Estimates of Weights in Gauss-Type Quadrature

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1. Introduction. It may readily be verified that the angular distance $\Delta\theta = \theta_{i+1,n} - \theta_{i,n}$ between the zeros $\theta_{i,n}$ of the Legendre polynomial $P_n(\cos\theta)$ in $\cos\theta$ is roughly constant for large n. From the quadrature formula itself the weights may be estimated to a corresponding degree of accuracy. Direct asymptotic estimates of the weights corresponding to $\cos\theta = 0$ in the $(2n + 1)$-point Gaussian quadrature are all available from Stirling's formula in the cases considered below. We here replace the P_n by C_n^λ, the Gegenbauer polynomials (effectively, tesseral harmonics or ultraspherical polynomials) of order $\lambda \gtrdot 0$, and the H_n in the single limiting set of Hermite polynomials. Explicit formulas are derived: but the estimates for the general weights have a precision limited by the corresponding precision of the estimates of the zeros.

2. The Quadrature Formula. The Lagrange interpolation formula

$$(1) \qquad f(x) = \sum_i \frac{P(x)f(x_i)}{P'(x_i)(x - x_i)}, \qquad P(x_i) = 0,$$
$$P'(x_i) \neq 0, \qquad i = 1, 2, \cdots, n,$$

algebraically valid for polynomials f of degree $\nu < n$, the degree of P, has a rather limited direct use in polynomial approximation theory. Combined with various restrictions on P to be in a basis of a set of polynomials with suitable properties, it becomes more useful.

Let $P^*(x)$ be of degree $n + 1$, so that $P^*(x) = axP(x) - bP(x) - cP_*(x)$ for constants a, b, and c, P_* representing a polynomial of degree $\nu < n$. We set

$$K(x, t) = K(t, x) = \frac{P^*(x)P(t) - P^*(t)P(x)}{x - t},$$

a polynomial of degree n in x for each t, so that

$$K(x, t) = aP(x)P(t) + cK_*(x, t),$$

K_* being defined in terms of P and P_* exactly as K is determined by P^* and P. In particular, $K(x, x) = P(x)P^{*\prime}(x) - P^*(x)P'(x)$; and (1) is modified to become

$$(2) \qquad f(x) = \sum_i \frac{K(x, x_i)}{K(x_i, x_i)} f(x_i).$$

A suitable normalization with respect to a fixed integrable weight function w, essentially positive over the interval I of integration, is

$$\int_I K(x, x_i)w(x)\, dx = 1,$$

so that (2) becomes

$$(3) \qquad \int_I f(x)w(x)\, dx = \sum_i f(x_i)W_i,$$

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where

$$W_i = \frac{1}{K(x_i, x_i)} \tag{4}$$

is the formula for the weights.

From the above,

$$K_n(x, t) = \sum_{i=0}^{n} a_j p_j(x) p_j(t),$$

the indices j indicating the degrees of the polynomials p_j. Referring to (1), for example, we set

$$p_n(t) \doteq k_n t^n - \sum_{i=0}^{n-1} c_{j,n} p_j(t), \qquad n = 1, 2, 3, \cdots,$$

where

$$p_0(t) = k_0 > 0, \qquad \int_I w(t)\, dt = \frac{1}{k_0^2},$$

$$\int_I p_n(t) p_j(t) w(t)\, dt = 0, \qquad 0 \leqq j < n, \tag{5}$$

and

$$\int_I \{p_n(t)\}^2 w(t)\, dt = 1.$$

The inductive definition is complete if we assume $k_n > 0$. Indeed, for an arbitrary polynomial P,

$$P(t) = \sum_{j=0}^{n} a_j k_j t^j = \sum_{j=0}^{n} a_{j,n} p_j(t), \tag{5'}$$

the $a_{j,n}$ being determined uniquely by the a_j and k_j, where $\int_I k_j t^j p_j(t) w(t)\, dt = 1$, so that

$$x p_n(x) = \frac{k_n}{k_{n+1}} p_{n+1}(x) + b_n p_n(x) + \frac{k_{n-1}}{k_n} p_{n-1}(x) + \sum_{j=0}^{n-2} b_{j,n} p_j(x) \tag{6}$$

in any case, with $b_{j,n} = 0$ by (5). Then

$$\begin{aligned} K_n(x, t) &= \sum_{j=0}^{n} p_j(x) p_j(t) \\ &= \frac{k_n}{k_{n+1}} \frac{p_{n+1}(x) p_n(t) - p_n(x) p_{n+1}(t)}{x - t}, \quad \text{and} \\ K_n(x, x) &= \sum_{j=0}^{n} \{p_j(x)\}^2 \\ &= \frac{k_n}{k_{n+1}} \{p'_{n+1}(x) p_n(x) - p_n'(x) p_{n+1}(x)\} \\ &= \int_I \{K_n(x, t)\}^2 w(t)\, dt, \end{aligned} \tag{7}$$

these being the standard Christoffel formulae (see [1]).

If f is of degree $2n - 1$ or less, the quotient Q of f by p_n is uniquely determined, with remainder $p_*(t) = f(t) - Q(t)p_n(t)$ of degree $n - 1$ or less. Then, if $p_n(x_i) = 0$, n being fixed,

$$
\begin{aligned}
&\int_I p_*(t)w(t)\,dt = \int_I f(t)w(t)\,dt, \quad \text{by (5), and} \\
(8)\qquad & \\
&\int_I f(t)w(t)\,dt = \sum_i W_i f(x_i),
\end{aligned}
$$

as before. (The formulae (7) guarantee the separation of n distinct zeros in I.)

3. Sums of Squares. The Cesàro-one sums

$$\sigma_n(x, t) = \frac{1}{n}\sum_{j=0}^{n-1} K_j(x, t)$$

are expressed in the way suggested by Christoffel's method as follows:

$$
\begin{aligned}
n(x - t)^2\sigma_n(x, t) &= \sum_{j=0}^{n-1} \frac{k_j}{k_{j+1}} (b_j - b_{j+1})\{p_{j+1}(x)p_j(t) + p_{j+1}(t)p_j(x)\} \\
(9)\qquad &+ \frac{k_{n-1}}{k_{n+1}} \{p_{n+1}(x)p_{n-1}(t) + p_{n-1}(x)p_{n+1}(t)\} \\
&- 2\left(\frac{k_{n-1}}{k_n}\right)^2 p_n(x)p_n(t) + 2\sum_{j=0}^{n-1} p_j(x)p_j(t)\left\{\left(\frac{k_j}{k_{j+1}}\right)^2 - \left(\frac{k_{j-1}}{k_j}\right)^2\right\},
\end{aligned}
$$

where $b_j = \int_I t\{p_j(t)\}^2 w(t)\,dt$ and $k_{-1} = 0$.

Beginning with $k_2(b_1 - b_0)/k_1 = c_{1,2}$, we see that $b_j = b_{j+1}$ for all j if and only if w is symmetric over I. After a translation, we may assume in this case that the $p_i(t)$ are alternately even and odd polynomials. We assume that this condition holds in the sequel.

Let

$$\Lambda_j(x) = \frac{k_{j-1}}{k_j} p_{j-1}(x) - \frac{k_j}{k_{j+1}} p_{j+1}(x),$$

so that

$$4\frac{k_{j-1}}{k_{j+1}} p_{j+1}(x)p_{j-1}(x) = x^2\{p_j(x)\}^2 - \{\Lambda_j(x)\}^2.$$

Then, for suitable constants c_n, we set

$$
\begin{aligned}
L_n(x) &= (c_n^2 - x^2)\{p_n(x)\}^2 + \{\Lambda_n(x)\}^2 \\
(10)\qquad &= 4\sum_{j=0}^{n-1} \{p_j(x)\}^2\left\{\left(\frac{k_j}{k_{j+1}}\right)^2 - \left(\frac{k_{j-1}}{k_j}\right)^2\right\} + \left\{c_n^2 - 4\left(\frac{k_{n-1}}{k_n}\right)^2\right\}\{p_n(x)\}^2.
\end{aligned}
$$

To make this formulation of sums of squares useful, the weight function w is further restricted.

4. Gegenbauer Polynomials. See [1].

The expansion of $\rho^{-2\lambda} = (1 - 2rt + r^2)^{-\lambda}$ as a power series in r,

$$(1 - rz)^{-\lambda}(1 - r\bar{z})^{-\lambda} = \sum_{j=0}^{\infty} C_j^{\lambda}(t)r^j,$$

subject to

$$z + \bar{z} = 2t = 2\cos\theta, \qquad z\bar{z} = 1, \qquad 0 \leqq r < 1,$$

determines the Gegenbauer polynomials C_n^λ of order $\lambda > 0$. If y is any successively differentiable function of ρ,

$$r^2 \frac{\partial^2 y}{\partial r^2} + \frac{\partial^2 y}{\partial t^2} = r^2 \frac{d^2 y}{d\rho^2}.$$

In the above case, $y = \rho^{-2\lambda}$, so $d^2y/d\rho^2 + ((2\lambda + 1)/\rho)(dy/d\rho) = 0$, and so

$$r^2 \frac{\partial^2 y}{\partial r^2} + (2\lambda + 1) r \frac{\partial y}{\partial r} + (1 - t^2) \frac{\partial^2 y}{\partial t^2} = (2\lambda + 1) t \frac{\partial y}{\partial t}.$$

Comparing coefficients in the power series, we have

$$(11) \qquad \frac{d}{dt}\left\{(1 - t^2)^{\lambda+1/2} \frac{dC_n^\lambda(t)}{dt}\right\} = -n(n + 2\lambda)(1 - t^2)^{\lambda-1/2} C_n^\lambda(t).$$

Multiplying by $C_j^\lambda(t)$, alternating the indices n and j, and subtracting, then integrating from $t = -1$ to $t = 1$, we have

$$C_j^\lambda(t) = \sqrt{h_j} p_j(t),$$

the $\{p_j\}$ being orthogonal (with property (5)) with respect to w,

$$w(t) = (1 - t^2)^{\lambda-1/2}.$$

Here,

$$\int_{-1}^{+1} \{C_j^\lambda(t)\}^2 w(t)\, dt = h_j,$$

easily calculated explicitly. From the definition above, using the series and the binomial theorem,

$$C_n^\lambda (\cos\theta) = \sum_{j=0}^{n} \binom{\lambda + j - 1}{j} \binom{\lambda + n - j - 1}{n - j} \cos(\overline{n - 2j}\theta),$$

so

$$|C_n^\lambda(t)| \leqq C_n^\lambda(1) = \binom{2\lambda + n - 1}{n}, \qquad -1 \leqq t \leqq 1,$$

if $\lambda > 0$.

We may make direct use of the Christoffel formulae (7), comparison of terms in a linear expansion, and induction, to obtain

$$2h_n k_0^2(n + \lambda) = \lambda \binom{n + 2\lambda - 1}{n},$$

$$4\left(\frac{k_{n-1}}{k_n}\right)^2 = \frac{n(n + 2\lambda - 1)}{(n + \lambda)(n - 1 + \lambda)},$$

and

$$\lambda!^2 2^{2\lambda}(n+\lambda)\{C_n^{\lambda}(t)\}^2 = \pi\binom{2\lambda+n-1}{n}\lambda(2\lambda)!\{p_n(t)\}^2. \tag{12}$$

Also,

$$\frac{k_{n-1}}{k_n}\,p_{n-1}(0) = -\frac{k_n}{k_{n+1}}\,p_{n+1}(0),$$

so that

$$\lim_{n\to\infty}\{p_{2n}(0)\}^2 = \frac{2}{\pi} \tag{13}$$

and

$$\lim_{n\to\infty} p_n(1)(n+\lambda)^{-2\lambda} = \sqrt{\frac{2}{\pi}}\,\frac{2^{\lambda}2!}{(2\lambda)!},$$

the relative errors in the corresponding approximations being of (order) $O(1/(n+\lambda)^2)$ uniformly in n for fixed λ by Stirling's formula.

We set $z = (1-t^2)^{\lambda/2}p_n(t)$, and find

$$\frac{dz}{dt} = (n+\lambda)(1-t^2)^{\lambda/2-1}\Lambda_n(t),$$

using (6) and (11). If

$$L_n(t) = \{p_n(t)\}^2(1-t^2) + \{\Lambda_n(t)\}^2,$$

(11) becomes

$$\frac{d}{dt}\{L_n(t)(1-t^2)^{\lambda-1}\} = -\frac{2\lambda(1-\lambda)}{n+\lambda}(1-t^2)^{\lambda-2}p_n(t)\Lambda_n(t). \tag{14}$$

From the above quadratic relation, and (6),

$$2\sqrt{(1-t^2)}\,|\,p_n(t)\Lambda_n(t)| \leqq L_n(t).$$

Differentiating the logarithm of L_n, and integrating, we have

$$\log\left\{\frac{L_n(t)}{L_n(0)}(1-t^2)^{\lambda-1}\right\} < \frac{|\lambda(1-\lambda)|}{n+\lambda}\,\frac{|t|}{\sqrt{(1-t^2)}}, \qquad 0<|t|<1.$$

In particular, $\lim_{n\to\infty} L_n(t)(1-t^2)^{\lambda-1} = 2/\pi$, $-1<t<1$.

However, relation (10) now reads as follows:

$$L_n(t) = -2\sum_{j=0}^{n-1}\frac{\lambda(1-\lambda)\{p_j(t)\}^2}{(j-1+\lambda)(j+\lambda)(j+1+\lambda)} - \frac{\lambda(1-\lambda)\{p_n(t)\}^2}{(n+\lambda-1)(n+\lambda)},$$

whence

$$\begin{aligned} L_n(t)(1-t^2)^{\lambda-1} &= \{p_n(t)\}^2(1-t^2)^{\lambda} + \{\Lambda_n(t)\}^2(1-t^2)^{\lambda-1} \\ &= \frac{2}{\pi} - \frac{\lambda(1-\lambda)\{p_n(t)\}^2(1-t^2)^{\lambda-1}}{(n+\lambda)(n+\lambda+1)} \\ &\quad + 2\sum_{j=n+1}^{\infty}\frac{\lambda(1-\lambda)\{p_j(t)\}^2(1-t^2)^{\lambda-1}}{(j-1+\lambda)(j+\lambda)(j+1+\lambda)}. \end{aligned} \tag{15}$$

The maximum of $z^2 = \{p_n(t)\}^2(1 - t^2)^\lambda$ in any subinterval of I with endpoints $t = x_i$ or $t = \pm 1$, corresponds only to $\Lambda_n(t) = 0$, so that if

$$(n + \lambda)(n + \lambda + 1)(1 - t^2) \geqq \frac{|\lambda(1 - \lambda)|}{\epsilon},$$

$$p_n(t)(1 - t^2)^\lambda(1 \pm \epsilon) < \frac{2}{\pi},$$

and, otherwise,

$$p_n(1)(1 - t^2)^\lambda$$

is uniformly bounded, by (12) and (13).

On the other hand, if $p_n(x_i) = 0$,

$$\{\Lambda(x_i)\}^2(1 - x_i^2)^{\lambda-1} = \frac{2}{\pi} + \frac{2\lambda(1 - \lambda)}{1 - x_i^2} \sum_{j=n+1}^{\infty} \frac{\{p_j(x_i)\}^2(1 - x_i^2)^\lambda}{(j - 1 + \lambda)(j + \lambda)(j + 1 + \lambda)},$$

where

$$\sum_{j=n+1}^{\infty} \frac{\{p_j(x)\}^2(1 - x^2)^\lambda}{(j - 1 + \lambda)(j + \lambda)(j + 1 + \lambda)} < \frac{1 + \epsilon_n'}{\pi(n + \lambda)^2}$$

and $\lim_{n\to\infty} \epsilon_n' = 0$, if $| \pm 1 + x | > \delta$, any fixed positive number. That is, if $| \pm 1 + x_i | > \delta$,

$$\frac{1}{W_i} = K_n(x_i, x_i) = \frac{\Lambda_n(x_i)p_n'(x_i)}{2} = \frac{n + \lambda}{2} \frac{\{\Lambda_n(x_i)\}^2}{1 - x_i^2}, \tag{16}$$

and for such zeros $x = x_i$,

$$W_i \cong \frac{\pi}{n + \lambda} (1 - x_i^2)^\lambda, \tag{17}$$

with a relative-error estimate

$$\frac{|\lambda(1 - \lambda)|}{(n + \lambda)^2(1 - x_i^2)} \tag{18}$$

for both upper and lower bounds.

If n is an odd number, and $x_i = 0$, we easily compute

$$\frac{1}{W_i} = \frac{n + \lambda}{\pi}\left\{1 + \frac{\lambda(1 - \lambda)}{2n^2} + \frac{\lambda(1 - \lambda)^2}{n^3} + \cdots\right\},$$

using Stirling's formula, for the corresponding median weight W_i. The precision of the estimate here is easily controlled; but in the general case the sums of squares seem difficult to handle with precision.

5. Spacing of Zeros. Let $v = p_n'(t)/p_n(t)$. Using (11), we find

$$(1 - t^2) \frac{dv}{dt} = (2\lambda + 1)tv - n(n + 2\lambda) - (1 - t^2)v^2.$$

Combining this with the Christoffel formulae, using induction and the result $| p_n(t) | \leqq p_n(1)$, we have

$$v \leqq \frac{p_n'(1)}{p_n(1)} = \frac{n(n + 2\lambda)}{2\lambda + 1} \quad \text{if} \quad x_n < t \leqq 1,$$

$x = x_n$ being the zero of $p_n(t)$ nearest $t = 1$. Since $(p_n(x_n) - p_n(1))/(x_n - 1) < p_n'(1)$, we have $x_n < 1 - (2\lambda + 1)/(n(n + 2\lambda))$.

In general, if we set $t = \sin \phi$, the equivalent differential relation

$$-\frac{d}{d\phi}\left\{\arctan\left[\frac{\Lambda_n(t)}{p_n(t)\sqrt{(1 - t^2)}}\right]\right\}$$
$$= n + \lambda + \frac{\lambda(1 - \lambda)}{n + \lambda}\frac{\{p_n(t)\}^2}{L_n(t)}, \qquad x_i < t < x_{i+1},$$

gives us the necessary information concerning the spacing of the zeros. We have

$$\pi = \Delta \arctan\left[\frac{\Delta_n(t)}{p_n(t)\sqrt{(1 - t^2)}}\right] = (n + \lambda)\Delta\phi_i + \frac{\lambda(1 - \lambda)}{n + \lambda}\int_{\phi_i}^{\phi_{i+1}}\frac{\{p_n(t)\}^2}{L_n(t)}\,d\phi,$$

where $x_i = \sin \phi_i$ and $\Delta\phi_i = \phi_{i+1} - \phi_i$.

6. Hermite Polynomials. From the defining formulas, we easily obtain

$$\left(\frac{d}{dt}\right)^m \{C_n^{\lambda}(t)\} = 2^n \binom{\lambda + m - 1}{m} C_{n-m}^{\lambda+m}(t)$$

by induction on m. Among other results, relations between the tesseral harmonics of Legendre,

$$P_n^{(m)}(t) = (1 - t^2)^{m/2}\left(\frac{d}{dt}\right)^m \{P_n(t)\},$$

$$P_n(t) = C_n^{\lambda}(t) \quad \text{for} \quad \lambda = \frac{1}{2},$$

and the Gegenbauer polynomials follow. Formally, the trigonometric basis is given by $\lambda = 0$ and $\lambda = 1$.

If $t^2 = s^2/2\lambda$, s being fixed, and $\lambda \to \infty$, we have

$$w(t) \to e^{-s^2/2}.$$

For the bounded n and s,

$$C_n^{\lambda}(t) \underset{n}{\rightarrow} H_n(s), \quad \text{if} \quad \lambda \to \infty,$$

the corresponding Hermite polynomial.

Let

$$\frac{d}{dt}\{H_n(t)e^{-t^2/2}\} = -H_{n+1}(t)e^{-t^2/2}, \qquad H_0(t) = 1,$$

for $n = 0, 1, 2, \cdots$. Then

$$H_n'(t) = nH_{n-1}(t),$$

by Leibnitz' rule for successive differentiation. It follows immediately that

$$H_n(x) = \sum_{j<(n+1)/2}\binom{n}{2j}(-1)^j C_j x^{n-2j}$$

for a single set of coefficients $\{C_j\}$. Since

$$tH_n(t) = nH_{n-1}(t) + H_{n+1}(t)$$

from the pair of relations given above, we have the Christoffel formulae

$$H_n(x, t) = \sum_{j=0}^{n} \frac{H_j(x)H_j(t)}{j!} = \frac{H_{n+1}(x)H_n(t) - H_n(x)H_{n+1}(t)}{n!(x - t)}$$

and

$$H_n(x, x) = \sum_{j=0}^{n} \frac{H_j^2(x)}{j!} = \frac{(n + 1)H_n^2(x) - nH_{n+1}(x)H_{n-1}(x)}{n!}$$

$$= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} H_n^2(x, t)e^{-t^2/2}\, dt.$$

To arrive at the last result, we make use of

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+t^2)/2}\, dxdt = 2\pi,$$

or the limits given above. Since

$$\sqrt{(2\pi)}e^{-x^2/2} = \int_{-\infty}^{\infty} e^{-t^2/2+ixt}t^n\, dt$$

we have

$$\sqrt{(2\pi)}H_n(x)e^{-x^2/2} = (-i)^n \int_{-\infty}^{\infty} e^{-t^2/2+ixt}t^n\, dt.$$

Let $z = e^{-t^2/4}H_n(t)$, so that

$$\frac{dz}{dt} = e^{-t^2/4}\left\{nH_{n-1}(t) - \frac{t}{2} H_n(t)\right\}$$

and

$$\frac{d^2z}{dt^2} = -z\left(n + \frac{1}{2} - \frac{t^2}{4}\right).$$

Then

$$(tz)^2 - 4\left(\frac{dz}{dt}\right)^2 = 4ne^{-t^2/2}H_{n-1}(t)H_{n+1}(t),$$

so that

$$e^{-x^2/2}\left\{\sum_{j=0}^{n-1} \frac{H_j^2(x)}{j!} + \frac{1}{2}\frac{H_n^2(x)}{n!}\right\} = \sum_{j=0}^{n-1} \frac{H_j^2(0)}{j!} + \frac{1}{2}\frac{H_n^2(0)}{n!} - \frac{1}{2}\int_0^x te^{-t^2/2}\frac{H_n^2(t)}{n!}\, dt$$

from the Christoffel formula. We do not obtain different results from the formulation of the Cesàro-one sums, in this case. We define

$$L_n(x) = \frac{1}{\sqrt{n}}\left\{\sum_{j=0}^{n-1} \frac{H_j^2(x)}{j!} + \frac{1}{2}\frac{H_n^2(x)}{n!}\right\},$$

so that

$$\lim_{n\to\infty} L_n(0) = \sqrt{\frac{2}{\pi}}.$$

Then, also,

$$L_n(x)e^{-x^2/2} = L_n(0) - \frac{1}{2\sqrt{n}}\int_0^x t\,\frac{H_n^2(t)}{n!}\,e^{-t^2/2}\,dt,$$

so here

$$\sqrt{n}L_n(t)e^{-t^2/2} = \frac{1}{n!}\left\{\left(\frac{dz}{dt}\right)^2 + \left(n + \frac{1}{2} - \frac{t^2}{4}\right)z^2\right\},$$

and

$$\lim_n L_n(x)e^{-x^2/2} = \sqrt{\frac{2}{\pi}}.$$

The formula for the weights W_i corresponding to $H_n(x_i) = 0$ becomes

$$\frac{1}{W_i} = H_n(x_i, x_i) = \sqrt{n}L_n(x_i),$$

so

$$W_i \cong \sqrt{\frac{\pi}{2n}}\,e^{-x_i^2/2},$$

with a relative error estimate

$$\frac{x_i^2}{2n - \delta} \quad \text{if} \quad x_i^2 < 2(1 + \delta).$$

If we consider the Fourier sine expansion over the interval $(a, a + \pi/k)$ between zeros $x = a, x = b = a + \pi/k$, of $H_n(x)e^{-x^2/4}$, we have

$$\int_a^b \left\{\left(\frac{dz}{dt}\right)^2 - k^2z^2\right\} dt > 0.$$

Now

$$\int_a^b \left\{\left(\frac{dz}{dt}\right)^2 - \left(n + \frac{1}{2} - \frac{t^2}{4}\right)z^2\right\} dt = 0,$$

so that

$$b - a > \frac{2\pi}{\sqrt{(4n + 2 - a^2)}}.$$

Otherwise, $dz/dt < 0$ if $t^2 \geqq 4n + 2$. We cannot have $z = 0$ there, since $z > 0$ if $t \to \infty$ for fixed n. Then

$$b^2 < 4n + 2.$$

We may point out that the estimates, for Cesàro-one and related sums, remain useful in establishing convergence properties of the expansions of functions (e.g., of bounded variation) as series of orthogonal polynomials.

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1. A. Erdélyi et al., *Higher Transcendental Functions*, Vol. II, McGraw-Hill, New York, 1953, Chapter 10, p. 174. MR **15**, 419.