Computer Use in Continued Fraction Expansions*

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Abstract. In this study, the use of computers is demonstrated for the rapid expansion of a general regular continued fraction with rational elements for $\sqrt{C} + L$, where C and L are rational numbers, C positive. Formulas for the expansion are derived. Conditions for the periodicity are considered. A Fortran program for the algorithms is given, as well as sample continued fraction expansions. Up to the present, practically all studies have been concerned with continued fractions with partial numerators ± 1 and partial denominators positive integers, due to difficulties in calculation. But now the use of computers makes possible the study of a much greater variety of continued fraction expansions. ■

1. Introduction. A general regular continued fraction has been defined [3] as a finite or infinite continued fraction

$$(1.1) \qquad b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2 +} \cdots, \qquad a_{n+1}, b_n \neq 0, n = 0, 1, \cdots,$$

with *real* numerical elements such that

$$(1.2) \qquad b_n \geqq 1 + |a_{n+1}|, \qquad |a_{n+1}| \geqq 1, n = 0, 1, \cdots,$$

or with *integral* elements such that

$$(1.3) \qquad |a_n| = 1, \quad b_n \geqq 1, \quad b_n + a_{n+1} \geqq 1, \qquad n = 1, 2, \cdots.$$

It has been shown in [3] that a continued fraction satisfying the latter conditions can always be transformed into one satisfying (1.2). A general regular continued fraction converges. The continued fraction expansion (1.1) for a real number F_0 is accomplished by a sequence of linear transformations

$$(1.4) \qquad F_n = b_n + \frac{a_{n+1}}{F_{n+1}}, \quad F_n \geqq 1, \qquad n = 0, 1, \cdots.$$

In [3] it was shown that, if

$$(1.5) \qquad |F_n - b_n| < 1, \quad \text{that is, } F_{n+1} > |a_{n+1}|, \qquad n = 0, 1, \cdots,$$

a general regular continued fraction expansion satisfying (1.2) converges to the generating number F_0. An expansion that satisfies (1.3) is semiregular (or regular if $a_{n+1} = 1$) and always converges to the generating number (cf. [6]).

In [4] the author studied general regular continued fraction expansions with real rational numerical elements for $\sqrt{C} + L$ with C and L rational numbers, C positive. Conditions were also found for general regular expansions (1.1) for $\sqrt{C} + L$ to be periodic. Simultaneously, the approximations x_n to $\sqrt{C} + L$ given by an extension of Newton's formula,

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Received July 2, 1968, revised September 25, 1968.

* Presented to the American Mathematical Society. This research was sponsored in part by the National Science Foundation.

$$x_n = \frac{x_i \cdot x_j + C - L^2}{x_i + x_j - 2L}, \tag{1.6}$$

where x_i and x_j are certain previous approximations to the value $\sqrt{C} + L$, were studied. Let $x_i = X_i/Y_i$ denote the ith approximant of (1.1). In [4] a complete classification was given concerning which ones of the approximants X_i/Y_i are also approximations to $\sqrt{C} + L$ given by formula (1.6).

In this study, the use of computers is demonstrated for the rapid expansion of a general regular continued fraction with rational elements for $\sqrt{C} + L$. The use of computers is especially convenient here since the same operations are repeated many times, and computation without computers is extremely arduous.

Furthermore, practically all studies up to this time have been concerned with the elements $a_i = \pm 1$, b_i positive integers. But now the use of computers makes a study of continued fractions with rational elements no more difficult, since it takes almost no time to work out many examples.

In Section 2, formulas for the expansion of $\sqrt{C} + L$ into a general regular continued fraction (1.1) with rational elements are derived. These are then modified for application to a computer program. In Section 3, conditions for the periodicity of these expansions are discussed. In Section 4 a summary of the calculation procedure is given, and sample problems computed for a particular class of general regular continued fraction expansions for $\sqrt{C} + L$ (Table 1). These are given in order to show how easily general regular continued fraction expansions can be generated for a given binomial quadratic surd.

A Fortran program for the expansion of $\sqrt{C} + L$ into a general regular continued fraction appears in the microfiche section of this issue of the journal.

2. Mathematical Description.

I. The first problem is the derivation of formulas for the computation of the expansion (1.1) for $\sqrt{C} + L$. For a given binomial quadratic surd $\sqrt{C} + L$, one writes

$$\sqrt{C} + L = \frac{\sqrt{D} + P_0}{Q_0}. \tag{2.1}$$

The continued fraction (1.1) is generated by a sequence of linear transformations

$$F_n = b_n + \frac{a_{n+1}}{F_{n+1}} = \frac{\sqrt{D} + P_n}{Q_n}, \qquad n = 0, 1, \cdots. \tag{2.2}$$

The elements must satisfy (1.2) or (1.3), and (1.5). The transformation (2.2) can be written

$$\frac{\sqrt{D} + P_n}{Q_n} = b_n + \frac{a_{n+1}Q_{n+1}}{\sqrt{D} + P_{n+1}}, \qquad n = 0, 1, \cdots. \tag{2.3}$$

From this formula, the recurrence relations

$$D - P_{n+1}^2 = a_{n+1}Q_nQ_{n+1}, \tag{2.4}$$

and

$$
(2.5)\qquad
\begin{aligned}
P_{n+1} &= b_n Q_n - P_n\,, \\
Q_{n+1} &= \frac{b_n(P_n - P_{n+1}) + a_n Q_{n-1}}{a_{n+1}} = \frac{D - P_{n+1}^2}{a_{n+1}Q_n}\,, \qquad n = 0, 1, \cdots, \\
Q_{-1} &= \frac{D - P_0^{\,2}}{Q_0} \qquad (a_0 = 1)\,,
\end{aligned}
$$

were derived in [4].

In the continued fractions considered, the a_{n+1}, b_n, Q_{n-1}, and P_n, $n = 0, 1, \cdots$, are rational numbers. Consequently, it is more convenient in the computation with computers for one to consider the continued fraction

$$
(2.6)\qquad \frac{B_0}{D_0} + \frac{A_1/C_1}{B_1/D_1} + \frac{A_2/C_2}{B_2/D_2} + \cdots
$$

for $\sqrt{C} + L = (\sqrt{D} + P_0/R_0)/(Q_0/S_0)$. This is generated by the sequence of linear transformations

$$
(2.7)\qquad \frac{\sqrt{D} + P_n/R_n}{Q_n/S_n} = \frac{B_n}{D_n} + \frac{(A_{n+1}/C_{n+1})\cdot(Q_{n+1}/S_{n+1})}{\sqrt{D} + P_{n+1}/R_{n+1}}\,, \qquad n = 0, 1, \cdots.
$$

Thus, for the continued fraction (2.6), the recurrence relations (2.4) and (2.5) become

$$
(2.8)\qquad
\begin{aligned}
\frac{P_{n+1}}{R_{n+1}} &= \frac{B_n Q_n R_n - D_n P_n S_n}{D_n R_n S_n}\,, \\
\frac{Q_{n+1}}{S_{n+1}} &= \frac{[B_n C_n S_{n-1}(P_n R_{n+1} - P_{n+1}R_n) + A_n D_n Q_{n-1} R_n R_{n+1}]C_{n+1}}{A_{n+1} C_n D_n R_n R_{n+1} S_{n-1}} \\
&= \frac{C_{n+1} S_n (D R_{n+1}^2 - P_{n+1}^2)}{A_{n+1} Q_n R_{n+1}^2}\,, \qquad n = 0, 1, \cdots, \\
\frac{Q_{-1}}{S_{-1}} &= \frac{(D R_0^{\,2} - P_0^{\,2}) S_0}{Q_0 R_0^{\,2}}\,.
\end{aligned}
$$

Semiregular continued fractions (continued fractions that satisfy (1.3)) are not considered here, since semiregular expansions for $\sqrt{C} + L$ have been treated by Perron [6], Goncalves [5], and many others. In fact, the author in [2] gave an Algol program for the expansion of $\sqrt{C} + L$ into a regular continued fraction. With slight modifications, the expansion into a semiregular continued fraction could be similarly programmed. Consequently, only general regular continued fractions (2.6) that satisfy the conditions

$$
(2.9)\qquad
\begin{aligned}
&\frac{B_n}{D_n} \geqq 1 + \left|\frac{A_{n+1}}{C_{n+1}}\right|, \qquad \left|\frac{A_{n+1}}{C_{n+1}}\right| \geqq 1, \qquad C_{n+1} > 0\,, \qquad D_n > 0\,, \\
&\left|F_n - \frac{B_n}{D_n}\right| < 1\,, \quad \text{i.e. } F_{n+1} > \left|\frac{A_{n+1}}{C_{n+1}}\right|, \qquad n = 0, 1, \cdots,
\end{aligned}
$$

are treated here. As in the case of semiregular continued fractions, *given* are the positive integers C_{n+1} and the positive or negative integers A_{n+1}, $n = 0, 1, \cdots$. Also given are the integers P_0, Q_0, and D ($R_0 = S_0 = 1, A_0 = C_0 = 1$), and the generating number $F_0 > 1$. The positive integers B_n and D_n are computed from (2.9) with *certain specific rules concerning their unique values.* As discussed in [3], [4], the continued fraction (2.6) can always be transformed into one for which $|A_{n+1}/C_{n+1}| > 1$, $B_n/D_n > 1$, so it is henceforth assumed that these conditions hold.

If $(\sqrt{D} + P_0)/Q_0$ is negative, one computes (2.6) for $-(\sqrt{D} + P_0)/Q_0$, and then multiplies (2.6) by -1. Furthermore, $\sqrt{D}$ is taken as the positive root, since, if one is given $(-\sqrt{D} + P_0)/Q_0$, one uses the equal value $(\sqrt{D} - P_0)/(-Q_0)$.

If the expansion is periodic, one notes that one has obtained a complete period p when the values of P_i/R_i, Q_i/S_i are repeated. One must of course start with a periodic sequence (of period r) of the A_{n+1}/C_{n+1}. Then $p = r \cdot k$, where k is a positive integer. (This is analogous to the case of semiregular continued fractions when one is given the $C_{n+1} = 1$, and the A_{n+1} are a given periodic sequence such that $|A_{n+1}| = 1$.)

However, it may very well happen that there is a fore-period of t terms, as in the expansion

$$(2.10)\qquad \begin{gathered} \frac{G_0}{H_0} + \frac{M_1/N_1}{G_1/H_1} + \cdots + \frac{M_t/N_t}{G_t/H_t + K} = \frac{X_t + K \cdot X_{t-1}}{Y_t + K \cdot Y_{t-1}}, \\ K = \left(\frac{A_1/C_1}{B_1/D_1} + \cdots + \frac{A_p/C_p}{B_p/D_p}\right) + \left(\frac{A_1/C_1}{B_1/D_1} + \cdots + \frac{A_p/C_p}{B_p/D_p}\right) + \cdots, \end{gathered}$$

where the periodic part does not begin until the $(t + 1)$th term, and X_t, Y_t, denote the numerator and denominator, respectively, of the tth approximant. This expansion is called *mixed-periodic.*

II. This portion of the mathematical description of the problem is concerned with additional formulas that one can use for further information about general regular expansions for $\sqrt{C} + L$. These are analogous to the computations of the author in [2] on regular continued fraction expansions.

The recurrence relations for the X_n and Y_n, the numerator and denominator, respectively, of the nth approximant of (2.6), are given by

$$(2.11)\qquad \begin{gathered} X_n = B_n X_{n-1}/D_n + A_n X_{n-2}/C_n, \quad Y_n = B_n Y_{n-1}/D_n + A_n Y_{n-2}/C_n, \\ n = 1, 2, \cdots, X_{-1} = 1, X_0 = B_0/D_0, Y_{-1} = 0, Y_0 = 1. \end{gathered}$$

Also of interest is the application of formula (1.6) for the approximants of the expansions considered here. For that purpose it takes the form

$$(2.12)\qquad \begin{aligned} x_{kp-s} &= \frac{Q_0^2 X_{kp-s}}{Q_0^2 Y_{kp-s}} \\ &= \frac{Q_0^2 X_{(k-r)p-s} X_{rp-1} + (D - P_0^2) Y_{(k-r)p-s} Y_{rp-1}}{Q_0^2 (X_{(k-r)p-s} Y_{rp-1} + X_{rp-1} Y_{(k-r)p-s}) - 2P_0 Q_0 Y_{(k-r)p-s} Y_{rp-1}}, \\ & k = 2, 3, \cdots, r = 1, 2, \cdots, k - r \geqq 1, p = 1, 2, \cdots, s \leqq p. \end{aligned}$$

There are numerous other formulas of this type that could also be applied to the approximants of general regular continued fractions on a computer (cf. [1], [4]).

Finally, it is of great importance for one to know how good an approximation to the value of $\sqrt{C} + L$ is an approximant X_n/Y_n. For that purpose, one can use the error formula (cf. [3])

$$(2.13)\qquad \frac{\left|\dfrac{A_1A_2\cdots A_n}{C_1C_2\cdots C_n}\right|}{Y_{n-1}(Y_n+Y_{n-1})} \leqq \left|F_0 - \frac{X^*_{n-1}}{Y^*_{n-1}}\right| \leqq \frac{\left|\dfrac{A_1A_2\cdots A_n}{C_1C_2\cdots C_n}\right|}{Y_{n-1}(Y_n-Y_{n-1})}.$$

Here X^*_{n-1}/Y^*_{n-1} refers to the approximant X_{n-1}/Y_{n-1} with all common factors removed from the numerator and denominator.

These three formulas have not been carried through in the Fortran program, since it has been the purpose of this paper to illustrate the usefulness of computers for expansions in general regular continued fractions, but the latter three formulas can likewise easily be applied to computers.

3. Conditions for the Periodicity of Expansion (2.6) for $\sqrt{C} + L$. It is well known (cf. [5] or [6]) that an infinite semiregular continued fraction (1.3), into which a binomial quadratic surd can be expanded, is always periodic provided the partial numerators a_{n+1}, $n = 0, 1, \cdots$, are a given periodic sequence. Consequently, semiregular continued fractions are not considered here, and the discussion is concerned only with those general regular continued fractions satisfying conditions (1.2).

In [4] it was shown that a convergent infinite continued fraction that is periodic of period p always represents a root of a quadratic equation provided the denominator of the $(p - 1)$th approximant of (2.6) is not zero. Furthermore, it was shown that a general regular continued fraction (2.6) into which a binomial quadratic surd can be expanded, with the A_{n+1} and C_{n+1} a given periodic sequence, and with definite rules for the unique values of the B_n/D_n, is always periodic provided the P_n/R_n in formulas (2.8) are integral.

That this is in general not the case can easily be seen if one writes down the successive values of R_n and S_n from formulas (2.8). It is seen that these values increase indefinitely, and it is only a fortunate cancellation of factors in the fractions P_n/R_n and Q_n/S_n that keeps the values of the numerators and denominators of these fractions bounded, with a consequent possibility for periodicity.

4. Fortran Program and Sample Problems for a Particular Class of General Regular Continued Fractions. In the microfiche section of this issue is given a Fortran program for a particular class of general regular continued fraction expansions for $\sqrt{C} + L$. In this class, the periodic sequence $A_{n+1} = -1$, $C_{n+1} = 1$ is chosen. Then the B_n/D_n are calculated so that in the first formula in (2.8) the P_{n+1}/R_{n+1} are integral, i.e. $R_{n+1} = 1$. This is done in the following way: The denominators D_n must be factors of Q_n. The factors $2, 3, \cdots, Q_n$ are tried in that order as factors, and D_n is set equal to the smallest one that is a factor. If none is a factor, then D_n is set equal to 1. The numerators B_n are then chosen as multiples of S_n, and so chosen that $|F_n - B_n/D_n| < 1$, and such that this difference is smallest, if there is a choice. Of course, since $A_{n+1} = -1$, $B_n/D_n > F_n$. Thus conditions (2.9) are satisfied.

It is clear that it is *not* always possible to find a B_n/D_n that satisfies all these conditions. Below, in Table 1, is an illustration of a binomial quadratic surd that cannot be expanded into such a continued fraction. Of course, it would always be possible for one to alter, for example, the choice of A_{n+1}/C_{n+1}, so that a general regular continued fraction expansion is possible for a given binomial quadratic surd.

TABLE 1. *Examples*

Example 1 $P = 40$ $Q = 39$ $D = 46$

N	A	C	B	D	P	R	Q	S	FN
0	1	1	4	3	40	1	39	1	1.19955
1	−1	1	117	14	12	1	98	39	7.47460
2	−1	1	28	15	9	1	195	14	1.13309
3	−1	1	65	42	17	1	1134	65	1.36318
4	−1	1	357	65	10	1	65	21	5.42198
5	−1	1	130	9	7	1	63	65	14.21986
6	−1	1	63	13	7	1	65	21	4.45275
7	−1	1	65	21	8	1	378	65	2.54193
8	−1	1	14	5	10	1	65	7	1.80733
9	−1	1	13	7	16	1	294	13	1.00738
10	−1	1	7	5	26	1	195	7	1.17680
11	−1	1	195	41	13	1	287	65	4.48031
12	−1	1	287	65	8	1	1170	287	3.62609
13	−1	1	65	41	10	1	861	65	1.26696
14	−1	1	1148	325	11	1	1625	287	3.14063
15	−1	1	975	287	9	1	2009	325	2.55314
16	−1	1	41	25	12	1	650	41	1.18473
17	−1	1	91	41	14	1	123	13	2.19650
18	−1	1	574	13	7	1	13	41	43.46733
19	−1	1	65	41	7	1	123	13	1.45667
20	−1	1	205	26	8	1	78	41	7.77020
21	−1	1	364	41	7	1	41	26	8.74001
22	−1	1	205	26	7	1	78	41	7.24456
23	−1	1	65	41	8	1	123	13	1.56236
24	−1	1	574	13	7	1	13	41	43.46733
25	−1	1	65	41	7	1	123	13	1.45667
26	−1	1	205	26	8	1	78	41	7.77020
27	−1	1	364	41	7	1	41	26	8.74001
28	−1	1	205	26	7	1	78	41	7.24456
29	−1	1	65	41	8	1	123	13	1.56236

Fore-period = 18 *Period* = 6

Example 2 $P = 9$ $Q = 5$ $D = 13$

N	A	C	B	D	P	R	Q	S	FN
0	1	1	13	5	9	1	5	1	2.52111
1	−1	1	40	3	4	1	3	5	12.67591
2	−1	1	8	5	4	1	5	1	1.52111
3	−1	1	40	3	4	1	3	5	12.67591
4	−1	1	8	5	4	1	5	1	1.52111

Period = 2

Example 3 $P = 51$ $Q = 23$ $D = 73$

N	A	C	B	D	P	R	Q	S	FN
0	1	1	60	23	51	1	23	1	2.58887

****NO CHOICE FOR B****

Furthermore, one could set down many different rules for the choice of the A_{n+1}/C_{n+1} and of the consequent computation of the B_n/D_n. The type of expansion given here is described in detail in order to illustrate how a general regular expansion is generated. But it must be borne in mind that many other types of such expansions are possible.

The Fortran program was written for the IBM 360 which was the machine available for these computations. Floating-point arithmetic was used in order to obtain the maximum number of digits. Special care was also used in order to avoid round-off error.

A lowest terms subroutine (LTU) was used to reduce to lowest terms all B_n/D_n and Q_n/S_n.

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