Some Integrals of the Arctangent Function

By M. L. Glasser

Integrals of the form $\int_0^\infty (\tan^{-1}cz)^{2n}R(z)dz$, where $R(z)$ is an even rational expression in z, occur in the theory of localized magnetic moments in metals. Since integrals of this general type do not appear to be tabulated, we present here a method for evaluation as well as some interesting related material.

By partial fraction decomposition all integrals of the above type may be reduced to the form

$$I_n(a) = \int_0^\infty (\tan^{-1} cz)^{2n}(z^2 + a^2)^{-1}dz\,, \tag{1}$$

where a is not required to be real. Also, by a simple change of variable, only the case $c = 1$ need be considered. By writing this as half the integral from $-\infty$ to ∞ and making the substitution $z = \tan \theta/2$ this may be brought into the form $2^{-(2n+1)}(a^2 + 1)^{-2} \int_{-\pi}^{\pi} \theta^{2n}(1 + \lambda \cos \theta)^{-1}d\theta$, where $\lambda = (a^2 - 1)/(a^2 + 1)$. In terms of $z = e^{i\theta}$, this may be written

$$I_n(a) = (-1)^{n+1}i2^{-2n}(a^2 - 1)^{-1} \int_{\Gamma_1} (z - z_0)^{-1}(z - z_1)^{-1} \ln^{2n} z dz \tag{2}$$

where Γ_1 is the contour $z = e^{i\theta}$, $-\pi < \theta < \pi$ and $z_0 = 1/z_1 = (1 - a)/(1 + a)$. For the moment we assume $0 < a < 1$ so $z_0 > 0$ and lies inside the unit circle. Closing Γ_1 by the loop Γ_2: $z = \rho e^{\pm i\pi}$, $0 < \rho < 1$, we trap the pole at z_0 and thus

$$I_n(a) = \frac{(-1)^n 2\pi}{2^{2n}(a^2 - 1)} \left\{\frac{\ln^{2n} z_0}{(z_0 - z_1)} + \frac{1}{2\pi i} \int_{\Gamma_2} \frac{\ln^{2n} z dz}{(z - z_0)(z - z_1)}\right\}. \tag{3}$$

The integral remaining in (3) is

$$J_n = \frac{1}{2\pi i} \int_0^1 \frac{(\ln x - i\pi)^{2n} - (\ln x + i\pi)^{2n}}{(x + z_0)(x + z_1)} dz\,. \tag{4}$$

This can be reduced to a sum of Kummer's Lambda functions [1]

$$\Lambda_{n+1}(x) = \int_0^x \frac{\ln^n|u|}{1 + u} du\,, \tag{5}$$

for example. However, since the resulting formula is somewhat unwieldy, the method will be illustrated by the cases $n = 1, 2$. We have

$$\begin{aligned} J_1 &= -2(z_1 - z_0)^{-1} \int_0^1 \ln x\{(x - z_0)^{-1} - (x + z_1)^{-1}\}dx \\ &= \frac{2z_0}{z_0^2 - 1} \left\{(1/2) \ln z_0 \ln\left[\frac{(1 + z_0)^2}{z_0}\right] + \mathrm{Li}_2\left(\frac{z_0}{z_0 + 1}\right) - \mathrm{Li}_2\left(\frac{1}{z_0 + 1}\right)\right\} \end{aligned} \tag{6}$$

Received July 26, 1967.

where $\mathrm{Li}_2(x)$ is the Euler dilogarithm [1]. Using the relation $\mathrm{Li}_2(x) + \mathrm{Li}_2(1 - x) = \pi^2/6 - \log x \log(1 - x)$, we find

$$I_1(a) = (\pi/4a)\left\{\frac{\pi^2}{6} - \ln^2\left(\frac{1+a}{2}\right) - 2\,\mathrm{Li}_2\left(\frac{1-a}{2}\right)\right\}. \tag{7}$$

Since both sides of this equation are analytic functions of a for $\mathrm{Re}\, a > 0$, the result is valid for all positive a. In addition, taking the limit $a \to 0$ leads to the known result $I_1(0) = \pi \ln 2$. The dilogarithm can be evaluated in closed form for a number of special cases [1], which leads to the apparently new results

$$\begin{aligned} I_1(\sqrt{5}) &= (\pi/4\sqrt{5})\left\{(3\pi^2/10) - 2\ln^2\left(\frac{1+\sqrt{5}}{2}\right)\right\}, \\ I_1(3) &= (\pi/12)\{(\pi^2/3) - \ln^2 2\}, \\ I_1(\sqrt{5} - 2) &= \frac{\pi}{4(\sqrt{5}-2)}\left\{\frac{\pi^2}{30} + \ln^2\left(\frac{\sqrt{5}+1}{2}\right)\right\}, \\ I_1(\sqrt{5} + 2) &= \frac{\pi}{4(\sqrt{5}+2)}\left\{\frac{11\pi^2}{30} - 5\ln^2\left(\frac{1+\sqrt{5}}{2}\right)\right\}. \end{aligned} \tag{8}$$

Other than for the trivial case $a = 1$, these are the only real values of a for which $I_1(a)$ may be expressed in elementary terms. The derivative of $I_1(a)$ is related to entry 3.813(5) of Gradshteyn and Ryzhik's tables [2] so (7) could also be obtained from that result by integration.

The case $n = 2$ leads to

$$\int_0^\infty \frac{(\tan^{-1} z)^4}{z^2 + a^2}\,dz = (\pi/4a)\left\{\frac{\pi^4}{40} + \pi^2\,\mathrm{Li}_2\left(\frac{a-1}{a+1}\right) - 6\,\mathrm{Li}_4\left(\frac{a-1}{a+1}\right)\right\} \tag{9}$$

in terms of tabulated functions [1]. Unfortunately, the tetralogarithm cannot be evaluated in closed form for many special values. The case $a \to 0$ leads to

$$\int_0^{\pi/2} x^4 \csc^2 x\,dx = \frac{\pi^3}{2}\ln 2 - \frac{9\pi}{4}\zeta(3). \tag{10}$$

This integral can be obtained from [2, Eq. 3.748.2, p. 418], which leads to

$$\int_0^{\pi/2} x^{p+1} \csc^2 x\,dx = (p + 1)(\pi/2)^p\left\{p^{-1} - 2\sum_{k=1}^{\infty}(p + 2k)^{-1}2^{-2k}\zeta(2k)\right\} \tag{11}$$

in terms of the Riemann Zeta function. Thus, for $p = 3$ we have the interesting relation

$$\zeta(3) = \frac{4\pi^2}{9}\left\{\sum_{k=1}^{\infty}\zeta(2k)/(4^k(2k + 3)) + \frac{1}{2}\ln 2 - \frac{1}{6}\right\}. \tag{12}$$

In a similar way we can sum the series $\sum_{k=1}^{\infty}\zeta(2k)2^{-2k}/(2k + p)$ for any odd p.

The method used here can also be extended to arbitrary integrals of the form $\int_0^\infty (\tan^{-1} z)^n R(z)\,dz$ where n and R are not required to be even. When symmetry is not invoked, however, the Cauchy principal part rather than the residue is involved.

Finally, it is emphasized that a is not restricted to real values, so that cases such as $R(z) = (1 + z^6)^{-1}$ may also be treated. The polylogarithms of complex argument have been studied in detail [1] and we also obtain results as

$$(13)\quad \int_0^\infty \frac{(\tan^{-1} z)^2}{(z^2-3)-4i}\,dz = \frac{\pi(1+2i)}{20}\left\{\frac{7\pi^2}{48} - \frac{1}{4}\ln^2 2 + \frac{\pi\ln 2}{4} - 2i\beta(2)\right\}$$

where $\beta(2)$ is Catalan's constant $0.915965\cdots$. Taking the real and imaginary parts of both sides of (13) gives

$$(14)\quad \begin{aligned} \int_0^\infty \frac{(\tan^{-1} z)^2}{z^4-6z^2+25}\,dz &= \frac{\pi}{40}\left\{\frac{7\pi^2}{48} - \frac{1}{4}\ln^2 2 + \frac{\pi\ln 2}{4} - \beta(2)\right\}, \\ \int_0^\infty \frac{z^2(\tan^{-1} z)^2}{z^4-6z^2+25}\,dz &= \frac{\pi}{8}\left\{\frac{7\pi^2}{48} - \frac{1}{4}\ln^2 2 + \frac{\pi\ln 2}{4} + \beta(2)\right\}. \end{aligned}$$

These are only a few of the special cases which can be expressed in closed form.

Acknowledgement. The author wishes to thank Drs. V.E. Wood and R.P. Kenan for helpful comments.

Battelle Memorial Institute
Columbus, Ohio 43201

1. L. Lewin, *Dilogarithms and Associated Functions,* Macdonald, London, 1958. MR **21** #4264.

2. I. S. Gradshteyn & I. M. Ryzhik, *Table of Integrals, Series and Products,* translated from Russian, Academic Press, New York, 1965. MR **33** #5952.