

# Error Estimates for the Clenshaw-Curtis Quadrature

By M. M. Chawla

**1. Introduction.** Clenshaw and Curtis [1] have proposed a quadrature scheme based on the "practical" abscissas  $x_i = \cos(i\pi/n)$ ,  $i = 0(1)n$  and they have also discussed the estimation of error of the quadrature formula. Elliott [2] has discussed the estimation of truncation errors in the two Chebyshev series approximations for a function, one based on the practical abscissas and the other on the "classical" abscissas  $x_i = \cos(2i+1)\pi/(2n+2)$ ,  $i = 0(1)n$ . Elliott also obtains asymptotic error estimates for the Lagrangian quadrature formulas based on these two sets of points. Recently, Fraser and Wilson [3] have discussed the estimation of error of the Clenshaw-Curtis quadrature and they give a simple formula for the calculation of the error in terms of the function-values.

In the present note we obtain error estimates for the Clenshaw-Curtis quadrature applied to functions analytic on the interval of integration  $[-1, 1]$ . We also obtain error estimates for the quadrature formula based on the classical abscissas.

**2. The Clenshaw-Curtis Quadrature Formula.** Let  $\Psi_n(x)$  denote the Lagrangian interpolation polynomial for  $f(x)$  at the practical abscissas  $x_i = \cos(i\pi/n)$ ,  $i = 0(1)n$ , and let  $\psi_n(x)$  denote the error of interpolation

$$(1) \quad \psi_n(x) = f(x) - \Psi_n(x).$$

If

$$(2) \quad \Psi_n(x) = \sum_{k=0}^{n''} B_{k,n} T_k(x)$$

where  $T_k(x) = \cos(k \arccos x)$ , Chebyshev polynomial of the first kind of degree  $k$ , and the double prime on the summation sign indicates that the first and the last terms are to be halved, then the coefficients  $B_{k,n}$  are given by

$$(3) \quad \begin{aligned} B_{k,n} &= \frac{2}{n} \sum_{i=0}^{n''} f(x_i) T_k(x_i) \\ &= \frac{2}{n} \sum_{i=0}^{n''} f(x_i) T_i(x_k), \end{aligned}$$

since  $T_k(x_i) = T_i(x_k)$ , and  $x_i = \cos(i\pi/n)$ ,  $i = 0(1)n$ . An elegant method for the evaluation of the coefficients  $B_{k,n}$  is described by Clenshaw [4].

Let  $C$  be a closed contour enclosing the interval  $[-1, 1]$  in its interior and let  $f(z)$  be regular within  $C$ . Since the practical abscissas are the zeros of the polynomial  $T_{n+1}(x) - T_{n-1}(x)$ , the error  $\psi_n(x)$  of the Lagrange interpolation for  $f(x)$  at these abscissas can be expressed by a contour integral (Davis [5, Theorem 3.6.1]) as

---

Received September 29, 1967.

$$(4) \quad \psi_n(x) = \frac{[T_{n+1}(x) - T_{n-1}(x)]}{2\pi i} \int_C \frac{f(z)dz}{(z-x)[T_{n+1}(z) - T_{n-1}(z)]}$$

for  $x \in [-1, 1]$ . If  $n$  is even, the integration of (2) gives

$$(5) \quad \int_{-1}^1 f(x)dx \simeq \int_{-1}^1 \Psi_n(x)dx = \sum_{j=0}^{n/2} \frac{(-2)B_{2j,n}}{4j^2 - 1}.$$

Substituting for  $B_{2j,n}$  from the first of the relations (3), the Clenshaw-Curtis approximate integration formula can be rewritten as

$$(6) \quad \int_{-1}^1 f(x)dx \simeq \sum_{i=0}^{n/2} \lambda_i f(x_i),$$

where the weights  $\lambda_i$  are given by

$$(7) \quad \lambda_i = \frac{2}{n} \sum_{j=0}^{n/2} \frac{(-1)^j T_{2j}(x_i)}{4j^2 - 1} \quad \text{for } i = 0(1)n.$$

The error of the Clenshaw-Curtis quadrature formula is given by

$$(8) \quad E_n(\Psi) = \int_{-1}^1 \psi_n(x)dx = \frac{1}{\pi i} \int_C \frac{[Q_{n+1}^*(z) - Q_{n-1}^*(z)]}{[T_{n+1}(z) - T_{n-1}(z)]} f(z)dz,$$

where we have put

$$(9) \quad Q_n^*(z) = \frac{1}{2} \int_{-1}^1 \frac{T_n(x)dx}{z - x}.$$

Equation (9) defines  $Q_n^*(z)$  as a single-valued analytic function in the  $z$ -plane with the interval  $[-1, 1]$  deleted.

**2.1. The Quadrature Formula Based on the Classical Abscissas.** Let  $\Phi_n(x)$  denote the Lagrange interpolation polynomial for the abscissas  $x_i = \cos(2i+1)\pi/(2n+2)$ ,  $i = 0(1)n$  which are the zeros of  $T_{n+1}(x)$ . The computation of the polynomial  $\Phi_n(x)$  has been discussed in detail by Elliott [2].

To describe the corresponding quadrature formula, let

$$(10) \quad \Phi_n(x) = \sum_{k=0}^{n'} A_{k,n} T_k(x),$$

where the prime on the summation sign indicates that the first term is to be halved. The coefficients  $A_{k,n}$  are given by

$$(11) \quad A_{k,n} = \frac{2}{n+1} \sum_{i=0}^n f(x_i) T_k(x_i) \quad \text{for } i = 0(1)n,$$

where  $x_i = \cos(2i+1)\pi/(2n+2)$  for  $i = 0(1)n$ . Now, the integration of (10) gives the quadrature formula based on these abscissas as

$$(12) \quad \begin{aligned} \int_{-1}^1 f(x)dx &\simeq \int_{-1}^1 \Phi_n(x)dx \\ &\simeq \sum_{k=0}^{n'} A_{k,n} \left( \int_{-1}^1 T_k(x)dx \right). \end{aligned}$$

But

$$\begin{aligned} \int_{-1}^1 T_k(x) dx &= \frac{2}{1-k^2} \quad \text{if } 1-k \text{ is odd} \\ &= 0 \quad \text{if } 1-k \text{ is even.} \end{aligned}$$

Since  $n$  is even, putting  $k = 2m$ , we get

$$(13) \quad \int_{-1}^1 f(x) dx \simeq \sum_{m=0}^{n/2} \frac{(-2) A_{2m,n}}{4m^2 - 1}.$$

Substituting for  $A_{2m,n}$  from (11), the above approximate integration formula can be put in the alternative form,

$$(14) \quad \int_{-1}^1 f(x) dx \simeq \sum_{i=0}^n \mu_i f(x_i),$$

where the weights  $\mu_i$  are given by

$$(15) \quad \mu_i = \frac{2}{n+1} \sum_{j=0}^{n/2} \frac{(-2) T_{2j}(x_i)}{4j^2 - 1}, \quad i = 0(1)n.$$

Let  $\phi_n(x) = f(x) - \Phi_n(x)$  denote the error of interpolation for  $f(x)$  at the classical abscissas. Then,  $\phi_n(x)$  may be expressed in terms of a contour integral as

$$(16) \quad \phi_n(x) = \frac{T_{n+1}(x)}{2\pi i} \int_C \frac{f(z) dz}{(z-x) T_{n+1}(z)}.$$

The error  $E_n(\Phi)$  for the quadrature formula (14) is given by

$$(17) \quad E_n(\Phi) = \int_{-1}^1 \phi_n(x) dx = \frac{1}{\pi i} \int_C \frac{Q_{n+1}^*(z)}{T_{n+1}(z)} f(z) dz.$$

**3. A Lemma for  $Q_n^*(z)$ .** Introduce the ellipse  $\mathcal{E}_\rho$  in the  $z$ -plane by

$$z = \frac{1}{2}(\xi + \xi^{-1}), \quad \xi = \rho e^{i\theta}, \quad 0 \leq \theta \leq 2\pi$$

with foci at  $z = \pm 1$  and whose sum of semi-axes is  $\rho$  ( $\rho > 1$ ).

We establish the following lemma.

LEMMA. For  $z \in \mathcal{E}_\rho$ ,

$$(18) \quad Q_n^*(z) = \xi^{-n-1} \sum_{k=-[n/2]}^{\infty} \frac{\sigma_{n,n+2k+1}^*}{\xi^{2k}},$$

where

$$\sigma_{n,n+2k+1}^* = 2(n+2k+1)/(2n+2k+1)(2k+1),$$

$[k] = \text{greatest integer} \leq k$ .

*Proof.* In (9), setting  $x = \cos \theta$  and transforming to the  $\xi$ -plane, we get

$$(19) \quad Q_n^*(z) = \xi^{-1} \int_0^\pi \frac{\cos(n\theta) \sin \theta d\theta}{1 - 2 \cos \theta \xi^{-1} + \xi^{-2}},$$

since  $T_n(\cos \theta) = \cos n\theta$ . Now,

$$(20) \quad \frac{\sin \theta}{\xi} [1 - 2 \cos \theta \xi^{-1} + \xi^{-2}]^{-1} = \sum_{m=1}^{\infty} \frac{\sin m\theta}{\xi^m}.$$

The last series converges uniformly and absolutely for  $0 \leq \theta \leq \pi$  and for all  $|\xi| \geq \rho > 1$ . Substituting (20) in (19),

$$Q_n^*(z) = \sum_{m=1}^{\infty} \frac{\sigma_{nm}^*}{\xi^m},$$

where

$$\begin{aligned} \sigma_{nm}^* &= \int_0^\pi \cos n\theta \sin m\theta d\theta \\ &= \frac{2m}{m^2 - n^2} \quad \text{if } m - n \text{ is odd} \\ &= 0 \quad \text{if } m - n \text{ is even.} \end{aligned}$$

The result follows by putting  $m - n = 2k + 1$  and observing that  $k \geq -[n/2]$  for  $n, m = 1, 2, 3, \dots$ .

From the above lemma we deduce

COROLLARY 1. For  $z \in \mathcal{E}_\rho$ ,

$$(21) \quad Q_{n+1}^*(z) = \xi^{-n} \sum_{k=-[(n-1)/2]}^{\infty} \frac{\sigma_{n+1,n+2k}^*}{\xi^{2k}}$$

where

$$\sigma_{n+1,n+2k}^* = 2(n+2k)/(2n+2k+1)(2k-1)$$

and

$$\sigma_{n+1,n+2k}^* \leq 2n/(2n+1).$$

COROLLARY 2. For  $z \in \mathcal{E}_\rho$ ,

$$(22) \quad Q_{n+1}^*(z) - Q_{n-1}^*(z) = \xi^{-n} \sum_{k=-[(n-1)/2]}^{\infty} \frac{\lambda_{nk}^*}{\xi^{2k}}$$

where

$$\lambda_{nk}^* = 8n(n+2k)/[4(n+k)^2 - 1][4k^2 - 1]$$

and

$$\lambda_{nk}^* \leq 8n^2/(4n^2 - 1).$$

*Proof.* Subtracting (18) with  $n$  replaced by  $n-1$  from (21),

$$Q_{n+1}^*(z) - Q_{n-1}^*(z) = \xi^{-n} \sum_{k=-[(n-1)/2]}^{\infty} \frac{\lambda_{nk}^*}{\xi^{2k}}$$

where

$$\begin{aligned} \lambda_{nk}^* &= \sigma_{n+1,n+2k}^* - \sigma_{n-1,n+2k}^* \\ &= \frac{8n(n+2k)}{[4(n+k)^2 - 1](4k^2 - 1)}. \end{aligned}$$

Also,

$$\lambda_{nk}^* \leq |\lambda_{n,0}| = \frac{8n^2}{4n^2 - 1}.$$

**4. Error Estimates.** We now obtain error estimates for the Clenshaw-Curtis quadrature formula for all functions analytic on  $[-1, 1]$ . Simultaneously, we shall obtain estimates for  $E_n(\Phi)$ .

Let  $f(x)$  be analytic on  $[-1, 1]$ . Then, for some  $\rho > 1$ ,  $f$  can be continued analytically so as to be single valued and regular in the closure of  $\mathcal{E}_\rho$ . In (8), taking the contour to be an ellipse  $\mathcal{E}_\rho$ , we have

$$(23) \quad |E_n(\Psi)| \leq \frac{1}{\pi} \int_{\mathcal{E}_\rho} \frac{|Q_{n+1}^*(z) - Q_{n-1}^*(z)| |f(z)| |dz|}{|T_{n+1}(z) - T_{n-1}(z)|}.$$

Now, for  $n$  even, from (22) we have

$$\begin{aligned} |Q_{n+1}^*(z) - Q_{n-1}^*(z)| &\leq \rho^{-n} \left( \frac{8n^2}{4n^2 - 1} \right) \sum_{k=-(n/2)+1}^{\infty} \rho^{-2k} \\ &\leq \left( \frac{8n^2}{4n^2 - 1} \right) (\rho^2 - 1)^{-1}, \end{aligned}$$

and

$$\frac{|dz|}{|T_{n+1}(z) - T_{n-1}(z)|} \leq \rho^{-1} (\rho^n - \rho^{-n})^{-1} |d\xi|.$$

Making use of these results, from (23) we obtain the following theorem.

**THEOREM 1.** *Let  $f(x)$  be analytic on  $[-1, 1]$  and be continuable analytically so as to be single valued and regular in the closure of an ellipse  $\mathcal{E}_\rho$  with foci at  $z = \pm 1$  and whose sum of semiaxes is  $\rho$  ( $\rho > 1$ ). Then, for  $n$  even,*

$$(24) \quad |E_n(\Psi)| \leq \left( \frac{16n^2}{4n^2 - 1} \right) \frac{M(\rho)}{(\rho^2 - 1)(\rho^n - \rho^{-n})}$$

where  $M(\rho) = \max_{z \in \mathcal{E}_\rho} |f(z)|$  (or equivalently on  $|\xi| = \rho$ ).

4.1. We next obtain an estimate for  $E_n(\Phi)$ . From (21), we obtain for  $n$  even,

$$(25) \quad \begin{aligned} |Q_{n+1}^*(z)| &\leq \rho^{-n} \left( \frac{2n}{2n+1} \right) \sum_{k=-(n/2)+1}^{\infty} \rho^{-2k} \\ &\leq \left( \frac{2n}{2n+1} \right) (\rho^2 - 1)^{-1}. \end{aligned}$$

Now, taking the contour to be an ellipse  $\mathcal{E}_\rho$  in (17), we have

$$(26) \quad |E_n(\Phi)| \leq \frac{1}{\pi} \int_{\mathcal{E}_\rho} \frac{|Q_{n+1}^*(z)| |f(z)| |dz|}{|T_{n+1}(z)|}.$$

Employing (25), we obtain the following theorem from (26).

**THEOREM 2.** *Let  $f(x)$  satisfy the regularity conditions of Theorem 1. Then, for  $n$  even,*

$$(27) \quad |E_n(\Phi)| \leq \left( \frac{4n}{2n+1} \right) \frac{(\rho + \rho^{-1})}{(\rho^2 - 1)} \frac{M(\rho)}{(\rho^{n+1} - \rho^{-n-1})}.$$

*Remarks.* (i) From (24) and (27) it would appear that the estimate for  $E_n(\Phi)$  is nearly half of that for  $E_n(\Psi)$  for large  $n$  and  $\rho \gg 1$ . For small values of  $n$  and  $\rho$  near 1, the estimate (27) is still less than the estimate (24).

(ii) The estimates (24) and (27) obtained for the ellipse will be reasonably reliable for large  $\rho$ , while these estimates are poor if  $\rho$  is near 1.

(iii) For fixed  $n$  and varying  $\rho$ , a "least conservative" upper bound can be established for these estimates for some  $\rho^*$  ( $1 < \rho^* \leq \rho$ ). However, observe that if  $f(z)$  is entire,  $\rho^*$  will be a value of  $\rho$  for which the right side of (24) or (27) is a minimum.

**5. Example.** We illustrate the error estimate (24) for the Clenshaw-Curtis quadrature for the function  $f(x) = 1/(x + 4)$ , and compare the estimates obtained with those given by Fraser and Wilson [3].

We select  $\rho = 7$  for which  $f(z) = 1/(z + 4)$  is regular within the closed ellipse  $\mathcal{E}_7$ . Now, on  $\mathcal{E}_\rho$ ,

$$|f(z)| \leq M(\rho) = \frac{2\rho}{(4 + (15)^{1/2} - \rho)(\rho - (4 - (15)^{1/2}))}.$$

Thus,  $M(7) \doteq 2.33333347$ .

The estimate (24) for the error of the Clenshaw-Curtis quadrature applied to this function is given by

$$(28) \quad |E_n(\Psi)| \leq \left( \frac{n^2}{4n^2 - 1} \right) \frac{M(7)}{(7^n - 7^{-n})}.$$

The exact value of  $\int_{-1}^1 dx/(4 + x) \doteq 0.5108\ 2562$ .

TABLE I

$n$	<i>estimated error</i>		<i>actual error</i>		<i>estimates of Fraser-Wilson</i>	
2	0.0042	3456	0.0002	8549	0.0166	6667
4	0.0000	8230	0.0000	0125	0.0002	6882
8	0.0000	00027	0.0000	0000	0.0000	0007

The error estimated by (28) is compared in Table I with the actual error and the estimates given by Fraser and Wilson [3].

Department of Mathematics  
Indian Institute of Technology  
Hauz Khas, New Delhi-29  
India

1. C. W. CLENSHAW & A. R. CURTIS, "A method for numerical integration on an automatic computer," *Numer. Math.*, v. 2, 1960, pp. 197-205. MR 22 #8659.

2. D. ELLIOTT, "Truncation errors in two Chebyshev series approximations," *Math. Comp.*, v. 19, 1965, pp. 234-248. MR 31 #5313.

3. W. FRASER & M. W. WILSON, "Remarks on the Clenshaw-Curtis quadrature scheme," *SIAM Rev.*, v. 8, 1966, pp. 322-327. MR 34 #3784.

4. C. W. CLENSHAW, *Chebyshev Series for Mathematical Functions*, National Physical Laboratory Mathematical Tables, Vol. 5, Department of Scientific and Industrial Research, H. M. S. O., London, 1962. MR 26 #362.

5. P. J. DAVIS, *Interpolation and Approximation*, Blaisdell, New York, 1963, pp. 67-68, 311-312. MR 28 #393.