TECHNICAL NOTES AND SHORT PAPERS

Inversion of the N-Dimensional Laplace Transform

By Bruce S. Berger

The inversion of the one dimensional Laplace transform in terms of series expansions of orthogonal functions has been considered by several authors [2], [3], [4], [7], [9]. Erdélyi [2] constructs expansions in terms of trigonometric functions and Legendre polynomials and suggests expansions in terms of Jacobi and ultraspherical polynomials. Papoulis [7] gives expansions in terms of the sine function and Legendre polynomials of the form

$$F(u) = g(e^{-\sigma u}) \sum_{n=0}^{\infty} b_n P_{2n}^{(\lambda)}(e^{-\sigma u}),$$

where σ is a positive parameter, $\lambda = \frac{1}{2}$ or 1, and g depends on the choice of λ. These formulae are closely related to the corresponding expansions of Erdélyi. In the following, an expansion is constructed for the inversion of the n-dimensional Laplace transform in terms of the ultraspherical polynomials in which the coefficients are computed recursively. Numerical results indicate that the formula developed here converges more rapidly than Papoulis' [7] trigonometric formula.

The 2-dimensional Laplace transform is defined by

$$f(p_1, p_2) = \int_0^{\infty} \int_0^{\infty} \exp(-p_1 u_1 - p_2 u_2) F(u_1, u_2)\, du_1\, du_2 . \tag{1}$$

Conditions on $F(u_1, u_2)$ which insure convergence of the integrals for the two dimensional case are discussed in [8]. Let $\sigma_i > 0$ and consider the change of variable given by

$$u_i = -\frac{1}{\sigma_i} \ln(1 - x_i^2). \tag{2}$$

Let

$$F(x_1, x_2) \equiv F\left\{\left[-\frac{1}{\sigma_1} \ln(1 - x_1^2)\right], \quad \left[-\frac{1}{\sigma_2} \ln(1 - x_2^2)\right]\right\}, \tag{3}$$

$$p_i = (m_i + \lambda_i + \tfrac{1}{2})\sigma_i \qquad \text{where} \quad m_i = 0, 1, 2, \cdots . \tag{4}$$

Substituting into Eq. (1) gives

$$\begin{aligned} &f[(m_1 + \lambda_1 + \tfrac{1}{2})\sigma_1, \quad (m_2 + \lambda_2 + \tfrac{1}{2})\sigma_2] \\ &\qquad = \frac{4}{\sigma_1\, \sigma_2} \int_0^1 \int_0^1 \prod_{i=1}^{2} x_i (1 - x_i^2)^{m_i} (1 - x_i^2)^{\lambda_i - 1/2} F(x_1, x_2)\, dx_1\, dx_2 . \end{aligned} \tag{5}$$

Assume that $F(x_1, x_2)$ may be expanded in a double series of odd ultraspherical

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polynomials. Then

$$F(x_1, x_2) = \sum_{\beta_1=0}^{\infty} \sum_{\beta_2=0}^{\infty} C_{\beta_1\beta_2} P_{2\beta_1+1}^{(\lambda_1)} P_{2\beta_2+1}^{(\lambda_2)} . \tag{6}$$

Substituting Eq. (6) into Eq. (5) changing the order of summation and integration, utilizing the orthogonality of the ultraspherical polynomials and noting that

$$x(1 - x^2)^m = \sum_{n=0}^{m} \frac{P_{2n+1}^{(\lambda)}(x)}{h_{2n+1}^{(\lambda)}} A(\lambda, m, 2n + 1), \tag{7}$$

where

$$h_{2n+1}^{(\lambda)} = 2^{1-2\lambda} \pi \{\Gamma(\lambda)\}^{-2} \frac{\Gamma(2n + 2\lambda + 1)}{(2n + \lambda + 1)\Gamma(2n + 2)} \tag{8}$$

and

$$A(\lambda, m, 2n + 1) = \frac{(-1)^n \Gamma(\frac{1}{2}) \Gamma(m + \lambda + \frac{1}{2}) \Gamma(n + \lambda + 1) m!}{\Gamma(\lambda)\Gamma(n + m + \lambda + 2) n! (m - n)!} \tag{9}$$

gives

$$\begin{aligned} f[(m_1 + \lambda_1 + \tfrac{1}{2})\sigma_1, (m_2 + \lambda_2 + \tfrac{1}{2})\sigma_2] \\ = \frac{1}{\sigma_1 \sigma_2} \sum_{\beta_1=0}^{m_1} \sum_{\beta_2=0}^{m_1} C_{\beta_1\beta_2} A(\lambda_1, m_1, 2\beta_1 + 1) A(\lambda_2, m_2, 2\beta_2 + 1). \end{aligned} \tag{10}$$

The coefficients $C_{\beta_1\beta_2}$ may be computed recursively. The form in which Eq. (9) appears was indicated by the referee. See [6, 16.3, Eq. 4] and [5, 4.4, Eq. 6]. These results are readily generalized to the case of n-variables.

Consider the following numerical examples.

$$\begin{aligned} F(u_1, u_2) &= 1 - J_0(u_1 u_2)^{1/2} \quad \text{for} \quad 0 \leqq u_1 \leqq a, 0 \leqq u_2 \leqq a \\ &= -J_0(u_1 u_2)^{1/2} \quad \text{for} \quad u_1 > a, u_2 > a. \end{aligned}$$

Then

$$\begin{aligned} f(p_1, p_2) = \frac{1}{p_1 p_2} [1 - \exp[-p_1 a] - \exp[-p_2 a] + \exp[-(p_1 + p_2)a]] \\ - \frac{1}{p_1 p_2 + \frac{1}{4}} . \end{aligned} \tag{11}$$

The results of the application of Eq. (10) to Eq. (11) are given in Table 1 for the cases $F(0.5, u_2)$ and $F(12.5, u_2)$ with $\sigma_1 = 0.114$, $\sigma_2 = 0.114$, $a = 18.0711$, $0 \leqq m_1 \leqq 8$, $0 \leqq m_2 \leqq 8$, $\lambda_1 = 5$, $\lambda_2 = 5$.

For comparison with Papoulis' trigonometric series consider the function:

$$\begin{aligned} F(u_1) &= \sin u_1 \quad \text{for} \quad 0 \leqq u_1 \leqq 10\pi \\ &= 0 \qquad \text{for} \quad 10\pi < u_1, \\ f(p_1) &= \frac{1}{p_1^2 + 1} (1 - \exp[-10\pi p_1]). \end{aligned} \tag{12}$$

TABLE 1
Two dimensional case

u_2	$F(u_1, u_2)$			
	$u_1 = 0.5$	Exact	$u_1 = 12.5$	Exact
1.0	.1217	.1211	1.409	1.385
2.0	.2352	.2348	1.178	1.178
3.0	.3423	.3413	.8405	.8167
4.0	.4416	.4409	.7157	.7003
5.0	.5350	.5339	.8004	.8069
6.0	.6224	.6206	.9989	1.002
7.0	.7027	.7012	1.187	1.168
8.0	.7764	.7761	1.278	1.246
9.0	.8457	.8455	1.249	1.227
10.0	.9126	.9096	1.132	1.137

TABLE 2
One dimensional case

u_1/π	$F(u_1)$			
	$0 \leqq m_1 \leqq 10$	$0 \leqq m_1 \leqq 15$	Papoulis	Exact
0.25	0.7073	0.7067	0.6434	0.7071
0.50	0.9985	0.9996	1.0264	1.0000
0.75	0.7094	0.7069	0.7108	0.7071
1.00	−0.0029	0.0007	0.0698	0.0000
1.25	−0.7087	−0.7079	−0.8933	−0.7071
1.50	−0.9869	−0.9999	−0.9667	−1.0000
1.75	−0.7061	−0.7046	−0.4092	−0.7071
2.00	−0.0456	−0.0033	0.1261	0.0000

The results of the application of Eq. (10) to Eq. (12) are given in Table 2 for the cases $0 \leqq m_1 \leqq 10$, $0 \leqq m_1 \leqq 15$ with $\sigma_1 = 0.2546$ and $\lambda_1 = 5$. The third column of Table 2 contains the values given by Papoulis' trigonometric expansion retaining 11 terms with $\sigma = 0.2546$.

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On the First Positive Zero of $P_{\nu-1/2}^{(-m)}(\cos\theta)$, Considered as a Function of ν

By R. D. Low

1. Introduction. Several years ago Pal [1], [2] published two papers in which he considered the roots of the equations $P_\nu^{(m)}(\mu) = 0$ and $(d/d\mu)\, P_\nu^{(m)}(\mu) = 0$ regarded as equations in ν.† In these equations m is an integer and $\mu = \cos\theta$. Among the roots which Pal computed and tabulated are those of the equation $P_\nu^{(2)}(\cos\theta) = 0$ for $\theta = \pi/12$, $\pi/6$, and $\pi/4$, and he lists as the first root in each case: 4.77, 2.26, and 1.52. In view of the fact that $P_\nu^{(2)}(\cos\theta) = \nu(\nu+2)(\nu^2-1)\cdot P_\nu^{(-2)}(\cos\theta)$, it must be assumed that the numbers just mentioned are respectively the first positive roots of the equation $P_\nu^{(-2)}(\cos\theta) = 0$ for $\theta = \pi/12$, $\pi/6$, and $\pi/4$, since the equation $P_\nu^{(2)}(\cos\theta) = 0$ has the roots -2, -1, 0, and 1 regardless of the value of θ. In any event it will be seen that the numbers 4.77, 2.26, and 1.52 are not roots at all in as much as they are *less than* the first element of a sequence of lower bounds to be exhibited below.

2. A Sequence of Lower Bounds. We restrict our attention to the function $P_{\nu-1/2}^{(-m)}(\cos\theta)$ in which $m = 1, 2, 3, \cdots$ because of the identity [3]

$$P_{\nu-1/2}^{(m)}(\cos\theta) = (-1)^m(\nu^2-\tfrac{1}{4})(\nu^2-\tfrac{9}{4})\cdots[\nu^2-(2m-1)^2/4]P_{\nu-1/2}^{(-m)}(\cos\theta),$$

which shows that the zeros of $P_{\nu-1/2}^{(m)}(\cos\theta)$ consist of $\pm\frac{1}{2}, \pm\frac{3}{2}, \cdots, \pm(m-\frac{1}{2})$, together with those of $P_{\nu-1/2}^{(-m)}(\cos\theta)$. It is known that $P_{\nu-1/2}^{(-m)}(\cos\theta)$, considered as a function of the complex variable ν, has infinitely many zeros which are all real and simple. Moreover, since $P_{\nu-1/2}^{(-m)}(\cos\theta)$ is an even function of ν which does not vanish for $\nu = 0$, only its positive zeros need be considered. Hence the purpose of the present investigation is to establish a sequence of lower bounds for the first positive zero of $P_{\nu-1/2}^{(-m)}(\cos\theta)$. In addition to the properties mentioned already, it is also known that $P_{\nu-1/2}^{(-m)}(\cos\theta)$ is an entire function of order unity. Hence if $\nu_{n,m}(\theta)$ denotes its nth positive zero, $P_{\nu-1/2}^{(-m)}(\cos\theta)$ can be expressed as an infinite product of the form

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† A trivial change in notation has been made; Pal uses n instead of ν.