MATHEMATICS OF COMPUTATION, VOLUME 24, NUMBER 110, APRIL, 1970

Chebyshev Iteration Methods for Integral Equations of the Second Kind

By T. W. Sag

Abstract. In this paper the numerical solution of Fredholm integral equations of the second kind using an iterative method in which the solution is represented by a Chebyshev series is discussed. A description of a technique of Chebyshev reduction of the norm of the kernel for use in cases when the iterations converge slowly or not at all is also given. Finally, the application of the methods to other types of second-kind equations is considered.

1. Introduction. Consider first the method of successive approximations (Tricomi [1]) for the solution of Fredholm equations of the second kind, which have the form

$$f(x) = g(x) + \lambda \int_a^b K(x, y) f(y)\, dy, \tag{1.1}$$

where $g(x)$ and the kernel $K(x, y)$ are known functions, λ is a known constant and $f(x)$ is to be determined. The method is to obtain successive approximations $f_i(x)$, $i = 1, 2, \ldots$ to the solution $f(x)$ from the equation

$$f_i(x) = g(x) + \lambda \int_a^b K(x, y) f_{i-1}(y)\, dy, \tag{1.2}$$

starting with the approximation $f_0(x) = g(x)$.

It can be shown that if $g(x)$ is an L_2-function, i.e., $\int_a^b g^2(x)\, dx$ exists, and $K(x, y)$ is an L_2-kernel, i.e., $\|K\|^2 = \int_a^b \int_a^b K^2(x, y)\, dy\, dx$ exists, then the successive approximations converge almost uniformly* to the unique function $f(x)$ satisfying Eq. (1.1) for all values of λ inside the circle $|\lambda| = 1/\|K\|$.

This classical iteration procedure may be approximated by a matrix iteration by replacing the iterates $f_i(x)$ by truncated Chebyshev series approximations and evaluating the integral in Eq. (1.2) by a quadrature formula. Details of this procedure are given in Section 2. It can be shown that the matrix iteration is equivalent to the classical iteration for an integral equation with the functions $g(x)$ and $K(x, y)$ perturbed. If the number of terms in the Chebyshev series approximations and the number of quadrature points is sufficiently large, then the perturbations will be small and so the condition for convergence of the matrix iteration will be almost the same as that for the classical iteration. The perturbations are caused by errors arising from the truncation of Chebyshev series, and from the quadrature formula, and so, if the matrix iteration converges, it gives a solution which differs from the true solution by a function depending on these errors.

Received February 26, 1968, revised June 23, 1969.

AMS Subject Classifications. Primary 4511, 4530, 6575; Secondary 6510, 6520, 6555.

Key Words and Phrases. Fredholm integral equations, iteration, Chebyshev series approximation, numerical quadrature, Chebyshev reduction of kernel, nonlinear integral equations.

* Convergence for all $x \in [a, b]$ for which $\int_a^b K^2(x, y)\, dy$ is finite.

In many cases the iteration does not converge, or converges too slowly for practical purposes. The former situation may occur when $|\lambda|\ \|K\| > 1$, and the latter when $|\lambda|\ \|K\|$ is not significantly less than unity. In these cases we can rewrite the integral equation as an equation with a 'reduced' kernel (i.e., a kernel with a smaller value of $\|K\|$) and solve the latter equation by iteration. Details of this technique are given in Section 6.

Before describing the Chebyshev iteration method, we mention briefly two methods which approximate the integral equation by a matrix equation, but solve the latter by a direct method. The first method, due to Fox and Goodwin [2], replaces the integral term of (1.1) by an n-point quadrature formula at n values of x in $[a, b]$. The contribution to the solution due to the error in the quadrature formula is computed by solving a sequence of matrix equations with a common matrix but with different right-hand sides.

The second method, due to Elliott [3], makes use of Chebyshev series. Equation (1.1) is normalized by a linear transformation of variables so that the range $[a, b]$ becomes $[-1, 1]$, the latter range being the most convenient for representation of functions by Chebyshev series. The function $f(x)$ is approximated by a truncated Chebyshev series of the form

$$f(x) \doteq \sum_{n=0}^{N}{}' a_n T_n(x), \tag{1.3}$$

where $T_n(x) = \cos(n \arccos x)$ is the Chebyshev polynomial of degree n of the first kind and $\sum'$ denotes a sum for which the first term is halved. A Chebyshev approximation of the form

$$K(x, y) \doteq \sum_{n=0}^{M}{}' b_n(x) T_n(y) \tag{1.4}$$

for the kernel $K(x, y)$ is then determined at $N + 1$ points $x = x_i$, $i = 0, \ldots, N$ in $[-1, 1]$. Using a formula for integrals of the form

$$\int_{-1}^{1} T_m(x) T_n(x)\, dx,$$

it is then possible to express the integral term of Eq. (1.1) as a series of the form

$$\int_{-1}^{1} K(x, y) f(y)\, dx \doteq \sum_{n=0}^{N}{}' a_n \beta_n(x) \tag{1.5}$$

at the points $x = x_i$, where the $\beta_n(x)$ are functions of the $b_n(x)$. Substituting Eqs. (1.3) and (1.5) in the integral equation at the $N + 1$ points x_i, $i = 0, \ldots, N$, yields the matrix equation

$$(T - \lambda B)\mathbf{a} = \mathbf{g}, \tag{1.6}$$

where T and B are matrices with elements

$$\left.\begin{aligned} T_{ij} &= \tfrac{1}{2} T_j(x_i) \\ B_{ij} &= \tfrac{1}{2} \beta_j(x_i) \end{aligned}\right\} \quad j = 0,\ i = 0, \ldots, N,$$

$$\left.\begin{aligned} T_{ij} &= T_j(x_i) \\ B_{ij} &= \beta_j(x_i) \end{aligned}\right\} \qquad j = 1, \ldots, N, \; i = 0, \ldots, N,$$

and **a** and **g** are vectors with ith components a_i and $g(x_i)$, respectively, for $i = 0, \ldots, N$. Equation (1.6) is solved directly to obtain the coefficients a_i. The Chebyshev series (1.3) may then be evaluated at any point x by use of a recurrence relation technique given by Clenshaw [4]. This process is more suitable for an automatic computer, and much faster and generally more accurate than polynomial interpolation between $N + 1$ function values. The latter process would be required to evaluate the solution at an arbitrary point x from the solution given by the method of Fox and Goodwin. The author prefers the use of Chebyshev series to function values because of the above, and also because it is easier to estimate the maximum error due to use of truncated Chebyshev series (see Section 4). The motivation for the use of an iterative rather than a direct method is that the former method can be applied with simple modifications (see Section 8) to other types of second kind equations. Also, it has been shown by the author [5] that the computing time required to set up the final matrix equation is approximately the same for Elliott's method as for the Chebyshev iteration method, and that in cases when $|\lambda| \, \|K\|$ is significantly smaller than unity the latter method produces a solution of specified accuracy in a shorter time.

2. Chebyshev Iteration Method. In this section we consider a Fredholm equation of the second kind which has been transformed by a linear transformation of variables from the form of (1.1) to the form

$$f(x) = g(x) + \lambda \int_{-1}^{1} K(x, y) f(y) \, dy. \tag{2.1}$$

In order that Chebyshev series may be used, we restrict $g(x)$ to the class of functions which are piecewise smooth in $[-1, 1]$ and $K(x, y)$ to the class of functions for which $\int_{-1}^{1} K(x, y) T_k(y) \, dy$, $k = 0, 1, 2, \ldots$ is piecewise smooth in $[-1, 1]$.

We choose an initial approximation $f_0(x)$ to $f(x)$, expand it in a truncated Chebyshev series, and then approximate the iterations

$$f_i(x) = g(x) + \lambda \int_{-1}^{1} K(x, y) f_{i-1}(y) \, dy, \qquad i = 1, 2, \ldots \tag{2.2}$$

as follows. We assume a Chebyshev expansion

$$f_{i-1}(x) = \sum_{j=0}^{N2}{}' a_j T_j(x) \tag{2.3}$$

has been obtained from the previous iteration, assume that the ith iterate has the form

$$f_i(x) = \sum_{j=0}^{N}{}' A_j T_j(x), \qquad N \geqq N2, \tag{2.4}$$

and calculate Chebyshev coefficients in the expansions

$$g(x) \doteq \sum_{j=0}^{N}{}' g_j T_j(x), \tag{2.5}$$

$$\int_{-1}^{1} K(x, y) T_k(y)\, dy \doteq \sum_{j=0}^{N}{}' \beta_{kj} T_j(x). \tag{2.6}$$

The Chebyshev coefficients of $f_0(x)$ and those for (2.5) and (2.6) are usually calculated by a curve fitting method of Lanczos [6]. A function $\phi(x)$ with an infinite Chebyshev expansion $\sum_{i=0}^{\infty} \alpha_i T_i(x)$ is approximated by a series of the form

$$\phi^{(N)}(x) = \sum_{i=0}^{N}{}' b_i T_i(x). \tag{2.7}$$

If $\phi^{(N)}(x) = \phi(x)$, then, by virtue of the orthogonality relations, satisfied by the Chebyshev polynomials for summation over the points $x_j = \cos j\pi/N$, we have

$$b_i = \bar{b}_i, \qquad i = 0, \ldots, N - 1; \qquad b_N = \tfrac{1}{2}\bar{b}_N, \tag{2.8a}$$

where

$$\bar{b}_i = \frac{2}{N} \sum_{j=0}^{N}{}'' \phi(x_j) T_i(x_j), \tag{2.8b}$$

where $\sum''$ indicates a sum with first and last terms halved. In this case we have $\alpha_i = b_i$, $i = 0, \ldots, N$, but in general the coefficients of $\phi^{(N)}(x)$ and $\phi(x)$ are related by the formula

$$\bar{b}_i = \alpha_i + \sum_{p=1}^{\infty} (\alpha_{2pN+i} + \alpha_{2pN-i}), \tag{2.9}$$

given by Clenshaw [4].

For (2.6) the function values required for evaluating the coefficients β_{kj} are evaluated by approximating the integral on the left-hand side by a quadrature formula.

On substituting (2.3), (2.4), (2.5) and (2.6) into (2.2), rearranging the resulting double sum, and equating coefficients of $T_j(x)$, we obtain

$$A_j = g_j + \lambda \sum_{k=0}^{N2}{}' \beta_{kj} a_k, \qquad j = 0, \ldots, N, \tag{2.10}$$

which may be written in the matrix form

$$\mathbf{A} = \mathbf{g} + \lambda B \mathbf{a}, \tag{2.10a}$$

where $\mathbf{A}$, $\mathbf{g}$ and $\mathbf{a}$ are vectors of the coefficients A_j, g_j and a_j, and B is the matrix with elements

$$B_{i0} = \tfrac{1}{2}\beta_{0i}; \qquad B_{ij} = \beta_{ji}; \qquad j = 1, \ldots, N2, \quad i = 0, \ldots, N.$$

We now consider the errors in the approximating processes used in obtaining the matrix equation. The truncation error in Chebyshev approximation of a function $\phi(x)$ is

$$\begin{aligned} R\phi(x) &= \phi(x) - \phi^{(N)}(x) \\ &= \sum_{i=0}^{\infty}{}' \alpha_i T_i(x) - \sum_{i=0}^{N}{}' b_i T_i(x) \\ &= \sum_{i=N+1}^{\infty} \alpha_i \big(T_i(x) - T_m(x)\big), \end{aligned}$$

where, from (2.9), $m = |i - 2jN|$ with j chosen so that $0 \leqq m \leqq N$. Since $|T_i(x) - T_m(x)| \leqq 2$ for $x \in [-1, 1]$, we have $|R\phi(x)| \leqq 2\sum_{i=N+1}^{\infty} |\alpha_i|$. If the function $\phi(x)$ has a rapidly convergent Chebyshev series and we can assume that $|\alpha_{N+k}| \leqq |a_N|/2^k$ for suitably chosen large N, then it follows that

$$|R\phi(x)| \leqq 2|\alpha_N| \sum_{i=1}^{\infty} 1/2^i = 2|\alpha_N|. \tag{2.11}$$

This assumption is not possible for functions which are even or odd or nearly so, as then alternate coefficients are zero or almost zero. To cover all types of functions, we replace $2|\alpha_N|$ in Eq. (2.11) by $|\alpha_{N-1}| + |\alpha_N|$ and use the inequality

$$|R\phi(x)| \leqq |\alpha_{N-1}| + |\alpha_N| \tag{2.12}$$

for estimating truncation error.

To evaluate the integrals on the left-hand side of (2.6) one could follow Elliott's method and approximate the kernel $K(x, y)$ by a truncated Chebyshev series of the form (1.4) and then use a formula to evaluate $\int_{-1}^{1} T_n(y)T_m(y)\,dy$ exactly. Alternatively one can evaluate the integrals using a quadrature formula of the type

$$\int_{-1}^{1} \phi(x)\,dx \doteq \sum_{i=1}^{M} w_i\phi(x_i). \tag{2.13}$$

The first method is essentially a special case of the second as the Chebyshev coefficients of $K(x, y)$ are calculated as linear combinations of values of $K(x, y)$. For M as in (1.4) and k as in (2.6) this method gives the exact value of the integral (2.6) if the integrand is a polynomial of degree $M + k$. Hence, one would expect about the same accuracy or better if one used a quadrature formula of type (2.13) which is exact for polynomials of degree $M + N$. If $M \geqq N$, then it can be expected that a Gaussian quadrature formula using M points, which is exact for polynomials of degree $2M - 1$, will give at least comparable accuracy to the above method.

A priori estimates of integration error are in general difficult to make whereas a posteriori estimates can be made in some cases if the integral is evaluated for several different numbers of points. For this purpose it has been found convenient to use a low order Gaussian formula over a number of equal subintervals of the interval of integration. In practice it is found that this type of formula gives comparable accuracy to a higher order Gaussian formula over the whole interval using the same total number of points. The method is convenient because firstly, for any specified number of subintervals the weights w_i and evaluation points x_i are readily obtainable from those of the basic Gaussian formula. Secondly, an estimate of the error in using a quadrature formula with M evaluation points can be computed by comparing values of the integrals obtained using M and $2M$ points respectively. The author has shown in [5] that if the basic Gaussian formula is exact for polynomials of degree $\leqq n$, then

$$I - I(M) \cong cM^{-(n+1)} \tag{2.14}$$

for sufficiently large M, where c is a constant, $I(M)$ denotes the approximate value of the integral using M points and I denotes the exact value. On applying (2.14), with M replaced by $2M$, the computable estimate

$$|I - I(M)| \approx \frac{2^{n+1}}{2^{n+1} - 1}|I(2M) - I(M)| \tag{2.15}$$

is obtained. Finally, the method can be used for nonlinear integral equations, whereas the method involving Chebyshev expansion of the kernel is not applicable in this case.

3. Error Analysis. In this section we derive an expression for the difference between the solution $f(x)$ of (2.1) and the ith computed iterate $f_i(x)$ of the Chebyshev iteration method of Section 2.

Let $\int_{-1}^{1} K(x, y)f_{i-1}(y)\,dy = Q_i(x) + E_i(x)$, where $Q_i(x)$ is the value of the integral given by a quadrature formula of type (2.13). Let

$$\bar{f}_i(x) = g(x) + \lambda \int_{-1}^{1} K(x, y)f_{i-1}(y)\,dy.$$

Then using the notation of Section 2 for truncated Chebyshev series and truncation error, we have

$$\begin{aligned} \bar{f}_i(x) - f_i(x) &= g(x) + \lambda \int_{-1}^{1} K(x, y)f_{i-1}(y)\,dy - g^{(N)}(x) - \lambda Q_i^{(N)}(x) \\ &= Rg(x) + \lambda RQ_i(x) + \lambda E_i(x) \\ &= R\bar{f}_i(x) + \lambda E_i^{(N)}(x). \end{aligned} \tag{3.1}$$

Also

$$\begin{aligned} f(x) - f_i(x) &= \bar{f}_i(x) - f_i(x) - (\bar{f}_i(x) - f(x)) \\ &= R\bar{f}_i(x) + \lambda E_i^{(N)}(x) + \lambda \int_{-1}^{1} K(x, y)(f(y) - f_{i-1}(y))\,dy \\ &= R\bar{f}_i(x) + \lambda E_i^{(N)}(x) + \lambda \int_{-1}^{1} K(x, y)(f(y) - f_i(y))\,dy \\ &\quad + \lambda \int_{-1}^{1} K(x, y)(f_i(y) - f_{i-1}(y))\,dy. \end{aligned} \tag{3.2}$$

Let $\eta_i = \max_{-1 \leqq x \leqq 1} |f(x) - f_i(x)|$ and suppose we have $|R\bar{f}_i(x)| \leqq et$, $|\lambda E_i^{(N)}(x)| \leqq ei$, $|f_i(x) - f_{i-1}(x)| \leqq eI$, and

$$\int_{-1}^{1} |K(x, y)|\,dy \leqq H.$$

Then we have

$$|f(x) - f_i(x)| \leqq et + ei + |\lambda| H(\eta_i + eI).$$

Hence

$$\eta_i \leqq et + ei + |\lambda| H(\eta_i + eI),$$

and so

$$\eta_i \leqq (et + ei + |\lambda| HeI)/(1 - |\lambda| H), \tag{3.3}$$

provided $1 - |\lambda| H > 0$.

4. Error Estimates. In this section we show how upper estimates et, ei, and eI for

Chebyshev truncation error, integration error, and iteration error (the difference between successive iterates) are obtained.

Suppose $\bar{f}_i(x)$ as defined in Section 3 has the Chebyshev series representation

$$\bar{f}_i(x) = \sum_{j=0}^{\infty}{}' \bar{A}_j T_j(x),$$

and that $E_i^{(N)}(x) = \sum_{j=0}^{\prime N} \varepsilon_j T_j(x)$. Then from (3.1) and (2.4) we have

$$\bar{f}_i^{(N)}(x) = \sum_{j=0}^{N}{}' (A_j + \lambda\varepsilon_j) T_j(x).$$

Hence from (2.9) it follows that

$$A_N + \lambda\varepsilon_N = \bar{A}_N + \sum_{p=1}^{\infty} \bar{A}_{(2p+1)N}, \tag{4.1a}$$

and

$$A_{N-1} + \lambda\varepsilon_{N-1} = \bar{A}_{N-1} + \sum_{p=1}^{\infty} (\bar{A}_{(2p+1)N-1} + \bar{A}_{(2p-1)N+1}). \tag{4.1b}$$

If the quadrature formula (2.13) is reasonably accurate we can assume that $\lambda\varepsilon_N$ and $\lambda\varepsilon_{N-1}$ are negligible compared with A_N and A_{N-1} respectively. Also, if $\bar{f}_i(x)$ has a rapidly convergent Chebyshev series, then on the right-hand side of (4.1a) we can neglect all terms in the sum, and on the right-hand side of (4.1b) we can neglect all terms in the sum except $\bar{A}_{N+1}$. Under these assumptions we have

$$A_N \cong \bar{A}_N, \qquad A_{N-1} \cong \bar{A}_{N-1} + \bar{A}_{N+1}.$$

If we make the additional assumption

$$|\bar{A}_{N+1}| \leqq \tfrac{1}{4}|\bar{A}_{N-1}|,$$

we have approximately, on using estimate (2.12),

$$|R\bar{f}_i(x)| \leqq \tfrac{4}{3}|A_{N-1}| + |A_N|.$$

For simplicity, we replace this by

$$|R\bar{f}_i(x)| \leqq et = |A_{N-1}| + |A_N|, \tag{4.2}$$

which is equivalent to allowing a slightly larger truncation error than the prescribed limit.

To obtain an upper estimate of integration error, we first determine the Chebyshev coefficients of a new ith iterate

$$f_i^*(x) = \sum_{j=0}^{N}{}' A_j^* T_j(x), \tag{4.3}$$

by using a quadrature formula with twice the number of points as were used for $f_i(x)$. Assuming the quadrature formula is exact for polynomials of degree $\leqq n$, we have from (2.15)

$$|\lambda E_i^{(N)}(x)| \approx \frac{2^{n+1}}{2^{n+1} - 1} |f_i^*(x) - f_i(x)|. \tag{4.4}$$

From (2.4), (4.3) and (4.4), and the fact that $|T_n(x)| \leqq 1$, we have approximately

$$|\lambda E^{(N)}(x)| \leqq \frac{2^{n+1}}{2^{n+1}-1} \sum_{j=0}^{N}{}' |A_j^* - A_j|, \tag{4.5}$$

but for simplicity we replace this by

$$|\lambda E^{(N)}(x)| \leqq ei = \sum_{j=0}^{N}{}' |A_j^* - A_j|. \tag{4.6}$$

For the iteration error we have

$$|f_i(x) - f_{i-1}(x)| = \left| \sum_{j=0}^{N}{}' (A_j - a_j) T_j(x) \right| \leqq \sum_{j=0}^{N}{}' |A_j - a_j| = eI. \tag{4.7}$$

5. Practical Procedure. In the practical application of the Chebyshev iteration method, during the first few iterations, the number of Chebyshev coefficients $N + 1$ is increased until the estimate (4.2) of Chebyshev truncation error becomes smaller than a prescribed limit.

The next iteration is carried out repeatedly with the number of quadrature points being doubled each time, until the integration error estimate (4.6) becomes smaller than the prescribed limit.

Iteration then proceeds with N and M fixed until the iteration error estimate (4.7) becomes smaller than the prescribed limit.

In the above the prescribed limit used is εM_f, where ε is a prescribed number and M_f is an estimate of $\max_{-1 \leqq x \leqq 1} |f_i(x)|$, where $f_i(x)$ is the current iterate.

In addition, limits are placed on N, M, and the number of iterations, and if any of these limits are exceeded, the value of the appropriate error estimate is stored and control passes to the next stage of the procedure. At the end of the computation ε is replaced by the largest of the error estimates if any of them exceeds ε.

6. Chebyshev Reduction of the Kernel. In this section we describe how the integral equation (2.1) can be written as an equation with a 'reduced' kernel (see Section 1), and how the latter equation can be solved by iteration.

The 'reduced' kernel is formed by subtracting a series of degenerate kernels from the kernel of the original equation. Given a set of linearly independent functions $Y_j(y)$, $j = 1, \ldots, M$, we can determine a set of functions $X_j(x)$, $j = 1, \ldots, M$, such that

$$\|KM\|^2 = \int_{-1}^{1} \int_{-1}^{1} KM^2(x, y)\, dy\, dx$$

is a minimum, where

$$KM(x, y) = K(x, y) - \sum_{j=1}^{M} X_j(x) Y_j(y).$$

As the Chebyshev iteration method gives the solution of the integral equation in terms of a Chebyshev series, it is convenient to reduce the kernel of the equation by subtracting a Chebyshev series of the form

$$\sum_{i=0}^{M-1} A_i(x) T_i(y),$$

with the functions $A_i(x)$ chosen to minimize

$$\int_{-1}^{1}\left(K(x, y) - \sum_{i=0}^{M-1} A_i(x)T_i(y)\right)^2 dy \tag{6.1}$$

for each x in $[-1, 1]$. This ensures that

$$\int_{-1}^{1}\int_{-1}^{1}\left(K(x, y) - \sum_{i=0}^{M-1} A_i(x)T_i(y)\right)^2 dy\,dx \tag{6.2}$$

is a minimum. To obtain the $A_i(x)$ we differentiate (6.1) with respect to each $A_i(x)$ for fixed x, and equate the derivatives to zero. This yields a system of linear equations for the $A_i(x)$. On solving this system it is found that the $A_i(x)$ are linear combinations of integrals of the form

$$\int_{-1}^{1} K(x, y)T_k(y)\,dy. \tag{6.3}$$

For example, for the case $M = 2$ we obtain

$$A_0(x) = \frac{1}{2}\int_{-1}^{1} K(x, y)\,dy; \qquad A_1(x) = \frac{3}{2}\int_{-1}^{1} K(x, y)T_1(y)\,dy.$$

From Eqs. (2.6) and (2.10a) we see that the coefficients of the truncated Chebyshev series for the $A_i(x)$ are linear combinations of elements of the matrix B defined in (2.10a).

If we let

$$KM(x, y) = K(x, y) - \sum_{i=0}^{M-1} A_i(x)T_i(y), \tag{6.4}$$

and let BM be the matrix corresponding to $KM(x, y)$ in the same way as B corresponds to $K(x, y)$, then it is found that the elements of BM are linear combinations of elements of B. For example, in the case $M = 2$ we find that

$$\begin{aligned} B2_{ij} &= B_{ij} + B_{i0}/(j^2 - 1), \qquad && j \text{ even},\\ &= B_{ij} + 3B_{i1}/(j^2 - 4), && j \text{ odd}. \end{aligned}$$

The quantity $\|KM\|^2$ (see (6.2)) is readily expressible in terms of $\|K\|^2$ and integrals of the form (6.3). For example, for $M = 2$ we obtain

$$\|K2\|^2 = \|K\|^2 - \int_{-1}^{1}\left\{\frac{1}{2}\left(\int_{-1}^{1} K(x, y)\,dy\right)^2 - \frac{3}{2}\left(\int_{-1}^{1} K(x, y)T_1(y)\,dy\right)^2\right\} dx.$$

We now consider the solution of the equation which results when the original Eq. (2.1) is rewritten in terms of the kernel $KM(x, y)$ of (6.4). The new equation is

$$f(x) = g(x) + \sum_{i=0}^{M-1} \lambda c_{i+1}A_i(x) + \lambda\int_{-1}^{1} KM(x, y)f(y)\,dy, \tag{6.5}$$

where the c_i are given by

$$c_i = \int_{-1}^{1} T_{i-1}(y)f(y)\,dy; \qquad i = 1, \ldots, M. \tag{6.6}$$

At this stage the c_i are undetermined, but if $\phi^{(i)}(x)$, $i = 0, \ldots, M$, are the solutions of the integral equations

$$\phi^{(i)}(x) = g^{(i)}(x) + \lambda \int_{-1}^{1} KM(x, y)\phi^{(i)}(y)\, dy, \qquad i = 0, \ldots, M, \tag{6.7}$$

where $g^{(0)}(x) = g(x)$, $g^{(i)}(x) = \lambda A_{i-1}(x)$, $i = 1, \ldots, M$, then

$$f(x) = \phi^{(0)}(x) + \sum_{i=1}^{M} c_i\phi^{(i)}(x) \tag{6.8}$$

is the solution of (6.5). Substituting (6.8) into (6.6), we obtain

$$c_i = \int_{-1}^{1} T_{i-1}(y)\phi^{(0)}(y)\, dy + \sum_{j=1}^{M} c_j \int_{-1}^{1} T_{i-1}(y)\phi^{(j)}(y)\, dy$$

and hence

$$\sum_{j=1}^{M} c_j\left(\delta_{ij} - \int_{-1}^{1} T_{i-1}(y)\phi^{(j)}(y)\, dy\right) = \int_{-1}^{1} T_{i-1}(y)\phi^{(0)}(y)\, dy, \qquad i = 1, \ldots, M. \tag{6.9}$$

To solve Eq. (6.5), we first solve the $M + 1$ integral equations (6.7) by the Chebyshev iteration method and then solve the M simultaneous linear equations (6.9) to obtain the c_i, and finally use (6.8) to obtain the required solution. The integrals of the type $\int_{-1}^{1} T_i(y)\phi^{(j)}(y)\, dy$, which occur in the coefficients of the linear equations (6.9), can be evaluated from a formula for $\int_{-1}^{1} T_i(y)T_j(y)\, dy$, after the Chebyshev coefficients of the functions $\phi^{(j)}(x)$ have been found by the Chebyshev iteration method.

The work involved in solving the integral equations (6.7) is not excessive as we need only calculate one matrix BM corresponding to the common kernel $KM(x, y)$, and, provided $\|KM\|$ is small, very few iterations will be required to obtain an accurate solution. If $\|KM\|$ can be made small using a small value of M, the work involved in solving the linear equations (6.9) is negligible. If a larger value of M is required to make $\|KM\|$ small, it may be better to solve the original integral equation (2.1) by a direct method such as Elliott's (see Section 1). In practice the author has restricted himself to reductions of order $M \leqq 3$.

Before carrying out any reduction of the kernel, it is possible to obtain approximate values for $\|KM\|$ for various values of M. From these values it is possible to compute theoretical estimates of the amount of work required to obtain the solution of the integral equation to a given accuracy for various orders of reduction. These estimates can then be used to select the optimal order of reduction.

7. Numerical Examples. Chebyshev reduction of the kernel by application of the Chebyshev iteration method was used to solve

(a) Love's equation (see Love [7]):

$$f(x) = 1 + \frac{1}{\pi}\int_{-1}^{1} \frac{f(y)\, dy}{1 + (x - y)^2};$$

(b) the Lichtenstein-Gershgorin equation (see [8]):

$$f(x) = 2\arctan(k \sin \pi x/(k^2(\cos \pi x + \cos^2 \pi x) + \sin^2 \pi x)) + \int_{-1}^{1} kf(y)\, dy/(k^2 + 1 - (k^2 - 1)\cos \pi(x + y)), \qquad k = 1.2,$$

(c) $$f(x) = e^{-(x+6)} + \int_{-1}^{1} (x+3)e^{(x+2)y-3} f(y)\,dy.$$

In examples (a) and (b) the solutions of the equations are even and odd functions respectively. The author has shown, [5], that it is possible to obtain the solution in these cases by computing only even or odd Chebyshev coefficients respectively. Only the odd columns of the iteration matrix BM are different for $M = 1$ and $M = 2$ and so for Eq. (a) the computations are the same as for these reductions. Similarly, for Eq. (b) the computations are the same for $M = 0$ and $M = 1$, and for $M = 2$ and $M = 3$.

For each example an accuracy limit of 10^{-6} was prescribed. The computations are summarized in Tables 1, 2, and 3. The computed solutions of Eqs. (a) and (b) were similar to those obtained by Elliott [3] who used the same accuracy limit. In Table 3 computed solutions of Eq. (c) are tabulated along with the exact solution e^x. For this equation certain error estimates did not fall below the prescribed limit; this is indicated in the summary of computation, and the values of the estimates are given below the summary. The maximum relative errors in the computed solutions are also given for comparison.

TABLE 1. *Love's equation* (a)

Order of Reduction	0	1	3
$\|KM\|^2$	0.2076	0.0042	0.0005
Final number of Chebyshev coefficients	7	8	9
Final number of Quadrature Points	24	30	30
Number of iterations	12	2	2
Average ratio of successive iteration error estimates	0.454	0.040	0.005

From the tables it appears that the effect of the higher order reductions in reducing the number of iterations is not as marked as one might expect from examining the norms $\|KM\|$ of the reduced kernels. This is because at least one or two iterations are required before the iterates settle down.

8. Application to Other Types of Second Kind Equations. The Chebyshev iteration method and the technique of Chebyshev reduction of the kernel have been applied to systems of Fredholm equations of the second kind and to Volterra equations. Only minor modifications to the methods given in Sections 2 and 6 are required. Details are given in [5].

TABLE 2. *Lichtenstein-Gershgorin equation* (b)

Order of Reduction	0	2
$\|KM\|^2$	0.00861	0.00357
Final number of Chebyshev Coefficients	16	16
Final number of Quadrature points	30	30
Number of iterations	4	4
Average ratio of successive iteration error estimates	0.091	0.037

Nonlinear integral equations have also been solved by a Chebyshev iteration method. The iteration equation in this case is not linear, but the practical procedure of Section 5 may still be used. A nonlinear equation of the form

$$f(x) = \int_{-1}^{1} K(x, y, f(x), f(y))\, dy \tag{8.1}$$

may be solved under certain conditions [1], [5] by an iteration of the form

$$f_i(x) = \int_{-1}^{1} K(x, y, f_{i-1}(x), f_{i-1}(y))\, dy, \tag{8.2}$$

starting with the approximation $f_0(x) \equiv 0$. Assuming the iterate $f_{i-1}(x)$ has been determined as a truncated Chebyshev series, the iteration (8.2) can be carried out by expanding the right-hand side into a truncated Chebyshev series by the Lanczos curve fitting method discussed in Section 2, and the use of a quadrature formula to evaluate the integrals. Values of $f_{i-1}(x)$ required for computing values of the integrand $K(x, y, f_{i-1}(x), f_{i-1}(y))$ may be obtained by the method of Clenshaw (see Section 1).

Effectively the same iteration may be carried out by representing the iterates $f_i(x)$ by their values at the points $x_k = \cos k\pi/N$, $k = 0, \ldots, N$, since these are the values used to calculate Chebyshev coefficients by the Lanczos method. With some manipulation it can be shown that the truncated Chebyshev series $\phi^{(N)}(x)$ of a function $\phi(x)$ may be expressed in the form

$$\phi^{(N)}(x) = \sum_{k=0}^{N}{}'' \alpha_k(x)\phi(x_k), \tag{8.3}$$

where

$$\begin{aligned} \alpha_k(x) &= (-1)^{k+1}(1 - x^2)^{1/2}\sin(N \arccos x)/(N(x - x_k)), && x \neq x_k, \\ &= 1, && x = x_k, \end{aligned}$$

TABLE 3. *Equation* (c)

Order of Reduction	0	1	2	3
$\|KM\|^2$	1.214	0.736	0.204	0.032
Final number of Chebyshev Coefficients	14	10	12	12
Final Number of Quadrature Points	96	72†	96‡	96
Number of iterations	30*	15	4	3
Average ratio of successive iteration error estimates	0.95	0.454	0.083	0.032

*Iteration error estimate 1.0×10^{-2}.
†Integration error estimate 3.6×10^{-4}.
‡Integration error estimate 5.8×10^{-5}.

	Computed Solution for reduction of order			
x	1	2	3	*True Solution*
−1.0	0.3678437	0.3678880	0.3678793	0.3678794
−0.8	0.4492850	0.4493394	0.4493288	0.4493290
−0.6	0.5487578	0.5488245	0.5488114	0.5488116
−0.4	0.6702540	0.6703358	0.6703198	0.6703200
−0.2	0.8186500	0.8187500	0.8187305	0.8187308
0.0	0.9999012	1.0000236	0.9999996	1.0000000
0.2	1.2212820	1.2214316	1.2214023	1.2214028
0.4	1.4916771	1.4918600	1.4918242	1.4918247
0.6	1.8219384	1.8221619	1.8221182	1.8221188
0.8	2.2253206	2.2255935	2.2255401	2.2255409
1.0	2.7180126	2.7183461	2.7182809	2.7182818
Maximum Relative Error	9.9×10^{-5}	2.4×10^{-5}	3.3×10^{-7}	

Using a formula of type (8.3), values of the iterate $f_{i-1}(x)$ required for evaluating the integrand may be computed in about the same time as that taken by Clenshaw's method. Chebyshev coefficients (apart from the $N - 1$st and Nth, which are required for estimating truncation error) need not be computed at all. Because of this the computing time per iteration is somewhat reduced.

It is not possible to use kernel reducing techniques on nonlinear integral equations. However, Anderson [9] describes some methods for accelerating the convergence of general iterative procedures, and these methods are very effective when applied to nonlinear integral equations.

The Chebyshev iteration described above was applied to the integral equation

$$f(x) = x + 0.5 \int_0^1 e^{-xy} f^2(y)\, dy, \tag{8.4}$$

which has also been solved by Haselgrove (unpublished work), who obtained a set of nonlinear algebraic equations by replacing the integral by a quadrature formula using 21 equally spaced points, and solved these by a method which he describes in [10]. An estimated accuracy of order 10^{-6} was achieved using 7 Chebyshev coefficients after 20 iterations. The final solution is tabulated together with Haselgrove's solution in Table 4.

TABLE 4

Solution of Eq. (8.4)

x	*Solution by Iteration Method*	*Haselgrove's Solution*
0.0	0.2791588	0.2793876
0.1	0.3608004	0.3609945
0.2	0.4437933	0.4439571
0.3	0.5280324	0.5281694
0.4	0.6134208	0.6135344
0.5	0.6998697	0.6999627
0.6	0.7872971	0.7873723
0.7	0.8756278	0.8756874
0.8	0.9647925	0.9648387
0.9	1.0547276	1.0547622
1.0	1.1453743	1.1453990

9. Acknowledgements. The author wishes to thank Dr. Joan Walsh of the University of Manchester for her helpful suggestions for the work and constructive criticism of the text of this paper.

Discipline of Applied Mathematics
School of Physical Sciences
Flinders University of South Australia
Bedford Park, South Australia 5042

1. F. G. Tricomi, *Integral Equations*, Pure and Appl. Math., vol. 5, Interscience, New York, 1957. MR **20** #1177.

2. L. Fox & E. T. Goodwin, "The numerical solution of non-singular linear integral equations," *Philos. Trans. Roy. Soc. London. Ser. A*, v. 245, 1953, pp. 501–534. MR **14,** 908.

3. D. Elliott, "A Chebyshev series method for the numerical solution of Fredholm integral equations," *Comput. J.*, v. 6, 1963/64, pp. 102–111. MR **27** #5386.

4. C. W. Clenshaw, *Chebyshev Series for Mathematical Functions*, National Physical Laboratory Mathematical Tables, vol. 5, H.M.S.O., London, 1962. MR **26** #362.

5. T. W. Sag, *Numerical Methods for the Solution of Integral Equations*, Ph.D. Thesis, University of Manchester, 1966.

6. C. Lanczos, *Tables of Chebyshev Polynomials* (*Introduction*), Nat. Bur. Standards Appl. Math. Series 9, U. S. Government Printing Office, Washington, D. C., 1952.

7. E. R. Love, "The electrostatic field of two equal circular co-axial conducting disks," *Quart. J. Mech. Appl. Math.*, v. 2, 1949, p. 430. MR **11,** 629.

8. J. Todd & S. E. Warschawski, *On the Solution of the Lichtenstein-Gershgorin Integral Equation in Conformal Mapping*. II: *Computational Experiments*, Nat. Bur. Standards Appl. Math. Ser., 42, U. S. Government Printing Office, Washington, D. C., 1955, pp. 31–44. MR **17,** 540.

9. D. G. Anderson, "Iterative procedures for non-linear integral equations," *J. Assoc. Comput. Mach.*, v. 12, 1965, pp. 547–560. MR **32** #1919.

10. C. B. Haselgrove, "The solution of non-linear equations and of differential equations with two-point boundary conditions," *Comput. J.*, v. 4, 1961/62, pp. 255–259. MR **23** #*B*3146.