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Midpoint Quadrature Formulas

By Seymour Haber

A family of quadrature formulas for the interval (0, 1) can be constructed in the following manner: For any positive integer n, we partition (0, 1) into subintervals $I_1, I_2, \cdots, I_n$ (I_1 being the leftmost, I_2 adjacent to it, etc.) of lengths $a_1, a_2, \cdots, a_n$, respectively. Now let x_k be the midpoint of I_k, for $k = 1, \cdots, n$, and take

$$a_1 f(x_1) + \cdots + a_n f(x_n) \tag{1}$$

as the approximation to $\int_0^1 f(x)dx$. The simplest of these rules is the "Euler's" or "midpoint" rule

$$\int_0^1 f(x)dx \approx f(\tfrac{1}{2}) .$$

We will refer to the members of this family as "midpoint quadrature formulas" and determine their properties. We first find their "degrees of precision"—that is, for any formula, the highest integer p such that the formula is exact for all polynomials of degree p or lower.

Theorem 1. *The degree of precision of a midpoint quadrature formula is* 1.

Proof. The formula is exact for constants, since necessarily $a_1 + a_2 + \cdots + a_n = 1$. To check the exactness of the formula for $f(x) = x$, we first note that

$$x_1 = \frac{a_1}{2}, x_2 = a_1 + \frac{a_2}{2}, \cdots, x_n = a_1 + \cdots + a_{n-1} + \frac{a_n}{2} . \tag{2}$$

So for the integral $\int_0^1 x\, dx$, (1) gives us

$$a_1(a_1/2) + a_2(a_1 + a_2/2) + \cdots + a_n(a_1 + \cdots + a_{n-1} + a_n/2) .$$

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But this is just

$$\tfrac{1}{2}(a_1^2 + a_2^2 + \cdots + a_n^2 + 2a_1a_2 + 2a_1a_3 + \cdots + 2a_{n-1}a_n),$$

or $\frac{1}{2}(a_1 + \cdots + a_n)^2$, which is $\frac{1}{2}$. Thus the degree of precision is at least one. To show it is no greater, we calculate error in integrating $x^2/2$ by the rule:

$$\int_0^1 \frac{x^2}{2}dx - \sum_{i=1}^n a_i \frac{x_i^2}{2} = \frac{1}{6} - \frac{1}{2}\sum_{i=1}^n a_i\left(a_1 + a_2 + \cdots + a_{i-1} + \frac{a_i}{2}\right)^2.$$

Multiplying out and collecting terms in the last sum, we obtain:

$$\sum_i a_i x_i^2 = \frac{1}{4}\sum_i a_i^3 + \sum_{i\neq j} a_i a_j^2 + 2\sum_{i\neq j\neq k} a_i a_j a_k,$$

where the indices of summation run from 1 to n.

Now

$$1 = (a_1 + \cdots + a_n)^3 = \sum_i a_i^3 + 3\sum_{i\neq j} a_i a_j^2 + 6\sum_{i\neq j\neq k} a_i a_j a_k,$$

so that

$$\frac{1}{3} - \sum_i a_i x_i^2 = \frac{1}{12}(a_1^3 + \cdots + a_n^3).$$

It follows that

$$\int_0^1 \frac{x^2}{2}dx - \sum_i a_i \frac{x_i^2}{2} = \frac{1}{24}(a_1^3 + \cdots + a_n^3) > 0, \tag{3}$$

which proves the theorem.

THEOREM 2. *The error of a midpoint quadrature formula, for an integrand with continuous second derivative, is given by*

$$\int_0^1 f(x)dx - \sum_i a_i f(x_i) = \frac{1}{24}(a_1^3 + \cdots + a_n^3)f''(\xi) \tag{4}$$

for some ξ *in* (0, 1).

Proof. By a general remainder theorem (see, e.g., [1]) the error may be written in the form

$$\int_0^1 f''(t)K(t)dt \tag{5}$$

where

$$K(t) = \frac{(1-t)^2}{2} - \sum_{x_i > t} a_i(x_i - t).$$

To derive (4) from (5) it is sufficient to show that $K(t)$ does not change sign in (0, 1); for then we may write

$$\int_0^1 f''(t)K(t)dt = f''(\xi)\int_0^1 K(t)dt,$$

and, taking $f(x) = x^2/2$, we see from (3) that

$$\int_0^1 K(t)dt = \frac{1}{24}(a_1^3 + \cdots + a_n^3) .$$

We shall show that, in fact, $K(t) \geqq 0$ for $t \in [0, 1]$.

For t between x_k and x_{k+1},

$$\begin{aligned} 2K(t) &= (1-t)^2 - 2\sum_{i=k+1}^{n} a_i(x_i - t) \\ &= (1-t)^2 - 2\sum_{i=k+1}^{n} a_i(1-t) + 2\sum_{i=k+1}^{n} a_i(1-x_i) . \end{aligned}$$

Now, in fact

$$2\sum_{i=k+1}^{n} a_i(1-x_i) = (a_{k+1} + a_{k+2} + \cdots + a_n)^2 .$$

To prove this by induction, we need only show that

$$2a_k(1-x_k) = a_k^2 + 2a_k(a_{k+1} + \cdots + a_n) ,$$

which follows directly from the fact that

$$x_k = 1 - a_n - a_{n-1} - \cdots - a_{k+1} - a_k/2 .$$

Therefore, in $[x_k, x_{k+1})$,

$$2K(t) = ((1-t) - (a_{k+1} + \cdots + a_n))^2 \geqq 0 ;$$

and it can similarly be shown that K is nonnegative in $[0, x_1]$ and $[x_{n,1}]$.

It is easy to see that, given n, the coefficient $(a_1^3 + \cdots + a_n^3)/24$ in (4) is least when $a_1 = a_2 = \cdots = a_n = 1/n$, so that for any n, the "best" midpoint quadrature rule is simply the repeated Euler's rule.

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