On a Generalization of the Midpoint Rule*

By Franz Stetter

I. Introduction. A modified midpoint rule for the approximate calculation of weighted integrals $\int_a^b p(x)f(x)dx$, where $p(x) \geqq 0$ is the weight function, has been recently proposed by Jagermann [1]. Although this formula reduces to the common midpoint rule in the particular case $p(x) \equiv 1$, in the general case of arbitrary weight functions the error does not vanish for all polynomials $\alpha + \beta x$. The purpose of this paper is to generalize the midpoint rule such that the formula is exact for polynomials of first degree and arbitrary weight function $p(x) \geqq 0$.

In view of practical calculations, the repeated midpoint rule is very useful because of its simplicity and small round-off error. Moreover, an error estimation does not require higher derivatives whose bounds are often not easy to obtain. For a comparison of the repeated midpoint rule to both Gaussian quadratures and "best" quadratures we refer to Stroud and Secrest [2].

II. Generalized Midpoint Rule. We assume that the weight function $p(x)$ does not identically vanish on any subinterval of $[a, b]$. Let

$$y = H(x) = \int_a^x p(t)dt\,, \qquad H(b) = 1\,, \tag{1}$$

and let the inverse function of H (which exists because $H(x)$ is monotonic increasing) be denoted by L:

$$x = L(y) = H^{-1}(y)\,. \tag{2}$$

For $i = 0, 1, \cdots, N - 1$, ($N \geqq 1$) we put

$$a_i = N \int_{i/N}^{(i+1)/N} L(y)dy = N \int_{x_i}^{x_{i+1}} tp(t)dt\,, \tag{3}$$

where $x_i = L(i/N)$. We now define the generalized rule by:

$$\int_a^b p(x)f(x)dx = \frac{1}{N} \sum_{i=0}^{N-1} f(a_i) + R_N\,. \tag{4}$$

Assuming $f \in C^2[a, b]$ the error R_N can be expressed by

$$R_N = \frac{1}{2}\left(\int_a^b x^2 p(x)dx - \frac{1}{N} \sum_{i=0}^{N-1} a_i^2\right) f''(\xi) = \frac{1}{2} C_N f''(\xi)\,, \qquad a < \xi < b\,. \tag{5}$$

Proof. Dividing $[a, b]$ into the subintervals $[x_i, x_{i+1}]$ we obtain for the error R_N

$$R_N = \sum_{i=0}^{N-1} \left\{ \int_{x_i}^{x_{i+1}} p(x)f(x)dx - \frac{1}{N} f(a_i) \right\}.$$

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By the Taylor series

$$f(x) = f(a_i) + (x - a_i)f'(a_i) + \tfrac{1}{2}(x - a_i)^2 f''(\xi_i)$$

and by (3) we get the expression

$$\begin{aligned} R_N &= \frac{1}{2}\sum_{i=0}^{N-1}\left\{\int_{x_i}^{x_{i+1}} (x - a_i)^2 p(x) f''(\xi_i)dx\right\} \\ &= \frac{1}{2}\left(\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}} (x - a_i)^2 p(x)dx\right) f''(\xi)\,. \end{aligned} \tag{6}$$

Furthermore, it follows from (3) that

$$\begin{aligned} \sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}} (x - a_i)^2 p(x)dx &= \sum_{i=0}^{N-1}\left\{\int_{x_i}^{x_{i+1}} x^2 p(x)dx - \frac{2}{N}a_i^2 + \frac{1}{N}a_i^2\right\} \\ &= \int_a^b x^2 p(x)dx - \frac{1}{N}\sum_{i=0}^{N-1} a_i^2\,. \end{aligned} \tag{7}$$

(6) and (7) yield the bound (5).

C_N can also be interpreted as the integration error of the function $f = x^2$. It may be noted that Jagermann's modification of the midpoint rule is obtained if the integral $N\int_{i/N}^{(i+1)/N} L(y)dy$ in (3) is approximated by the (ordinary) midpoint rule, i.e., by $L((2i + 1)/2N)$.

III. Examples.

(a) For $p(x) \equiv 1$ and $a = 0$, $b = 1$, we obtain $a_i = (2i + 1)/2N$ and, from (5), $C_N = 1/12N^2$ in accordance with the common midpoint rule.

(b) Let $p(x) = \pi^{-1}(1 - x^2)^{-1/2}$ and $a = -1$, $b = 1$. From $L(y) = -\cos\pi y$ it immediately follows that:

$$a_i = -\frac{2N}{\pi}\sin\frac{\pi}{2N}\cos\frac{2i+1}{2N}\pi \qquad (i = 0, \cdots, N - 1)$$

and

$$\begin{aligned} C_N &= \frac{1}{2} - \frac{1}{N}\sum_{i=0}^{N-1} a_i^2 = \frac{1}{2} \qquad \text{for } N = 1 \\ &= \frac{1}{2} - \frac{2N^2}{\pi^2}\sin^2\frac{\pi}{2N} \qquad \text{for } N \geqq 2\,. \end{aligned}$$

Obviously, $C_N = O(N^{-2})$.

(c) For the infinite interval $a = 0$, $b = \infty$ and the weight function $p(x) = e^{-x}$ we get from $L(y) = -\log(1 - y)$:

$$a_{N-1} = 1 + \log N\,,$$

$$a_i = 1 + \log N - (N - i - 1)\log(N - i - 1) - (N - i)\log(N - i)$$

for $i = 0, 1, \cdots, N - 2$. Numerically computed values of C_N

N	1	2	5	10	20	50
C_N	1.000	0.520	0.213	0.108	0.054	0.022

show that C_N goes to 0 with the order $O(N^{-1})$.

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1. D. JAGERMANN, "Investigation of a modified mid-point quadrature formula," *Math. Comp.*, v. 20, 1966, pp. 79–89. MR **32** #8499.

2. A. H. STROUD & D. SECREST, *Gaussian Quadrature Formulas*, Prentice-Hall, Englewood Cliffs, N. J., 1966. MR **34** #2185.