Consecutive Primes in Arithmetic Progression

By L. J. Lander and T. R. Parkin

A. Schinzel and W. Sierpiński [1] conjectured that there exist arbitrary long arithmetic progressions formed of consecutive prime numbers. Sierpiński stated in [2] that a progression of five consecutive primes had not yet been found. A direct computer search showed that the first such progression has the common difference $d = 30$ and begins with the prime 9,843,019. The first progression of six consecutive primes begins with 121,174,811 and also has $d = 30$. Up to the limit 3×10^8 there are 25 other progressions of five consecutive primes, all with $d = 30$; there are no other progressions of six consecutive primes.

The referee points out that recently a much larger quintuplet, beginning with 10000024493, and again having $d = 30$, was recorded [3], but without reference to Sierpiński's remark. The smaller set that we found, and the single sextuplet, may still be worth recording.

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Convergence of Successive Substitution Starting Procedures

By A. C. R. Newbery

The method of successive substitutions (also known as Picard's method) has been proposed [1], [2] as a means of initialising the numerical solution of the differential equation $x' = f(x, t)$. The method is capable of advancing the solution k steps at an average cost of k function-evaluations per step with a truncation error of order $O(h^{k+2})$. This makes it potentially one of the most efficient methods available for the purpose, and so it seems appropriate to study its numerical convergence properties. The method is based on k formulas of the form $x_r = x_0 + hL_r(x_0', x', \cdots, x_k')$, $r = 1, 2, \cdots, k$ where, L_r denotes a linear combination with known constant coefficients. The required coefficients are implicit in the corrector matrices published in [3]. For a given k, the coefficients in L_r are the entries in the rth column of the kth corrector matrix. For example with $k = 2$ we would obtain the formulas:

$$x_1 = x_0 + (h/24)(10x_0' + 16x_1' - 2x_2') \,, \quad x_2 = x_0 + (h/24)(8x_0' + 32x_1' + 8x_2') \,.$$

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We may use the modified Euler method (Euler with a single uniterated trapezoidal correction) to obtain first approximations to the x_i, $i = 1, \cdots, k$. These approximations are subject to truncation error $O(h^3)$ and they cost one derivative evaluation per step. Thereafter, each application of the successive substitution formulas augments the order of the truncation error by one, and it costs one further evaluation per step; consequently, $k - 1$ applications will yield a truncation error of $O(h^{k+2})$ even though the iterations may not have closed. If subsequent iterations are performed, the order of truncation error will not be augmented beyond $O(h^{k+2})$.

For the study of convergence we define $x_r^{(p)}$ to be the pth iterate on x_r. We may conveniently take $x_r^{(0)}$ to be that value which is defined by the Euler or modified Euler process. Thereafter, the iterative cycle will be defined by

$$(1) \qquad x_r^{(p+1)} = x_0 + h\beta_r f(x_0, t_0) + h \sum_{j=1}^{k} b_{rj} f(x_j^{(\tilde{p})}, t_j), \qquad r = 1, \cdots, k.$$

In this equation β_r is the rth element in the first row of the kth corrector matrix in [3]; b_{rj} is the (r, j)th element of a matrix B_k obtained from the corrector matrix by deletion of the first and last rows and transposition of the remaining k-square matrix. The superscript $\tilde{p}$ in (1) may be defined in various ways, but only two will be considered here:

(A) $p = p$, or

(B) if $j \geq r$ then $\tilde{p} = p$; else $\tilde{p} = p + 1$.

These two alternative definitions of $\tilde{p}$ give rise to iterations analogous to Gauss-Jacobi and Gauss-Seidel respectively, and we now study their convergence properties.

We assume that the partial derivative $f_x(x_i, t_i)$ is locally constant; we write $hf_x = q$ and $x_r^{(p+1)} - x_r^{(p)} = \delta_r^{(p)}$. If we then write Eq. (1) for two consecutive p-values and subtract, we obtain

$$(2) \qquad \delta_r^{(p)} = h \sum_{j=1}^{k} b_{rj}(f(x_j^{(\tilde{p})}, t_j) - f(x_j^{(\tilde{p}-1)}, t_j)) = q \sum_{j=1}^{k} b_{rj}\delta_j^{(\tilde{p}-1)}.$$

Let $d^{(p)}$ be a k-dimensional column vector whose rth component is $\delta_r^{(p)}$; then if the alternative (A) is chosen, (2) reduces to $d^{(p)} = qB_k d^{(p-1)}$. The process will therefore converge, provided that $|hf_x|$ is smaller than the reciprocal of the spectral radius of B_k. The critical value is indicated in the table below. If the alternative (B) is chosen, then $d^{(p)} = q(I - qL)^{-1}(U + D)d^{(p-1)}$, where $B_k = L + D + U$. The spectral radius of the matrix $G(q) = q(I - qL)^{-1}(U + D)$ is a function of q. By means of a systematic search procedure one may approximate the critical values, i.e., the smallest positive and largest negative values of q such that the spectral radius of $G(q)$ is equal to one. These critical values have been computed numerically and are tabulated below to three significant figures. Since one ordinarily operates well within the radius of convergence, we have also tabulated the range of q-values such that the convergence factor (i.e., the spectral radius of $G(q)$) does not exceed .1. This has been done only for method (B); to obtain the corresponding figure for method (A) one would simply divide the first-column entry by 10. It will be noted that the Jacobi-like method (A) has generally superior convergence properties to the Seidel-like method (B), and that the superiority becomes more marked with increasing k. In the case $k = 2$ the problem is (just) simple enough to work by hand. The exact values for the first-row tabular entries are

$\sqrt{3}$, $(3/2)(-3+\sqrt{5})$, $3\sqrt{2}/2$, $(3/190)(-15+(35)^{1/2})$, $(15-(15)^{1/2})/70$.

The over-all evidence suggests very strongly that in most practical situations method (A) is preferable to method (B).

TABLE

k	*Method* (A). $\lvert q\rvert$*-bound for convergence*	*Method* (B). *q-range for convergence*	*Method* (B). *q-range such that convergence factor* $\leq \cdot 1$
2	1.73	(−1.15 , 2.12)	(− .143 , .159)
3	1.43	(− .860 , 1.43)	(− .119 , .135)
4	1.33	(− .738 , 1.64)	(− .106 , .117)
5	1.21	(− .711 , 1.21)	(− .0994 , .102)
6	1.16	(− .687 , 1.50)	(− .0926 , .0866)
7	1.10	(− .576 , .813)	(− .0769 , .0686)
8	1.07	(− .493 , .475)	(− .0629 , .0517)

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A Polynomial Approximation Converging in a Lens-Shaped Region[1]

By Jay A. Leavitt

The Taylor series expansion of $y = 1/(1+x^2)$ about $x = 0$ has a radius of convergence $R = 1$, while the function itself is analytic for all real values of x. In order to represent $1/(1+x^2)$ by a Taylor series for values of x outside the interval $(-1, 1)$, it is necessary to expand about a point of nonsymmetry.

In practice, given an analytic function $f(x)$, one uses only its truncated Taylor series $T_n(x)$. The expansion of such a truncated series of order n, i.e. $T_n(x)$, about the point b provides a polynomial, say $V_n(z)$ where $z = x - b$, which is of order n. But $V_n(z)$ converges to $f(x)$ only in the original circle of convergence of the $T_n(x)$. Nevertheless, this property is used to produce a sequence of even polynomials, $U_n(x)$, which have real coefficients and which converge to $y = 1/(1+x^2)$ in a lens-shaped region that includes an extended interval of the real axis.

Let us expand $1/(x+i)$ about $x = (\lambda - 1)i$ and $1/(x-i)$ about $x = -(\lambda - 1)i$ and truncate; $\lambda \geq 1$ real.

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