \right).$$

Since the subscript  $m$  ranges over all positive integers, we need a generalisation of equation (3.3). It is not difficult to show that

$$(3.7) \quad \sum_{i=0}^N T_m(x_i) T_n(x_i) = \begin{cases} 0 & \text{for } m \neq 2p(N+1) \pm n, \\ \frac{(-1)^p(N+1)}{2} & \text{for } m = 2p(N+1) \pm n, \end{cases}$$

where  $p$  is a positive integer. Equation (3.6) then gives

$$(3.8) \quad A_{n,N} = a_n + \sum_{p=1}^{\infty} (-1)^p (a_{2p(N+1)-n} + a_{2p(N+1)+n})$$

for  $n = 0(1)N$ . In particular, we find

$$A_{N,N} = a_N - a_{N+2} + \dots,$$

which is not nearly as good an approximation to  $a_N$  as  $\frac{1}{2}B_{N,N}$  which has an error  $a_{3N} + \dots$ . Since

$$(3.9) \quad \phi_N(x) = f(x) - \Phi_N(x) = \sum_{n=0}^N (a_n - A_{n,N}) T_n(x) + \sum_{n=N+1}^{\infty} a_n T_n(x),$$

we find on using equation (3.8) that

$$(3.10) \quad \phi_N = \max_{-1 \leq x \leq 1} |\phi_N(x)| \leq 2 \sum_{n=N+1}^{\infty} |a_n| - \sum_{p=0}^{\infty} |a_{(2p+1)(N+1)}| < 2 \sum_{n=N+1}^{\infty} |a_n|,$$

which may be compared with equation (2.8) for  $\psi_N$ .

When the coefficients  $a_n$  converge rapidly, we have a simple estimate for  $\phi_N$ . From equation (3.9),

$$(3.11) \quad \phi_N(x) = a_{N+1}T_{N+1}(x) + a_{N+2}(T_N(x) + T_{N+2}(x)) + \dots,$$

so that, if  $|a_{N+2}|$ ,  $|a_{N+3}|$ , etc., are negligible compared with  $|a_{N+1}|$ , we have

$$(3.12) \quad \phi_N = |a_{N+1}|, \quad \text{approximately.}$$

A comparison of equation (2.11) and (3.12) indicates that in the case when  $f(x)$  possesses a rapidly convergent Chebyshev series expansion, the truncation error of  $\Phi_N(x)$  is half that of  $\Psi_N(x)$ .

In Section 6 we shall compare the two polynomial approximations  $\Psi_N(x)$  and  $\Phi_N(x)$  to  $f(x)$  in greater detail.

**4. Explicit Forms of the Truncation Errors.** The main purpose of this paper is to determine a priori the degree of the approximating polynomial  $\Phi_N(x)$  or  $\Psi_N(x)$  to a function  $f(x)$ , so that the maximum truncation error is less than some prescribed amount. In order to consider this error in more detail, we shall derive in this section explicit forms for  $\phi_N(x)$  and  $\psi_N(x)$  in terms of contour integrals.

So far, we have considered  $f(x)$  to be defined for  $-1 \leq x \leq 1$ . Let us continue this definition into the complex plane so that we consider  $f(z)$  defined for all  $z$ , and such that  $f(z)$  takes the value of  $f(x)$  on the basic interval  $[-1, 1]$ . First let us consider the truncation error  $p_N(x) = f(x) - P_N(x)$ . In a recent paper, Elliott [5] has shown that

$$(4.1) \quad a_n = \frac{1}{\pi i} \int_C \frac{f(z) dz}{\sqrt{(z^2 - 1)}(z + \sqrt{(z^2 - 1)})^n},$$

where  $C$  is a contour enclosing  $-1 \leq \text{Re } z \leq 1, \text{Im } z = 0$ , on and within which  $f(z)$  is regular, and  $|z + \sqrt{(z^2 - 1)}| > 1$  for all  $z$  except  $-1 \leq \text{Re } z \leq 1, \text{Im } z = 0$ . Substituting this expression for  $a_n$  into  $p_N(x) = \sum_{n=N+1}^{\infty} a_n T_n(x)$ , interchanging the integral and the sum, we find on evaluating the sum that

$$(4.2) \quad p_N(x) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-x)\sqrt{(z^2-1)}} \left\{ \frac{T_{N+1}(x)}{(z+\sqrt{(z^2-1)})^N} - \frac{T_N(x)}{(z+\sqrt{(z^2-1)})^{N+1}} \right\} dz.$$

To obtain  $\phi_N(x)$  as a contour integral, we first derive an expression for  $A_{n,N}$  as a contour integral. We start with Cauchy's formula, which states that if a function  $f(z)$  is regular on and within a contour  $C$ , then

$$(4.3) \quad f(x) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-x} dz,$$

where  $x$  is any point within  $C$ . If  $C$  is chosen so that it contains the interval  $-1 \leq$

$x \leq 1$ , then, from equation (3.2), we have

$$(4.4) \quad A_{n,N} = \frac{1}{\pi i(N+1)} \int_c f(z) \left\{ \sum_{i=0}^N \frac{T_n(x_i)}{z - x_i} \right\} dz, \quad \text{where}$$

$$x_i = \cos \frac{\pi(2i+1)}{2(N+1)}.$$

Now it can be shown (see Appendix 1) that

$$(4.5) \quad \sum_{i=0}^N \frac{T_n(x_i)}{z - x_i} = \frac{(N+1)U_{N-n}(z)}{T_{N+1}(z)},$$

where  $U_n(x)$  is the Chebyshev polynomial of the second kind of degree  $n$ . It is perhaps appropriate at this point to recall the definition of  $T_n(z)$  and  $U_n(z)$  for complex argument  $z$ . We have

$$(4.6) \quad T_n(z) = \frac{1}{2} \{ (z + \sqrt{z^2 - 1})^n + (z - \sqrt{z^2 - 1})^n \};$$

$$U_n(z) = \frac{1}{2\sqrt{z^2 - 1}} \{ (z + \sqrt{z^2 - 1})^{n+1} - (z - \sqrt{z^2 - 1})^{n+1} \}.$$

Combining equations (4.4) and (4.5) we have

$$(4.7) \quad A_{n,N} = \frac{1}{\pi i} \int_c \frac{f(z)U_{N-n}(z)}{T_{N+1}(z)} dz.$$

With this definition of  $A_{n,N}$ , the polynomial  $\Phi_N(x)$  can be expressed as

$$(4.8) \quad \Phi_N(x) = \frac{1}{\pi i} \int_c \frac{f(z)}{T_{N+1}(z)} \left\{ \sum_{n=0}^N U_{N-n}(z)T_n(x) \right\} dz$$

$$= \frac{1}{2\pi i} \int_c \frac{f(z)[T_{N+1}(z) - T_{N+1}(x)]}{(z-x)T_{N+1}(z)} dz$$

(see Appendix 1). It then follows immediately on using equation (4.3) that

$$(4.9) \quad \phi_N(x) = \frac{T_{N+1}(x)}{2\pi i} \int_c \frac{f(z) dz}{(z-x)T_{N+1}(z)}.$$

This equation is the starting point for the subsequent analysis. Before discussing the evaluation of this integral, let us derive a similar expression for  $\psi_N(x)$ . From equations (2.2) and (4.3),

$$(4.10) \quad B_{n,N} = \frac{1}{\pi i N} \int_c f(z) \left\{ \sum_{i=0}^N \frac{T_n(x_i)}{z - x_i} \right\} dz, \quad \text{where } x_i = \cos \frac{\pi i}{N}.$$

Again (see Appendix 1),

$$(4.11) \quad \sum_{i=0}^N \frac{T_n(x_i)}{z - x_i} = \frac{N[U_{N-n}(z) - U_{N-n-2}(z)]}{T_{N+1}(z) - T_{N-1}(z)},$$

which gives immediately that

$$(4.12) \quad B_{n,N} = \frac{1}{\pi i} \int_c \frac{f(z)[U_{N-n}(z) - U_{N-n-2}(z)]}{T_{N+1}(z) - T_{N-1}(z)} dz.$$

With this definition of  $B_{n,N}$  we find

$$\begin{aligned}
 \Psi_N(x) &= \frac{1}{\pi i} \int_C \frac{f(z)}{T_{N+1}(z) - T_{N-1}(z)} \left\{ \sum_{n=0}^N [U_{N-n}(z) - U_{N-n-2}(z)] T_n(x) \right\} dz \\
 (4.13) \quad &= \frac{1}{2\pi i} \int_C \frac{f(z) \{ [T_{N+1}(z) - T_{N-1}(z)] - [T_{N+1}(x) - T_{N-1}(x)] \}}{(z-x)[T_{N+1}(z) - T_{N-1}(z)]} dz,
 \end{aligned}$$

(see Appendix 1). Combining this equation with equation (4.3), we find the required result for  $\psi_N(x)$ ,

$$(4.14) \quad \psi_N(x) = \frac{[T_{N+1}(x) - T_{N-1}(x)]}{2\pi i} \int_C \frac{f(z) dz}{(z-x)[T_{N+1}(z) - T_{N-1}(z)]}.$$

It should be noted that equations (4.9) and (4.14) are particular examples of the general expression for the truncation error of Lagrangian interpolation (see, for example, [6, p. 42]). However, in the above derivation we have also obtained explicit expressions for the coefficients  $A_{n,N}$  and  $B_{n,N}$  in terms of contour integrals. Furthermore, equation (4.9) has been previously obtained by Lanczos [7], but Lanczos did not make very extensive use of it for obtaining a priori estimates of the truncation error, which is the purpose of this paper. In Sections 5 and 7 we shall consider the evaluation of these integrals when  $f(z)$  is a meromorphic function and an integral function, respectively.

**5.  $f(z)$  a Meromorphic Function.** Let us suppose first that  $f(z)$  has  $M$  simple poles at the points  $z_m$  ( $m = 1(1)M$ ) with residues  $\rho_m$ . We take our contour  $C$  initially as an ellipse  $E_\mu$  with foci at  $z = \pm 1$ , given by  $|z + \sqrt{z^2 - 1}| = \mu$  ( $> 1$ ). The constant  $\mu$  is chosen so that  $\mu < |z_m + \sqrt{z_m^2 - 1}|$  for all  $m$ . We now let  $\mu \rightarrow \infty$ . In order that the contour  $C$  have  $f(z)$  regular within it, we enclose each pole  $z_m$  by a circle  $\gamma_m$  of radius  $\epsilon_m$ . The complete contour, for very large  $\mu$ , is taken as the ellipse  $E_\mu$  described in the positive (counterclockwise) sense minus the  $M$  small circles  $\gamma_m$ . Within this contour  $f(z)$  is regular. We now let  $\mu \rightarrow \infty$  and  $\epsilon_m \rightarrow 0$  for all  $m$ . If  $f(z)$  is such that the integral around the large ellipse tends to zero as  $\mu \rightarrow \infty$ , we find by the theorem of residues that

$$(5.1) \quad \phi_N(x) = -T_{N+1}(x) \sum_{m=1}^M \frac{\rho_m}{(z_m - x)T_{N+1}(z_m)},$$

and

$$(5.2) \quad \psi_N(x) = -[T_{N+1}(x) - T_{N-1}(x)] \sum_{m=1}^M \frac{\rho_m}{(z_m - x)[T_{N+1}(z_m) - T_{N-1}(z_m)]}.$$

From these equations we may be able to evaluate the quantities  $\phi_N$  and  $\psi_N$  explicitly, or find asymptotic estimates for them. Under certain conditions we can let  $M \rightarrow \infty$  in these equations. This may be illustrated by means of an example.

Suppose we wish to determine  $N$  such that  $\Phi_N(x)$  and  $\Psi_N(x)$  approximate to the function

$$(5.3) \quad f(x) = \frac{k}{(k^2 + 1) - (k^2 - 1) \cos \pi x}, \quad k > 1$$

with an error of less than  $10^{-5}/2$ , say, when  $k = 1.2$ . Now the function  $f(z)$  possesses

simple poles at the points  $z_m, \bar{z}_m$ , where

$$(5.4) \quad \begin{aligned} z_m &= 2m + i\beta \quad \text{for } m = 0, \pm 1, \pm 2, \dots; \\ \beta &= \frac{1}{\pi} \operatorname{arccosh} \frac{k^2 + 1}{k^2 - 1}. \end{aligned}$$

The residue of  $f(z)$  at  $z_m$  is  $-i/2\pi$  for all  $m$ , and at  $\bar{z}_m$  is  $i/2\pi$  for all  $m$ .

Since  $f(x)$  is an even function of  $x$ , we shall consider the truncation errors  $\phi_{2N}(x)$  and  $\psi_{2N}(x)$ . First, from equation (5.1), we have that for large  $N$ , the maximum contribution to the truncation error arises from the pair of poles at  $z_0, \bar{z}_0$ . Thus, for large  $N$ ,

$$(5.5) \quad \phi_{2N}(x) \sim \frac{iT_{2N+1}(x)}{2\pi} \left\{ \frac{1}{(i\beta - x)T_{2N+1}(i\beta)} + \frac{1}{(i\beta + x)T_{2N+1}(-i\beta)} \right\}.$$

Defining  $\xi = \beta + \sqrt{(\beta^2 + 1)}$ , and writing  $T_{2N+1}(z) \sim \frac{1}{2}(z + \sqrt{z^2 - 1})^{2N+1}$ , we have

$$(5.6) \quad \phi_{2N}(x) \sim \frac{(-1)^{N+1} 2x T_{2N+1}(x)}{\pi \xi^{2N+1} (\beta^2 + x^2)}.$$

When  $N$  is large, it is not difficult to show (see next section) that

$$(5.7) \quad \phi_{2N} \sim \begin{cases} \frac{1}{\pi \beta \xi^{2N+1}} & \text{if } \beta \leq 1, \\ \frac{2}{\pi(1 + \beta^2)\xi^{2N+1}} & \text{if } \beta \geq 1. \end{cases}$$

With  $k = 1.2$ , i.e.  $\beta = 0.763$ , we find  $\phi_{2N} < 10^{-5}/2$  when  $N = 8$ .

We can proceed in a similar way to estimate  $\psi_{2N}$  for large  $N$ . Considering the contribution to the truncation error from the pair of poles at  $z_0, \bar{z}_0$ , we find

$$(5.8) \quad \psi_{2N}(x) \sim \frac{2(-1)^{N+1} x (T_{2N+1}(x) - T_{2N-1}(x))}{\pi(1 + \xi^2)\xi^{2N-1}(\beta^2 + x^2)},$$

so that

$$(5.9) \quad \psi_{2N} \sim \frac{1}{\pi \xi^{2N} (\beta^2 + 1)} \quad \text{for all } \beta.$$

Again, with  $k = 1.2$  we have  $\psi_{2N} < 10^{-5}/2$  when  $N = 8$ .

**6. Comparison of the Polynomial Approximations  $\Phi_N(x)$  and  $\Psi_N(x)$ .** In Section 3, we showed that when  $f(x)$  possesses a rapidly convergent Chebyshev series, then  $\Phi_N(x)$  has a maximum error which is approximately half that of  $\Psi_N(x)$ . In comparing the two polynomial approximations we shall say that one approximation is "better" than the other if the former has the smaller maximum deviation. We have found already one case in which  $\Phi_N(x)$  is better than  $\Psi_N(x)$ , and one immediately asks the question whether this is true in general. To demonstrate that this is not the case, let us consider the function for which the coefficients  $a_n$  are in geometric progression, i.e., we shall assume that  $a_n = t^n$ , where  $0 < t < 1$ . For  $t$  close to zero, the coefficients  $a_n$  converge fairly rapidly; for  $t$  close to 1, we have a slowly convergent series.

It is not difficult to show that the function  $(1 - t^2)/2(1 + t^2 - 2tx)$  has its Chebyshev coefficients given by  $a_n = t^n$ . This is a rational function possessing one simple pole on the real axis at  $z = (1 + t^2)/2t$ , with residue  $-(1 - t^2)/4t$ . Equation (5.1) gives immediately that

$$(6.1) \quad \phi_N(x) = \frac{t^{N+1}(1 - t^2)T_{N+1}(x)}{(1 + t^{2N+2})(1 + t^2 - 2tx)},$$

so that  $\phi_N$  is given by  $\phi_N(1)$  and we have

$$(6.2) \quad \phi_N = \frac{t^{N+1}(1 + t)}{(1 + t^{2N+2})(1 - t)} \sim a_{N+1} \frac{(1 + t)}{(1 - t)},$$

for  $N$  sufficiently large.

Proceeding similarly for  $\psi_N(x)$ , equation (5.2) gives

$$(6.3) \quad \psi_N(x) = \frac{t^{N+1}[T_{N+1}(x) - T_{N-1}(x)]}{(1 - t^{2N})(1 + t^2 - 2tx)}.$$

Writing  $x = \cos \theta$ , we have

$$(6.4) \quad \psi_N(x) = -\frac{2t^{N+1} \sin N\theta}{(1 - t^{2N})} \cdot \frac{\sin \theta}{(1 + t^2 - 2t \cos \theta)}.$$

The problem is now reduced to finding  $\psi_N$ . Obviously neither end point of the range gives  $\psi_N(x)$  to be a maximum, and we must find the turning points of  $\psi_N(x)$ . For large  $N$ , the problem is simplified; for  $\psi_N(x)$  must take its maximum value close to the point where  $S(\theta, t) = \sin \theta / (1 + t^2 - 2t \cos \theta)$  takes its maximum value. We find that the maximum value of  $S(\theta, t)$  is  $1/(1 - t^2)$ , occurring when  $\cos \theta = 2t/(1 + t^2)$ . Thus for large  $N$ , we can write

$$(6.5) \quad \psi_N \sim \frac{2t^{N+1}}{(1 - t^{2N})(1 - t^2)} \sim \frac{2a_{N+1}}{1 - t^2}.$$

Comparing equations (6.2) and (6.5) we find that  $\psi_N < \phi_N$  when  $t > \sqrt{2} - 1$ . This analysis leads us to the following conclusion: if  $f(x)$  is such that the coefficients in its Chebyshev series expansion converge quickly, then  $\Phi_N(x)$  is better than  $\Psi_N(x)$ ; if the coefficients are slowly convergent then  $\Psi_N(x)$  is better than  $\Phi_N(x)$ . We notice from equations (6.2) and (6.5) that in the limit as  $t \rightarrow 0$ ,  $\psi_N = 2\phi_N$ .

When  $f(x)$  is either an even or an odd function of  $x$ , we can show by arguments similar to those above that the  $\Psi_N(x)$  polynomial is always to be preferred to the  $\Phi_N(x)$  polynomial. Suppose first that  $f(x)$  is an *even* function of  $x$ , and suppose we have  $a_{2n} = t^{2n}$  with  $a_{2n+1} = 0$ , for  $0 < t < 1$ . This corresponds to the function  $f(x) = (1 - t^4)/2\{(1 + t^2)^2 - 4t^2x^2\}$ . We now find,

$$(6.6) \quad \phi_{2N}(x) = \frac{2t^{2N+2}(1 - t^2)}{(1 + t^{4N+2})} \cdot \frac{xT_{2N+1}(x)}{[(1 + t^2)^2 - 4t^2x^2]}, \quad \text{with}$$

$$\phi_{2N} = \frac{2t^{2N+2}}{(1 + t^{4N+2})(1 - t^2)} \sim \frac{2a_{2N+2}}{(1 - t^2)};$$

the maximum values of  $\phi_{2N}(x)$  occurring at  $x = 1$ . Proceeding similarly we find,

$$(6.7) \quad \psi_{2N}(x) = \frac{t^{2N+2}[T_{2N+2}(x) - T_{2N-2}(x)]}{(1 - t^{4N})[(1 + t^2)^2 - 4t^2x^2]},$$

and for large  $N$ , this gives

$$(6.8) \quad \psi_{2N} \sim \frac{2t^{2N+2}}{(1-t^{4N})(1-t^4)} \sim \frac{2a_{2N+2}}{1-t^4}.$$

Comparing equations (6.6) and (6.8) we have immediately that  $\psi_{2N} < \phi_{2N}$  for all  $t$  and not just for  $t > \sqrt{2} - 1$  as in the case of a general  $f(x)$ .

A similar result can be shown to be true when  $f(x)$  is an odd function of  $x$ .

We may sum up our results as in Table 1, which gives the better of  $\Phi_N(x)$  or  $\Psi_N(x)$  under the given conditions.

TABLE 1

	<i>Rapidly Convergent Coefficients <math>a_n</math></i>	<i>Slowly Convergent Coefficients <math>a_n</math></i>
general $f(x)$	$\Phi_N(x)$	$\Psi_N(x)$
$f(x)$ even	$\Psi_{2N}(x)$	$\Psi_{2N}(x)$
$f(x)$ odd	$\Psi_{2N+1}(x)$	$\Psi_{2N+1}(x)$

Only in one case does the  $\Phi_N(x)$  polynomial approximation give a smaller maximum error than the  $\Psi_N(x)$  polynomial approximation. On these grounds we suggest that  $\Psi_N(x)$  should always be used, although it should be pointed out that in the author's experience it is only rarely that one of the maximum truncation errors is more than twice the other one. There is one further computational point that should be noted here. Clenshaw [4] states that in using the  $\Psi_N(x)$  approximation, if nothing is known of a suitable value for  $N$ , then one can start with a small  $N$ , and keep doubling  $N$  until the required precision is reached. In computing  $\Psi_{2N}(x)$ , most of the intermediary results obtained in evaluating  $\Psi_N(x)$  may be used again. Finally we shall show in Section 8 that  $\Psi_N(x)$  is to be preferred to  $\Phi_N(x)$  if the quadrature method of Clenshaw and Curtis [2] is used.

**7.  $f(z)$  an Integral Function.** The results of Section 5 do not, of course, include the case when  $f(z)$  is an integral function of  $z$ . For such functions we propose to find asymptotic estimates of the contour integral for large  $N$ , by the method of steepest descents. This approach has been successfully used by Elliott and Szekeres [8] in a similar problem of estimating  $a_n$  (as given by equation (4.1)) for large  $n$ . The principal result we shall require is the following (see, for example, de Bruijn [9]). To find an estimate for  $\int_C \exp[\xi(z)] dz$ , we deform the contour  $C$  to pass through the "saddle points"  $\zeta$  defined by  $\xi'(\zeta) = 0$  (there may be more than one saddle point). Provided that the integral over the remaining part of  $C$  is negligible, the value of the contour integral is asymptotically equal to the contribution in the neighbourhood of the saddle point, and is given by

$$(7.1) \quad \sqrt{(2\pi)\alpha} |\xi''(\zeta)|^{-1/2} \exp[\xi(\zeta)],$$

where  $\alpha$  is a complex number defined by

$$(7.2) \quad |\alpha| = 1 \quad \text{and} \quad \arg \alpha = \frac{\pi}{2} - \frac{1}{2} \arg \xi''(\zeta).$$

Before applying this result we shall modify slightly the definitions of  $\phi_N(x)$  and  $\psi_N(x)$  as given by equations (4.9) and (4.14), respectively, for the case of  $N$  large. Now for large  $N$ , we have  $T_{N+1}(z) \sim (1/2)(z + \sqrt{(z^2 - 1)})^{N+1}$  from equation (4.6). Thus we can write

$$(7.3) \quad \phi_N(x) \sim \frac{T_{N+1}(x)}{\pi i} \int_c \frac{1}{(z-x)} \cdot \frac{f(z)}{(z + \sqrt{(z^2 - 1)})^{N+1}} dz,$$

and

$$(7.4) \quad \psi_N(x) \sim \frac{[T_{N+1}(x) - T_{N-1}(x)]}{2\pi i} \int_c \frac{1}{(z-x)} \frac{f(z)}{\sqrt{(z^2 - 1)}(z + \sqrt{(z^2 - 1)})^N} dz.$$

In order to apply the method of steepest descents, we express each integral in the form  $\int_c (z-x)^{-1} \exp[\xi(z)] dz$ . We assume that the function  $(z-x)^{-1}$  changes very little as the contour passes through a saddle point, and can for our purposes be taken as constant.

For  $\phi_N(x)$ , we have

$$(7.5) \quad \xi(z) = \log f(z) - (N+1) \log(z + \sqrt{(z^2 - 1)}),$$

so that the saddle points are given by solutions of

$$(7.6) \quad \sqrt{(z^2 - 1)} \frac{f'(z)}{f(z)} = N + 1,$$

and the second derivative  $\xi''(z)$  is given by

$$(7.7) \quad \xi''(z) = \frac{d}{dz} \left( \frac{f'(z)}{f(z)} \right) + \frac{(N+1)z}{(z^2 - 1)^{3/2}}.$$

If equation (7.6) has  $M$  roots  $\zeta_m$  for  $m = 1(1)M$ , then

$$(7.8) \quad \phi_N(x) \sim \frac{1}{i} \sqrt{\left(\frac{2}{\pi}\right)} T_{N+1}(x) \sum_{m=1}^M \frac{\alpha_m f(\zeta_m)}{(\zeta_m - x) |\xi''(\zeta_m)|^{1/2} (\zeta_m + \sqrt{(\zeta_m^2 - 1)})^N},$$

where  $|\alpha_m| = 1, \quad \arg \alpha_m = \frac{\pi}{2} - \frac{1}{2} \arg \xi''(\zeta_m),$

from which  $\phi_N$  may be estimated.

For  $\psi_N(x)$ , we proceed similarly. The saddle points are given by the solutions of the equation

$$(7.9) \quad \sqrt{(z^2 - 1)} \frac{f'(z)}{f(z)} = \frac{z}{\sqrt{(z^2 - 1)}} + N,$$

and the second derivative  $\xi''(z)$  is given by

$$(7.10) \quad \xi''(z) = \frac{d}{dz} \left( \frac{f'(z)}{f(z)} \right) + \frac{z^2 + 1}{(z^2 - 1)^2} + \frac{Nz}{(z^2 - 1)^{3/2}}.$$

If equation (7.9) has  $M$  roots  $\zeta_m$  for  $m = 1(1)M$ , then

$$(7.11) \quad \psi_N(x) \sim \frac{[T_{N+1}(x) - T_{N-1}(x)]}{i\sqrt{2\pi}} \sum_{m=1}^M \frac{\alpha_m f(\zeta_m)}{(\zeta_m - x) |\xi''(\zeta_m)|^{1/2} \sqrt{(\zeta_m^2 - 1)} (\zeta_m + \sqrt{(\zeta_m^2 - 1)})^N},$$

where

$$|\alpha_m| = 1, \quad \arg \alpha_m = \frac{\pi}{2} - \frac{1}{2} \arg \xi''(\zeta_m).$$

Again, with this expression for  $\psi_N(x)$ , we may estimate  $\psi_N$ .

These results may be simplified a little since we have assumed that  $N$  is large. For example, for both  $\phi_N(x)$  and  $\psi_N(x)$  we may assume that the saddle points are given by the same equation, viz.,

$$(7.12) \quad \sqrt{(z^2 - 1)} \frac{f'(z)}{f(z)} = N.$$

Under the same assumption of  $N$  large, we have in each case that the second derivative  $\xi''(z)$  is given approximately by

$$(7.13) \quad \xi''(z) = \frac{d}{dz} \left( \frac{f'(z)}{f(z)} \right) + \frac{Nz}{(z^2 - 1)^{3/2}}.$$

Thus the determination of one set of saddle points, and the evaluation of one set of second derivatives may be sufficient for estimating both  $\phi_N$  and  $\psi_N$ .

**8. The Quadrature Method of Clenshaw and Curtis.** We now consider the truncation error in a quadrature method recently proposed by Clenshaw and Curtis [2]. Their algorithm is briefly as follows. To evaluate the integral  $I(x)$ , where

$$(8.1) \quad I(x) = \int_{-1}^x f(t) dt \quad \text{for} \quad -1 \leq x \leq 1,$$

we choose an integer  $N$  and approximate  $f(t)$  for  $-1 \leq t \leq 1$  by the polynomial  $\Psi_N(t)$  (using the notation of this paper). Having calculated  $B_{n,N}$ , the coefficients  $\beta_n$  in the Chebyshev expansion of  $I(x)$  can be found from the relation

$$(8.2) \quad \beta_n = \frac{B_{n-1,N} - B_{n+1,N}}{2n} \quad \text{for} \quad n = 1(1)N + 1.$$

The coefficient  $\beta_0$  is found from the condition that  $I(-1) = 0$ . For any value of  $x$ , the Chebyshev series for  $I(x)$  can then be readily summed. We shall now find an asymptotic form of the error in  $I(1)$ , for large  $N$ .

From Sections 5 and 7, we find in each case that  $\psi_N(x)$  satisfies a relation of the form

$$(8.3) \quad \psi_N(x) = [T_{N+1}(x) - T_{N-1}(x)] \sum_{m=1}^M \frac{g_m}{(x - z_m)}.$$

If  $E_N(\Psi)$  denotes the error in  $I(1)$  when  $f(x)$  is approximated by the polynomial  $\Psi_N(x)$  then,

$$(8.4) \quad E_N(\Psi) = \int_{-1}^1 \psi_N(x) dx = \sum_{m=1}^M g_m (J_{N+1}(z_m) - J_{N-1}(z_m)),$$

where  $J_n(z)$  is defined by

$$(8.5) \quad J_n(z) = \int_{-1}^1 \frac{T_n(x)}{x - z} dx.$$

On writing  $x = \cos \theta$ , and integrating the resulting integral by parts, we find, for large  $n$ , that

$$(8.6) \quad J_n(z) \sim \frac{C(z)}{n^2} \left[ 1 + \frac{D(z)}{n^2} + \dots \right].$$

The coefficients  $C(z)$  and  $D(z)$  are defined by

$$(8.7) \quad \begin{aligned} C(z) &= \frac{2z}{z^2 - 1}, & D(z) &= \frac{z^2 + 5}{z^2 - 1}, & \text{for } n \text{ even;} \\ C(z) &= \frac{2}{z^2 - 1}, & D(z) &= \frac{2(2z^2 + 1)}{z^2 - 1}, & \text{for } n \text{ odd.} \end{aligned}$$

Substituting equations (8.6) and (8.7) into equation (8.4) we find

$$(8.8) \quad E_N(\Psi) \sim \frac{-4N}{(N^2 - 1)^2} \sum_{m=1}^M g_m C(z_m) \left[ 1 + \frac{2(N^2 + 1)D(z_m)}{(N^2 - 1)^2} \right],$$

i.e.,  $E_N(\Psi) \sim O(1/N^3)$  for large  $N$ .

It is of considerable interest to derive the corresponding expression for  $E_N(\Phi)$ , the error in  $I(1)$  if  $f(t)$  had been approximated by  $\Phi_N(t)$  instead of  $\Psi_N(t)$ . We find

$$(8.9) \quad E_N(\Phi) \sim \frac{1}{(N + 1)^2} \sum_{m=1}^M h_m C(z_m) \left[ 1 + \frac{D(z_m)}{(N + 1)^2} \right],$$

where we have assumed that  $\phi_N(x)$  is of the form

$$\phi_N(x) = T_{N+1}(x) \sum_{m=1}^M \frac{h_m}{(x - z_m)},$$

from the results for meromorphic and integral functions. Equation (8.9) shows that for large  $N$ ,  $E_N(\Phi) \sim O(1/N^2)$ .

Thus, when considering integration over the complete interval  $[-1, 1]$  the  $\Psi_N(x)$  polynomial approximation to  $f(x)$  is to be preferred to  $\Phi_N(x)$  since for large  $N$ , the truncation error is smaller. This has considerable importance in two problems considered so far in the literature. Clenshaw and Norton [10] have proposed a method using Chebyshev series for the numerical solution of boundary-value problems involving nonlinear ordinary differential equations. Elliott [11] has considered the numerical solution of Fredholm integral equations. In both cases collocation at the points  $x_i = \cos(\pi i/N)$  for  $i = 0(1)N$  was used. Certainly, as far as the latter application was concerned, the author based his choice of collocation points on computation expediency rather than any mathematical reasoning. The above analysis justifies this choice on the basis of minimising the truncation error.

**9. Conclusion.** In this paper we have considered the problem of finding a priori estimates of the truncation errors involved when a function  $f(x)$ , defined for  $-1 \leq x \leq 1$ , is approximated by polynomials  $\Phi_N(x)$  and  $\Psi_N(x)$ . These polynomials are Lagrangian interpolation polynomials obtained by collocation with  $f(x)$  at the points  $x_i = \cos(\pi(2i + 1)/2(N + 1))$  and  $x_i = \cos(\pi i/N)$  for  $i = 0(1)N$ , respectively. Such estimates have been obtained when  $f(z)$ , considered as a function of the complex variable  $z$ , is either a meromorphic or an integral function. One

of the results of the analysis is that the polynomial approximation  $\Psi_N(x)$  is to be preferred to  $\Phi_N(x)$  in many practical circumstances.

APPENDIX 1. In Section 4, we quoted without proof some results that will now be proved. The starting point is the identity

$$(A1) \quad \frac{T_N(z) - T_N(x)}{z - x} = U_{N-1}(z) + 2 \sum_{n=1}^{N-1} U_{N-n-1}(z)T_n(x).$$

This can be proved by multiplying each side of the equation by  $(z - x)$  and using the well-known recurrence relation for  $T_n(x)$  and  $U_n(x)$ , see [1]. With this relation the evaluation of the finite sums in equations (4.8) and (4.13) follow after a little algebra.

From these two equations we can obtain the results given in equations (4.5) and (4.11). First, for equation (4.5), we have

$$(A2) \quad \sum_{i=0}^N \frac{T_n(x_i)}{z - x_i} = - \sum_{i=0}^N \frac{T_n(z) - T_n(x_i)}{z - x_i} + T_n(z) \sum_{i=0}^N \frac{1}{z - x_i}.$$

Since the  $x_i$  are the zeros of  $T_{N+1}(x)$ , we have immediately that

$$(A3) \quad \sum_{i=0}^N \frac{1}{(z - x_i)} = \frac{T'_{N+1}(z)}{T_{N+1}(z)} = \frac{(N + 1)U_N(z)}{T_{N+1}(z)}.$$

Again from equation (A1), we have

$$(A4) \quad \sum_{i=0}^N \frac{T_n(z) - T_n(x_i)}{z - x_i} = (N + 1)U_{n-1}(z),$$

since  $\sum_{i=0}^N T_r(x_i) = 0$  for all  $r \neq 0$ . Combining equations (A2)–(A4), we find

$$(A5) \quad \sum_{i=0}^N \frac{T_n(x_i)}{(z - x_i)} = \frac{(N + 1)U_{N-n}(z)}{T_{N+1}(z)}.$$

We can proceed similarly to prove equation (4.11). As before, we write

$$(A6) \quad \sum_{i=0}'' \frac{T_n(x_i)}{z - x_i} = - \sum_{i=0}'' \frac{T_n(z) - T_n(x_i)}{z - x_i} + T_n(z) \sum_{i=0}'' \frac{1}{(z - x_i)}.$$

Since  $x_i$  are in this case the zeros of  $[T_{N+1}(z) - T_{N-1}(z)]$ , we have

$$\sum_{i=0}'' \frac{1}{(z - x_i)} = \frac{(N + 1)U_N(z) - (N - 1)U_{N-2}(z)}{T_{N+1}(z) - T_{N-1}(z)}$$

whence

$$(A7) \quad \sum_{i=0}'' \frac{1}{(z - x_i)} = \frac{N[U_N(z) - U_{N-2}(z)]}{T_{N+1}(z) - T_{N-1}(z)}.$$

Again, from equation (A1), since  $\sum_{i=0}'' T_r(x_i) = 0$  for all  $r \neq 0$ , we have

$$(A8) \quad \sum_{i=0}'' \frac{T_n(z) - T_n(x_i)}{(z - x_i)} = NU_{n-1}(z).$$

Combining equations (A6)–(A8), we find the required result,

$$(A9) \quad \sum_{i=0}'' \frac{T_n(x_i)}{(z - x_i)} = \frac{N[U_{N-n}(z) - U_{N-n-2}(z)]}{T_{N+1}(z) - T_{N-1}(z)}.$$

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