Rational Approximations to the Incomplete Elliptic Integrals of the First and Second Kinds*

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In this note we derive rational approximations (in Eqs. (20) and (21) below) to the integrals

$$F(\varphi, k) = \int_0^\varphi (1 - k^2 \sin^2 t)^{-1/2} dt, \tag{1}$$

and

$$E(\varphi, k) = \int_0^\varphi (1 - k^2 \sin^2 t)^{1/2} dt, \tag{2}$$

where k^2 is real and $0 < \varphi < \pi/2$, by obtaining the main diagonal Padé approximations to closely related functions. It is sufficient to consider the case $0 < k^2 < 1$, for if $k^2 > 1$,

$$\begin{gathered} F(\varphi, k) = k_1 F(\beta_1, k_1) \quad \text{and} \quad E(\varphi, k) = k_1[E(\beta_1, k_1) + (1 - k^2)^2 F(\beta_1, k_1)], \\ k_1 = 1/k \quad \text{and} \quad \beta_1 = \arcsin (k \sin \varphi), \end{gathered} \tag{3}$$

while if $k^2 < 0$,

$$\begin{gathered} F(\varphi, k) = (1 - k_2^2)^{1/2} F(\beta_2, k_2) \quad \text{and} \\ E(\varphi, k) = (1 - k_2^2)^{-1/2} \left[E(\beta_2, k_2) - \frac{k_2^2 \sin \beta_2 \cos \beta_2}{(1 - k_2^2 \sin^2 \beta_2)^{1/2}} \right], \\ k_2 = |k| (1 - k^2)^{-1/2} \quad \text{and} \quad \beta_2 = \arcsin \left[\left(\frac{1 - k^2}{1 - k^2 \sin^2 \varphi} \right)^{1/2} \sin \varphi \right]. \end{gathered} \tag{4}$$

Define $m = k^2$ and

$$\begin{gathered} a = \left[\frac{(2 - m)^2}{1 + m} \right]^{1/3} > 0, \qquad b = \left[\frac{(1 - 2m)^3}{(m - 2)(m + 1)} \right]^{1/3}, \\ c = \left[\frac{(1 + m)^2}{m - 2} \right]^{1/3} < 0, \qquad x = c + \frac{a - c}{\sin^2 \varphi}, \\ h = a \left[c + \frac{b(2m - 1)}{m - 2} \right] < 0, \quad g = 2m - 1, \qquad s = 2 \left[\frac{2 - m}{3a} \right]^{1/2}, \\ r(x) = x^3 + hx + g, \qquad v(x) = \frac{(x - c)^3 (x - a)}{x - b}, \\ I_1(x) = \int_x^\infty [r(t)]^{-1/2} dt \quad \text{and} \quad I_2(x) = \int_x^\infty [v(t)]^{-1/2} dt. \end{gathered} \tag{5}$$

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Then $a > b > c$ are the real roots of $r(z) = 0$ and it follows from [1] that

$$F(\varphi, k) = s^{-1}I_1(x) \quad \text{and} \quad E(\varphi, k) = s^{-1}I_2(x) . \tag{6}$$

Set

$$G_1(x) = [r(x)]^{1/2}I_1(x) , \qquad G_2(x) = \frac{[v(x)]^{1/2}}{2x} I_2(x) . \tag{7}$$

Then $G_l(x)$ $(l = 1, 2)$ satisfies the differential equation

$$r(x)\gamma_l(x)G_l'(x) - \delta_l(x)G_l(x) + r(x) = 0 , \tag{8}$$

where

$$\gamma_1(x) = 1 , \qquad \gamma_2(x) = 2x , \qquad \delta_1(x) = \tfrac{1}{2}(3x^2 + h) ,$$
$$\delta_2(x) = x^3 - 2(a + 2b)x^2 + (ab - bc - 3ac)x + 2abc .$$

For convenience, we make the transformations

$$z = 1/x , \qquad G_1(z) = x^{-1}[2 + x^2H_1(x)] , \qquad G_2(z) = H_2(z) . \tag{9}$$

Then (8) becomes

$$\eta_l(x)H_l'(x) + \rho_l(x)H_l(x) + \xi_l(x) = 0 , \qquad l = 1, 2 , \tag{10}$$

where

$$\eta_1(x) = x(1 + hx^2 + gx^3) , \qquad \eta_2(x) = 2\eta_1(x) , \qquad \rho_1(x) = \frac{5}{2} + \frac{3h}{2}x^2 + gx^3 ,$$
$$\rho_2(x) = 1 - 2(a + 2b)x + (ab - bc - 3ac)x^2 - 2gx^3 , \qquad \xi_1(x) = -2h - 3gx ,$$
$$\xi_2(x) = -1 - hx^2 - gx^3 , \qquad H_1(0) = \frac{4h}{5} \quad \text{and} \quad H_2(0) = 1 .$$

Main diagonal Padé approximations for the solution to (10) are readily computed by using the results of [2]. For completeness we list the recurrence relations which determine the main diagonal Padé approximations to $H_l(x)$, $l = 1, 2$. In the notation of [2], we have: for $l = 1$;

$$\begin{aligned} &y(x) = H_1(x) , \\ &y_0 = y(0) = 4h/5 , \\ &p_0 = p_2 = 0 , \qquad p_1 = 1 , \qquad p_3 = h , \qquad p_4 = g , \\ &q_0 = 5/2 , \qquad q_1 = 0 , \qquad q_2 = 3h/2 , \qquad q_3 = g , \\ &s_0 = -2h , \qquad s_1 = -3g , \qquad s_2 = s_3 = 0 , \end{aligned} \tag{11}$$

and for $l = 2$, $y(x) = H_2(x)$,

$$\begin{aligned} &y_0 = y(0) = 1 , \\ &p_0 = p_2 = 0 , \qquad p_1 = 2 , \qquad p_3 = 2h , \qquad p_4 = 2g , \\ &q_0 = 1 , \qquad q_1 = -2(a + 2b) , \qquad q_2 = ab - bc - 3ac , \qquad q_3 = -2g , \\ &s_0 = -1 , \qquad s_1 = 0 , \qquad s_2 = -h , \qquad s_3 = -g . \end{aligned} \tag{12}$$

Let

$$y_n = \frac{A_n}{B_n}, \qquad A_n = \sum_{k=0}^{n} a_{n,k}x^k, \qquad B_n = \sum_{k=0}^{n} b_{n,k}x^k \tag{13}$$

be the nth-order main diagonal Padé approximations to $y(x)$. Then A_n and B_n satisfy

$$A_n = (1 + \beta_n x)A_{n-1} + \alpha_n x^2 A_{n-2}. \tag{14}$$

The equations which determine α_n and β_n are

$$\alpha_n = -\tau_{n-1,1}\left[(-1)^n\alpha_{n-1,1}p_1 + \alpha_{n-1,2}u_1 + 2\sum_{j=3}^{n}\alpha_{n-1,j}\tau_{j-2,1}\right]^{-1}, \tag{15}$$

and

$$\begin{aligned}\beta_n = &-\left[\tau_{n-1,2} + \alpha_{n,2}u_2 + 2\sum_{j=3}^{n}\alpha_{n,j}(\tau_{j-2,2} + \beta_{j-1}\tau_{j-2,1})\right]\\ &\times\left[2\tau_{n-1,1} + \alpha_{n,2}u_1 + 2\sum_{j=3}^{n}\alpha_{n,j}\tau_{j-2,1}\right]^{-1}, \qquad n = 2, 3, 4\cdots,\end{aligned}$$

where

$$\begin{aligned}\tau_{n,k} = &\ \tau_{n-1,k+2} + 2\beta_n\tau_{n-1,k+1} + \alpha_n^2\tau_{n-2,k} + \beta_n^2\tau_{n-1,k} + (-1)^n\alpha_{n,1}p_{k+2}\\ &+ \alpha_{n,2}u_{k+2} + \alpha_{n,2}\beta_n u_{k+1}\\ &+ 2\sum_{j=3}^{n}\alpha_{n,j}[\tau_{j-2,k+2} + (\beta_n + \beta_{j-1})\tau_{j-2,k+1} + \beta_n\beta_{j-1}\tau_{j-2,k}],\\ &\qquad n = 2, 3, 4, \cdots, k = 1, 2, 3,\\ u_k = &\ 2y_0q_k + 2s_k + (a_{1,1} + b_{1,1}y_0)q_{k-1} + 2b_{1,1}s_{k-1} \qquad k = 1, 2, 3, 4,\\ \alpha_{k,j} = &\ \alpha_k\alpha_{k-1}\cdots\alpha_j,\ \alpha_{k,k} = \alpha_k,\ \alpha_{k-1,k} = 1 \quad\text{and}\quad \alpha_{k,j} = 0, \qquad k < j - 1.\end{aligned} \tag{16}$$

The starting values for computation are

$$\begin{aligned}\tau_{0,k} &= y_0q_k + s_k,\\ \tau_{1,k} &= -\alpha_1p_{k+2} + y_0q_{k+2} + s_{k+2} + (a_{1,1} + b_{1,1}y_0)q_{k+1}\\ &\quad + 2b_{1,1}s_{k+1} + a_{1,1}b_{1,1}q_k + b_{1,1}^2s_k, \qquad k = 1, 2, 3\end{aligned}$$

for $l = 1$,

$$\begin{aligned}\alpha_1 &= -6g/7, \qquad \beta_1 = b_{1,1}\\ a_{1,1} &= 6g/7 + \frac{56h^3}{225g}, \qquad b_{1,1} = \frac{14h^2}{45g};\end{aligned} \tag{17}$$

for $l = 2$,

$$\begin{aligned}\alpha_1 &= -2/3(a + 2b), \qquad \beta_1 = b_{1,1},\\ a_{1,1} &= \frac{4a^2 + 16b^2 + 25ab - 9bc - 27ac - 9h}{30(a + 2b)},\\ b_{1,1} &= \frac{-4a^2 - 16b^2 - 19ab + 3bc + 9ac - 3h}{10(a + 2b)}.\end{aligned} \tag{18}$$

In either case, we have

$$A_0 = y_0\,, \qquad A_1 = y_0 + a_{1,1}x\,, \qquad B_0 = 1\,, \qquad B_1 = 1 + b_{1,1}x\,. \tag{19}$$

Thus, rational approximations to the incomplete elliptic integrals of the first and second kind respectively are

$$F_n(\varphi, k) = \frac{[r(x)]^{1/2}}{s}\left[2x + \frac{A_n(1/x)}{xB_n(1/x)}\right] \tag{20}$$

and

$$E_n(\varphi, k) = \frac{2x[v(x)]^{-1/2}}{s}\,\frac{A_n(1/x)}{B_n(1/x)}\,. \tag{21}$$

In the special case, $k^2 = m = \frac{1}{2}$, the approximation (20) does not apply. However, since $g = 0$ in this case, (20) becomes

$$t(1 + ht)H_1'(t) + \tfrac{1}{4}(5 + 3ht)H_1(t) - h = 0\,, \qquad H_1(0) = 4h/5\,, \qquad t = x^2\,, \tag{22}$$

and $H_1(t) = (4h/5)\ {}_2F_1(1, 3/4; 9/4; -ht)$ is the solution to (22). Padé approximations to this hypergeometric function together with an error analysis are available in [3].

Numerical results indicate rapid convergence of the approximations (20) and (21). These approximations are evidently insensitive to changes in k^2 and are very powerful for $\varphi < \pi/3$. They weaken as φ approaches $\pi/2$; however, the Landen transformations

$$F(\varphi, k) = \frac{2}{1 + k}F(\varphi_1, k_1)\,,$$

$$E(\varphi, k) = (1 + k)E(\varphi_1, k_1) + (1 - k)F(\varphi_1, k_1) - k\sin\varphi\,,$$

where

$$k_1 = 2\sqrt{k}/(1 + k) \quad \text{and} \quad \varphi_1 = \tfrac{1}{2}\varphi + \tfrac{1}{2}\arcsin(k\sin\varphi)\,, \tag{23}$$

should reduce φ to the desirable range in all but the extreme cases. For example, if $k = \frac{1}{2}$ and $\varphi = \pi/2$ we have

$$F(\tfrac{1}{2}, \pi/2) = \tfrac{4}{3}F(2\sqrt{2}/3, \pi/3)\,. \tag{24}$$

The approximations $\frac{4}{3}F_n(2\sqrt{2}/3, \pi/3)$ are listed in Table I.

TABLE I

n	$\frac{4}{3}F_n$	
4	1.68579	32446
6	1.68575	05579
8	1.68575	03557
10	1.68575	03548
12	1.68575	03548

The true value is 1.68575 03548.

We present in Table II a tabulation of $\epsilon_n = |F(\varphi, k) - F_n(\varphi, k)|$ for a number of values of n, φ and k. The behavior of the error involved in approximating $E(\varphi, k)$

TABLE II

$k^2 = .25$

$\varphi \backslash n$	4	6	8	10	20
60°	1.89 (−5)*	4.1 (−7)	4.0 (−8)		
80°	7.33 (−2)	2.11 (−2)	7.11 (−3)	3.84 (−3)	8.1 (−7)

by $E_n(\varphi, k)$ is almost identical and so is omitted. In both tables $\epsilon_n < 1.0 \times 10^{-8}$ for $\varphi \leq 30°$ and $n \geq 4$ (k arbitrary) so that these values are not listed. No entry in the table signifies an error less than 1.0×10^{-8}.

$k^2 = .75$

$\varphi \backslash n$	4	6	8	10	20
60°	1.92 (−3)	4.7 (−7)			
80°	1.51 (−1)	2.55 (−2)	5.44 (−3)	1.13 (−3)	8.0 (−7)

* The number in parentheses indicates the power of ten by which the tabular entry is to be multiplied.

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