

# High Accuracy Gamma Function Values for Some Rational Arguments

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During the course of determining accurate Gaussian quadrature formulas for certain nonclassical weight functions, it was necessary to compute  $\Gamma(p/q)$  to 60D to obtain moments to high accuracy. In particular, we needed  $\Gamma(p/q)$  for  $p = 1, 2, \dots, q - 1$ ,  $2p \neq q$ , and  $q = 3, 4, 5, 8, 10$ . The quantities  $\Gamma(p/q)$ , which for special cases also arise in the computation of Bessel functions of fractional order or of the Airy functions, were calculated using an interpretive routine that treated floating-point numbers of 70 significant figures.

TABLE I

<i>p</i>	<i>q</i>	<i>Values of ln <math>\Gamma(p/q)</math></i>											
1	3	0.98542	06469	27767	06918	71740	36977	96139	17355	56496	38588	58542	34757
2	3	0.30315	02751	47523	56867	58628	17372	01103	56634	93171	97830	62455	32199
1	4	1.28802	25246	98077	45737	06104	40219	71729	59253	77565	11286	05504	99987
3	4	0.20328	09514	31295	37148	14329	71862	42969	97596	67314	98257	86480	73977
1	5	1.52406	38224	30784	52488	10564	93926	30219	25659	33737	40640	34751	04287
2	5	0.79667	78177	01783	76654	47359	62391	62647	40394	48412	45829	74362	09729
3	5	0.39823	38580	69234	89961	68542	20400	87768	42343	54029	05730	96991	15903
4	5	0.15205	96783	99837	58877	82926	02290	57038	88430	53038	49486	41798	82363
1	8	2.01941	83575	53796	34532	02905	21167	09958	99482	80952	13444	96051	31965
3	8	0.86307	39822	70647	46240	50890	94134	01549	53324	70629	34842	71350	12142
5	8	0.36082	94954	88940	18118	49576	85822	77948	78573	69120	20625	81717	15344
7	8	0.08585	87072	25334	32350	23655	83769	48770	22697	19125	68187	11123	48816
1	10	2.25271	26517	34205	95986	97016	46368	49511	86156	27222	29495	37650	41740
3	10	1.09579	79948	18075	52167	71681	42370	10727	84451	48450	76420	34066	38624
7	10	0.26086	72465	31666	51438	57324	17016	75957	81424	62162	12570	28993	34661
9	10	0.06637	62397	34742	97118	87167	39867	10858	42423	52059	36627	35802	53581

As in Sherry and Fulda [1], use is made of the asymptotic expansion

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$$(1) \ln \Gamma(p/q) \sim L(p/q, n) + \sum_{r=1}^{\infty} C_r / (n + p/q)^{2r-1} \quad (n \text{ a positive integer}),$$

where  $C_r = B_{2r}/2r(2r - 1)$ ,  $B_{2r}$  are Bernoulli numbers, and

$$(2) \quad L(p/q, n) = (n - 1/2 + p/q) \ln (nq + p) - (n + p/q) + 1/2 \ln (2\pi) + (1/2 - p/q) \ln q - \sum_{j=0}^{n-1} \ln (p + jq).$$

The function  $\Gamma(x)$  is then calculated from

$$(3) \quad \Gamma(x) = \exp (0.1m) \exp [\ln \Gamma(x) - 0.1m],$$

where  $m$  is taken to be the greatest integer in  $10 \ln \Gamma(x)$ . Mansell's tables [2] provided the values of  $(1/2) \ln (2\pi)$  and logarithms of integers; the numbers  $B_k$  are found in Davis [3].

With the above procedure, we obtained Tables 1 and 2 giving values of  $\ln \Gamma(p/q)$  and  $\Gamma(p/q)$  to 60 decimal places.

TABLE 2

<i>p</i>	<i>q</i>	Values of $\Gamma(p/q)$
1	3	2.67893 85347 07747 63365 56929 40974 67764 41286 89377 95730 11009 50428
2	3	1.35411 79394 26400 41694 52880 28154 51378 55193 27266 05679 36983 94022
1	4	3.62560 99082 21908 31193 06851 55867 67200 29951 67682 88006 54674 33378
3	4	1.22541 67024 65177 64512 90983 03362 89052 68512 39248 10807 06112 30119
1	5	4.59084 37119 98803 05320 47582 75929 15200 34341 09998 29340 30177 88853
2	5	2.21815 95437 57688 22305 90540 21907 67945 07705 66501 77146 95822 41978
3	5	1.48919 22488 12817 10239 43333 88321 34228 13205 99038 75992 47353 38680
4	5	1.16422 97137 25303 37363 63209 38268 45869 31419 61768 89118 77529 84894
1	8	7.53394 15987 97611 90469 92298 41215 13362 46104 19588 14907 59409 83128
3	8	2.37043 61844 16600 90864 64735 04176 65250 98874 00803 35892 49877 75127
5	8	1.43451 88480 90556 77563 60197 39456 42313 66322 07772 20666 73307 70680
7	8	1.08965 23574 22896 95125 23767 55102 89297 11478 70067 76756 51205 13704
1	10	9.51350 76986 68731 83629 24871 77265 40219 25505 78626 08837 73430 50001
3	10	2.99156 89876 87590 62831 25165 15904 91779 11128 06024 92171 51127 44120
7	10	1.29805 53326 47557 78568 11711 79152 81161 77841 41170 55394 62479 21645
9	10	1.06862 87021 19319 35489 73053 35694 48077 81698 38785 06097 31790 49371

The calculated values were checked by means of the identity

$$(4) \quad \Gamma(x) \Gamma(1 - x) = \pi / \sin (\pi x).$$

In all cases, the two sides agreed to 64 decimal places. Also, the values of  $\Gamma(1/3)$ ,  $\Gamma(2/3)$ ,  $\ln \Gamma(1/3)$ , and  $\ln \Gamma(2/3)$  agree with the 35D calculations of Sherry and Fulda [1].

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1. M. E. SHERRY & S. FULDA, "Calculation of gamma functions to high accuracy," *Math. Comp.*, v. 13, 1959, pp. 314-315. MR 21 #7603.
2. W. E. MANSELL, *Tables of Natural and Common Logarithms to 110 Decimals*, Roy. Soc. Math. Tables, v. 8, Cambridge Univ. Press, New York, 1964. MR 29 #3675.
3. H. T. DAVIS, *Tables of the Higher Mathematical Functions*. Vol. II, Principia Press of Trinity University, San Antonio, Texas, 1935; rev. ed., 1963.