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1. R. C. Bose, "On the application of the properties of Galois fields to the construction of hyper-Graeco-Latin squares," *Sankhyā*, v. 3, 1938, pp. 323–338.
2. R. C. Bose & K. R. Nair, "On complete sets of Latin squares," *Sankhyā*, v. 5, 1941, pp. 361–382. MR **4**, 33.
3. A. D. Keedwell, "On the order of projective planes with characteristic," *Rend. Mat. e Appl. Ser.* V, v. 22, 1963, pp. 498–530.
4. W. Ledermann, *Introduction to the Theory of Finite Groups*, Oliver and Boyd, Edinburgh and Interscience, New York, 1949; 2nd ed., 1953. MR **10**, 427; MR **14**, 945.
5. H. B. Mann, "The construction of orthogonal Latin squares," *Ann. Math Statist.*, v. 13, 1942, pp. 418–423. MR **4**, 184; 340.

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Approximate Integration Formulas for Ellipses

By Nancy Lee and A. H. Stroud

1. Introduction. Here we give some approximate integration formulas of the form

$$(1)\qquad I(f) \equiv \iint_{E_B} \frac{f(x, y)}{\sqrt{((x-c)^2+y^2)}\,\sqrt{((x+c)^2+y^2)}}\,dxdy \simeq \sum_{i=1}^{N} A_i f(x_i, y_i),$$

$$(2)\qquad J(f) \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(x, y) f(x, y)\,dxdy \simeq \sum_{i=1}^{N} A_i f(x_i, y_i),$$

$$w(x, y) \equiv \frac{D(x, y)\exp[-aD^2(x, y)]}{\sqrt{((x-c)^2+y^2)}\,\sqrt{((x+c)^2+y^2)}},$$

$$D(x, y) \equiv \sqrt{((x-c)^2+y^2)} + \sqrt{((x+c)^2+y^2)}.$$

Here E_B is the interior of the ellipse with foci at $(\pm c, 0)$, semiminor axis B, and semimajor axis $\sqrt{(c^2+B^2)}$. In $w(x, y)$, a is a positive constant. For both of these integrals we give integration formulas exact for all polynomials of degree $\leq k$, $k = 3, 5, 7$. These formulas are somewhat similar to formulas given by Hammer and Stroud [1] for a circle and square and were found by similar methods.

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We have not encountered integrals of the form $I(f)$ and $J(f)$ in any practical problem but we believe that approximate integration formulas for these integrals will be useful since the weight functions in them become infinite at the points $(\pm c, 0)$. As a hypothetical example, the formulas we give here might be useful in problems in chemistry or physics which involve integrals of the form

$$\iint_{E_B} G(x, y)\, dxdy,$$

where $G(x, y)$ is related to the repulsive force on a free particle p due to two fixed particles located at $(\pm c, 0)$ under the assumption that the repulsive force on p becomes infinite as p approaches one of the fixed particles.

By transforming from rectangular to confocal elliptical coordinates, formulas for the integrals $I(f)$ and $J(f)$ can also be constructed by combinations of one-dimensional formulas. In this way one can obtain formulas of degree $2h - 1$ using h^2 points for $h = 1, 2, 3, \cdots$. Formulas of this type for $I(f)$ have been discussed by Page [2] and will not be described here.

2. Description of the Formulas. We give two formulas for each of the degrees 3, 5, 7 for each of the integrals $I(f)$ and $J(f)$. The formulas are given in terms of the monomial integrals $I_{j,k}$. Here $I_{j,k}$ denotes either $I(x^j y^k)$ or $J(x^j y^k)$, $j, k = 0, 1, 2, \cdots$.

If at least one of the integers j or k is odd, then

$$I(x^j y^k) = J(x^j y^k) = 0.$$

The values of $I(x^j y^k)$, j and k both even, are given by

$$I(x^{2n} y^{2m}) = B\left(\frac{2n+1}{2}, \frac{2m+1}{2}\right) \sum_{k=0}^{n} \binom{n}{k} c^{2n-2k} g_{m+k},$$

where

$$g_n = A \sum_{k=0}^{n-1} \frac{(-1)^{2k} c^{2k} B^{2n-2k-1} P_{n,k}}{n(n-1)\cdots(n-k)} + \frac{(-1)^n c^{2n} P_{n,n} L}{n!},$$

$$A = \sqrt{(c^2 + B^2)}, \quad L = 2 \log_e \left(\frac{A+B}{c}\right), \quad P_{n,k} = \left(\frac{2n-1}{2}\right) \cdots \left(\frac{2n-2k+1}{2}\right).$$

Here $B(r, s)$ is the beta function $\Gamma(r)\Gamma(s)/\Gamma(r+s)$.

Thus $I(1) = \pi L$.

The values of $J(x^j y^k)$ for j and k both even are

$$J(x^{2n} y^{2m}) = a^{-.5} e^{-4ac} B\left(\frac{2n+1}{2}, \frac{2m+1}{2}\right) \sum_{k=0}^{n} \binom{n}{k} c^{2n-2k} (4a)^{-m-k}\, \Gamma\left(\frac{2m+2k+1}{2}\right).$$

In Table 1 we give numerical values of the constants in the formulas for $I(f)$ for $B = 1$, $c = 1$, and in Table 2 numerical values for $J(f)$ for $c = 1$, $a = 1/4$.

TABLE 1

Formulas for $I(f)$, $B = 1$, $c = 1$

Formula 3a	
$u = 1.141174027799650$	$v = 0.549798291853001$
$A_1 = 1.384458393024340$	
Formula 3b	
$u = 0.806931893571098$	$v = 0.388766100454037$
$A_1 = 1.384458393024340$	
Formula 5a	
$u = 1.092499536304484$	$A_1 = 1.191157269603166$
$\lambda = 0.627903814268268$	$\eta = 0.657867463793369$
$A_0 = 1.221592995357823$	$A_2 = 0.483481509383302$
Formula 5b	
$v = 0.803909065610874$	$A_1 = 0.398360298916025$
$\lambda = 1.012197157907448$	$\eta = 0.302513408229517$
$A_0 = 1.221592995357823$	$A_2 = 0.879879994726872$
Formula 7a	
$u_1 = 1.246009745849288$	$A_1 = 0.438093548819901$
$u_2 = 0.780689798095836$	$A_2 = 0.971706127419465$
$v_1 = 0.895112350759653$	$A_3 = 0.163122086390356$
$v_2 = 0.394559771541860$	$A_4 = 0.541777751729327$
$\lambda = 0.900546669274181$	$\eta = 0.557659666410276$
$A_5 = 0.327108635844816$	
Formula 7b	
$u_1 = 1.271634501705140$	$A_1 = 0.366749953216258$
$u_2 = 0.821020360201681$	$A_2 = 1.007804731117124$
$v_1 = 0.910429182160393$	$A_3 = 0.144612593026746$
$v_2 = 0.440808525551630$	$A_4 = 0.489797261280969$
$\lambda = 0.900546669274181$	$\eta = 0.557659666410276$
$A_0 = 0.211469951435905$	$A_5 = 0.327108635844816$

The formulas are:

Formula 3a, 4 points, degree 3:

Point	*Coefficient*
$(\pm u, 0)$	A_1
$(0, \pm v)$	A_1

$$u^2 = \frac{2I_{20}}{I_{00}}, \qquad v^2 = \frac{2I_{02}}{I_{00}}, \qquad A_1 = \frac{I_{00}}{4};$$

Formula 3b, 4 points, degree 3:

$(\pm u, \pm v)$	A_1

$$u^2 = \frac{I_{20}}{I_{00}}, \qquad v^2 = \frac{I_{02}}{I_{00}}, \qquad A_1 = \frac{I_{00}}{4};$$

TABLE 2

Formulas for $J(f)$, $c = 1$, $a = 0.25$

Formula 3a

$u = 1.224744871391589$ $v = 0.707106781186548$

$A_1 = 1.024236695866873$

Formula 3b

$u = 0.866025403784439$ $v = 0.500000000000000$

$A_1 = 1.024236695866873$

Formula 5a

$u =$	1.243163121016122	$A_1 =$	0.810017256208442
$\lambda =$	0.790569415042095	$\eta =$	1.060660171779821
$A_0 =$	1.566479652502276	$A_2 =$	0.227608154637083

Formula 5b

$v =$	1.374368541872554	$A_1 =$	0.147884442718746
$\lambda =$	1.172603939955857	$\eta =$	0.456435464587638
$A_0 =$	1.566479652502276	$A_2 =$	0.558674561381931

Formula 7a

$u_1 =$	1.917739116886260	$A_1 =$	0.054743310430066
$u_2 =$	0.934449448785687	$A_2 =$	1.179794929237429
$v_1 =$	1.854770545973768	$A_3 =$	0.023083772103826
$v_2 =$	0.617009547822385	$A_4 =$	0.594185347729922
$\lambda =$	1.244989959798873	$\eta =$	1.024695076595960

$A_5 = 0.098333016116251$

Formula 7b

$u_1 =$	1.975911856128909	$A_1 =$	0.044100110335543
$u_2 =$	0.955805485502959	$A_2 =$	1.159574673340265
$v_1 =$	1.901481972888572	$A_3 =$	0.019625337242702
$v_2 =$	0.661716141789722	$A_4 =$	0.535916870607671
$\lambda =$	1.244989959798873	$\eta =$	1.024695076595960
$A_0 =$	0.185180735950124	$A_5 =$	0.098333016116251

Formula 5a, 7 points, degree 5:

$$\begin{array}{ll} (0, 0) & A_0 \\ (\pm u, 0) & A_1 \\ (\pm\lambda, \pm\eta) & A_2 \end{array}$$

$$u^2 = \frac{I_{40}I_{04} - I_{22}^2}{I_{20}I_{04} - I_{02}I_{22}}, \qquad \lambda^2 = \frac{I_{22}}{I_{02}}, \qquad \eta^2 = \frac{I_{04}}{I_{02}},$$

$$A_1 = \frac{(I_{20}I_{04} - I_{02}I_{22})^2}{2I_{04}(I_{40}I_{04} - I_{22}^2)}, \qquad A_2 = \frac{I_{02}^2}{4I_{04}}, \qquad A_0 = I_{00} - 2A_1 - 4A_2;$$

Formula 5b, 7 points, degree 5:

$$\begin{array}{ll} (0, 0) & A_0 \\ (0, \pm v) & A_1 \\ (\pm\lambda, \pm\eta) & A_2 \end{array}$$

$$v^2 = \frac{I_{40}I_{04} - I_{22}^2}{I_{40}I_{02} - I_{20}I_{22}}, \qquad \lambda^2 = \frac{I_{40}}{I_{20}}, \qquad \eta^2 = \frac{I_{22}}{I_{20}},$$

$$A_1 = \frac{(I_{40}I_{02} - I_{20}I_{22})^2}{2I_{40}(I_{40}I_{04} - I_{22}^2)}. \qquad A_2 = \frac{I_{20}^2}{4I_{40}} \qquad A_0 = I_{00} - 2A_1 - 4A_2;$$

Formula 7a, 12 points, degree 7:

$(\pm u_1, 0)$	A_1
$(\pm u_2, 0)$	A_2
$(0, \pm v_1)$	A_3
$(0, \pm v_2)$	A_4
$(\pm\lambda, \pm\eta)$	A_5

$$\lambda^2 = \frac{I_{42}}{I_{22}}, \qquad \eta^2 = \frac{I_{24}}{I_{22}}, \qquad A_5 = \frac{I_{22}^3}{4I_{42}I_{24}}.$$

u_1^2, u_2^2 are roots of $u^4 + c_1u^2 + c_0 = 0$, where

$$c_0 = [[I_{20} - 4A_5\lambda^2][I_{60} - 4A_5\lambda^6] - [I_{40} - 4A_5\lambda^4]^2]/D_1,$$

$$c_1 = [[I_{20} - 4A_5\lambda^2][I_{40} - 4A_5\lambda^4] - k_1[I_{60} - 4A_5\lambda^6]]/D_1,$$

$$D_1 = k_1[I_{40} - 4A_5\lambda^4] - [I_{20} - 4A_5\lambda^2]^2,$$

$$A_1 = [I_{20} - 4A_5\lambda^2 - k_1u_2^2]/[2(u_1^2 - u_2^2)],$$

$$A_2 = [I_{20} - 4A_5\lambda^2 - k_1u_1^2]/[2(u_2^2 - u_1^2)],$$

$$k_1 = (2/3)[I_{00} - 4A_5].$$

v_1^2, v_2^2 are roots of $v^4 + d_1v^2 + d_0 = 0$, where

$$d_1 = [[I_{02} - 4A_5\eta^2][I_{06} - 4A_5\eta^6] - [I_{04} - 4A_5\lambda^4]^2]/D_2,$$

$$d_0 = [[I_{02} - 4A_5\eta^2][I_{04} - 4A_5\eta^4] - k_2[I_{06} - 4A_5\eta^6]]/D_2,$$

$$D_2 = k_2[I_{04} - 4A_5\eta^4] - [I_{02} - 4A_5\eta^2]^2,$$

$$A_3 = [I_{02} - 4A_5\eta^2 - k_2v_2^2]/[2(v_1^2 - v_2^2)],$$

$$A_4 = [I_{02} - 4A_5\eta^2 - k_2v_1^2]/[2(v_2^2 - v_1^2)],$$

$$k_2 = (1/3)[I_{00} - 4A_5].$$

Formula 7b, 13 points, degree 7:

$(0, 0)$	A_0
$(\pm u_1, 0)$	A_1
$(\pm u_2, 0)$	A_2
$(0, \pm v_1)$	A_3
$(0, \pm v_2)$	A_4
$(\pm\lambda, \pm\eta)$	A_5

The parameters in this formula are determined by the same equations as the parameters in Formula 7a except we use

$$k_1 = 0.65[I_{00} - 4A_5], \qquad k_2 = 0.30[I_{00} - 4A_5],$$

$$A_0 = I_{00} - 2(A_1 + A_2 + A_3 + A_4) - 4A_5.$$

3. Concluding Remarks. We can obtain formulas similar to those given here for any region (and weight function) which has the same symmetries as the ellipse. We need only substitute the appropriate monomial integrals $I_{2n,2m}$ in the expressions given.

It should also be noted that the formulas of degree 7 are not unique. Similar formulas can be obtained by choosing different values for the quantities k_1 and k_2. Various 12-point formulas are obtained by choosing k_1 and k_2 to satisfy

$$k_1 + k_2 = I_{00} - 4A_5 .$$

Although there is this free parameter in the 12-point formulas we believe it is not possible to obtain a formula of degree 7 using fewer points.

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1. P. C. Hammer & A. H. Stroud, "Numerical evaluation of multiple integrals. II," *MTAC*, v. 12, 1958, pp. 272–280. MR **21** #970.

2. L. G. Page, "Numerical integration over ellipses," Master's Thesis, Univ. of Kansas, Lawrence, Kan., 1964.

Improved Asymptotic Expansion for the Exponential Integral with Positive Argument

By Donald van Zelm Wadsworth

The usual asymptotic approximation to the exponential integral can be markedly improved, for the case with positive real argument, by adding a simple correction term as shown below. Similar results for the error function with imaginary argument (essentially the same as Dawson's function) are given in [1].*

By definition, the exponential integral with positive real argument is

$$\mathrm{Ei}(x) = -\int_{-x}^{\infty} t^{-1}e^{-t}\,dt = -\int_L t^{-1}e^{-t}\,dt - i\pi.$$

The line integral along the real axis from $-x$ to ∞ is a Cauchy principal value since there is a pole at the origin. The path of integration L goes from $-x$ to ∞, passing above the origin. Repeated partial integration of the infinite integral yields $\mathrm{Ei}\,(x) = E_n(x) + e_n(x)$, where

$$E_n(x) = x^{-1}e^x \sum_0^{n-1} m!x^{-m}$$

is the asymptotic approximation for the interval $(n - \frac{1}{2}) \leqq x < (n + \frac{1}{2})$, and

$$e_n(x) = -(-)^n n! \int_L t^{-n-1}e^{-t}\,dt - i\pi$$

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* The correction term derived in [1] could also be obtained, in a less direct fashion, from the Chebyshev polynomial expansions for Dawson's function given in [2].