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On Iteration Procedures for Equations of the First Kind, $Ax = y$, and Picard's Criterion for the Existence of a Solution*

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Abstract. Suppose that the (not identically zero) linear operator A, on a real Hilbert space H to itself, is compact, selfadjoint, and positive semidefinite; that y is a vector of H which is perpendicular to the null space of A; and that μ is a real number such that $0 < \mu < 2/||A||$. Then, the "iteration scheme" $x_{n+1} = x_n + \mu(y - Ax_n)$, $n = 0, 1, 2, \cdots$, yields a strongly convergent sequence of vectors $\{x_n\}_{n=0}^{\infty}$ if and only if "Picard's criterion" for the existence of a solution of $Ax = y$ holds (i.e., if and only if y is perpendicular to the null space of A, and $\sum_{k=1}^{\infty} (y, u_k)^2/\lambda_k^2 < \infty$, where the u_k and the λ_k are the orthonormalized eigenvectors, and the corresponding eigenvalues, of A, respectively). An analogous result holds when A is only required to be compact.

Introduction. The purpose of this paper is to show that various iteration procedures for solving linear Fredholm integral equations of the first kind,

$$y(t) = \int_a^b K(s, t)x(s)\, ds, \tag{1}$$

are equivalent to Picard's [1] fundamental necessary and sufficient condition for the existence of a solution (for a relevant discussion of equations of the first kind, see Smithies [2, pp. 164–166]). Iteration procedures for the solution of Fredholm integral equations of the first kind have been given by Landweber [3] and Fridman [4].

Instead of employing the language of integral equations, the following discussion will be phrased in the terminology of Hilbert space. In this context, (1) may be viewed as

$$y = Ax, \tag{2}$$

where y is a given vector in a real infinite-dimensional Hilbert space H, and A is a linear operator (that is, additive and homogeneous) on H to itself. In Section 1, A ($\neq 0$) will be supposed to be compact (that is, A maps bounded sets into compact sets, in the sense of strong convergence), selfadjoint (that is, A coincides with its adjoint A^*), and positive semidefinite (that is, $(Au, u) \geqq 0$ for all $u \in H$).

The "family" of iterative procedures, under consideration in Section 1, is described by Eq. (10) of Section 1 (see subsection (c_1) of that section). Given the vector y,

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which is perpendicular to the null space of A, and a number μ (which is suitably restricted, that is to say, specifically, such that one has $0 < \mu < 2/\lambda_1$, where $\lambda_1 = ||A||$), one is asked to construct, by recurrence, the sequence of vectors $\{x_n\}_{n=0}^{\infty}$, where $x_{n+1} = x_n + \mu(y - Ax_n)$, for $n = 0, 1, \ldots$. In Section 2, A ($\neq 0$) is only assumed to be compact. The "family" of iterative procedures, under consideration in Section 2, is described in subsection (c_2) of Section 2. Given the vector y, which is perpendicular to the null space of A^*, and a number μ (such that $0 < \mu < 2/\lambda_1$, where $\lambda_1^2 = ||A^*A||$), one is asked to construct, by recurrence, the sequence of vectors $\{x_n\}_{n=0}^{\infty}$, where $x_{n+1} = x_n + \mu A^*(y - Ax_n)$, for $n = 0, 1, \ldots$. In each instance, it is shown that the convergence of the iteration scheme is equivalent to Picard's fundamental necessary and sufficient condition for the existence of a solution (see subsection (d_1) of Section 1, and subsection (d_2) of Section 2, for a statement of "Picard's criterion"). It is solely the proof of the equivalence which is the purpose of the paper; the related question of the numerical instability of the solution (the variation of the solution with y) is not considered, although, admittedly, it is of paramount importance in numerical applications; consequently, no numerical applications have been carried out.

Essential use is made, in Section 2, of the fact that the operators A^*A and AA^* are compact, selfadjoint, and positive semidefinite (that is, they satisfy the hypotheses required of the operator A in Section 1). For this reason, it will be supposed, throughout the remainder of this introduction, that A ($\neq 0$) is compact, selfadjoint, and positive semidefinite.

In these circumstances, A is known to possess a countable set of eigenvalues, $\lambda_1 \geqq \lambda_2 \geqq \lambda_3 \geqq \ldots \geqq 0$, with $\lambda_1 > 0$, together with corresponding eigenvectors $u_1, u_2, u_3, \ldots$ in H, such that $Au_i = \lambda_i u_i$ and

$$\begin{aligned}(u_i, u_j) &= 0, \qquad i \neq j,\\ &= 1, \qquad i = j,\end{aligned}$$

for $i, j = 1, 2, 3, \ldots$. Then, Picard's necessary and sufficient condition for the existence of a solution of (2) may be phrased as follows: given a vector $y \in H$, there is a vector x such that $Ax = y$ if and only if

$$\sum_{k=1}^{\infty} \frac{(y, u_k)^2}{\lambda_k^2} < \infty \tag{3}$$

and $(y, u) = 0$ for all u such that $Au = 0$. It should be noticed that a typical difficulty, occurring throughout the discussion, is first encountered here. Namely, if the number of nonzero eigenvalues is finite, then the sum in (3) is to be understood to be only a *finite* sum, taken only over the nonzero eigenvalues. For simplicity, in order to avoid this difficulty, *it will be assumed throughout that the number of nonzero eigenvalues is infinite*, so that every $\lambda_k > 0$ ($k = 1, 2, \cdots$). However, all the arguments can be readily modified to take into account the case when the number of nonzero eigenvalues is finite.

As customary, $R(A)$ and $\eta(A)$ will denote the range of A and the null space of A, respectively; i.e.,

$$R(A) = \{z \in H \mid z = Au \text{ for some } u \in H\}$$

and $\eta(A) = \{z \in H \mid Az = 0\}$. It is known that (see, e.g., Taylor [5, p. 332, Theorem

6.2-G]) the entire space H is the direct sum of the closure of $R(A)$ and the null space of A; that is, for every $x \in H$, there exist unique vectors $x_{\bar{R}} \in \operatorname{cl}(R(A))$ and $x_N \in \eta(A)$ such that $x = x_{\bar{R}} + x_N$ and $(x_{\bar{R}}, x_N) = 0$. In symbols,

$$H = \operatorname{cl}(R(A)) \oplus \eta(A). \tag{4}$$

In the terminology just introduced, Picard's condition, which consists of two parts, may be interpreted as follows. Suppose that $y \in H$ satisfies the second part of Picard's condition, which states that $(y, u) = 0$ for all u such that $Au = 0$; or, in other words $y \perp \eta(A)$. Then, according to (4), this already means that $y \in \operatorname{cl}(R(A))$. If y also satisfies (3), which is the first part of Picard's condition, then, by Picard's theorem, y must belong to $R(A)$. Therefore, the net effect of requiring (3), over and beyond requiring the second part of Picard's condition, is to "transfer" y from $\operatorname{cl}(R(A))$ to $R(A)$. Thus, according to Picard, the sum (3) is divergent only when both $y \in \operatorname{cl}(R(A))$ and $y \notin R(A)$—if $y = y_{\bar{R}} + y_N$, then

$$\sum_{k=1}^{\infty} \frac{(y, u_k)^2}{\lambda_k^2} = \sum_{k=1}^{\infty} \frac{(y_{\bar{R}}, u_k)^2}{\lambda_k^2}.$$

In Section 1, frequent appeal will be made to the following two (essentially known) lemmas, in the precise form in which they are stated below. In order to make the paper as self-contained as possible, and to simplify the reading of the later arguments, the proofs will be given in full here, for the convenience of the reader.

LEMMA 1. *Suppose $x \in H$. Then there exist uniquely determined real numbers $\{c_k\}_{k=1}^{\infty}$ and a unique vector $x_N \in \eta(A)$, such that*

$$\sum_{k=1}^{\infty} c_k^2 < \infty$$

and

$$x = x_N + \sum_{k=1}^{\infty} c_k u_k.$$

Proof. First, from (4), there exist unique vectors $x_{\bar{R}} \in \operatorname{cl}(R(A))$ and $x_N \in \eta(A)$, such that

$$x = x_N + x_{\bar{R}}.$$

Thus, it only remains to show that

$$x_{\bar{R}} = \sum_{k=1}^{\infty} (x_{\bar{R}}, u_k) u_k.$$

Since, by Bessel's inequality,

$$\sum_{k=1}^{\infty} (x_{\bar{R}}, u_k)^2 \leqq (x_{\bar{R}}, x_{\bar{R}}) < \infty,$$

the vector

$$w = \sum_{k=1}^{\infty} (x_{\bar{R}}, u_k) u_k = \lim_{n\to\infty} \sum_{k=1}^{n} (x_{\bar{R}}, u_k) u_k$$

belongs to H; and, in particular, $w \in \operatorname{cl}(R(A))$, because w is the strong limit of the sequence of vectors

$$\left\{\sum_{k=1}^{n} (x_{\bar{R}}, u_k)u_k\right\}_{n=1}^{\infty}$$

in $R(A)$. For positive integers m and n, in view of the orthogonality of the eigenvectors, one has

$$\begin{aligned}\left(\sum_{k=1}^{n} (x_{\bar{R}}, u_k)u_k, u_m\right) &= \sum_{k=1}^{n} (x_{\bar{R}}, u_k)(u_k, u_m), \\ &= 0, \qquad n < m, \\ &= (x_{\bar{R}}, u_m), \qquad m \leqq n.\end{aligned}$$

Upon taking the strong limit, as $n \to \infty$, and using the continuity of the scalar product, it follows that

$$(w, u_m) = (x_{\bar{R}}, u_m);$$

that is,

$$(w - x_{\bar{R}}, u_m) = 0 \tag{5}$$

for every $m = 1, 2, \cdots$. Therefore $w - x_{\bar{R}} \in \mathrm{cl}\,(R(A))$.

Recall that

$$\lambda_i = \max_{v \neq 0;\, v \perp u_1, \cdots, u_{i-1}} \frac{(Av, v)}{(v, v)}, \tag{6}$$

for $i = 2, 3, \cdots$. From (5) and (6) it follows that $0 \leqq (A(w - x_{\bar{R}}), w - x_{\bar{R}}) \leqq \lambda_i(w - x_{\bar{R}}, w - x_{\bar{R}})$ for $i = 2, 3, \cdots$. Since $\lambda_i \to 0$ as $i \to \infty$, this gives that $(A(w - x_{\bar{R}}), w - x_{\bar{R}}) = 0$. Thus, using the "generalized Schwarz inequality" (see F. Riesz-B. Sz.-Nagy [6, p. 262]), for *any* $z \in H$, one has

$$|(A(w - x_{\bar{R}}), z)|^2 \leqq (A(w - x_{\bar{R}}), w - x_{\bar{R}})(Az, z) = 0;$$

so that

$$A(w - x_{\bar{R}}) = 0.$$

In other words, $w - x_{\bar{R}} \in \eta(A)$. But, as was seen earlier, from (5), both w and $x_{\bar{R}}$ belong to $\mathrm{cl}\,(R(A))$; and hence, also $w - x_{\bar{R}} \in \mathrm{cl}\,(R(A))$. Consequently, (4) gives that $w - x_{\bar{R}} = 0$, the desired conclusion.

LEMMA 2. *Suppose that* $\{\Gamma_k\}_{k=1}^{\infty}$ *is a sequence of nonnegative real numbers satisfying*

$$\sum_{k=1}^{\infty} \Gamma_k^2 < \infty;$$

and that $\{\gamma_{k,n}\}_{k,n=1}^{\infty}$ *is a double sequence of real numbers satisfying the two conditions*

$$\lim_{n\to\infty} \gamma_{k,n} = 0 \quad \textit{for } k = 1, 2, \cdots,$$

and

$$|\gamma_{k,n}| \leqq \Gamma_k \quad \textit{for } k, n = 1, 2, \cdots.$$

Then, for each $n = 1, 2, \cdots$, *the sum*

$$x_n = \sum_{k=1}^{\infty} \gamma_{k,n} u_k$$

is a vector in H, *and the sequence* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to zero.*

Proof. That $x_n \in H$ follows from the inequality

$$\sum_{k=1}^{\infty} \gamma_{k,n}^2 \leqq \sum_{k=1}^{\infty} \Gamma_k^2 < \infty .$$

Then, for *any* positive integer m,

$$(x_n, x_n) = \sum_{k=1}^{m} \gamma_{k,n}^2 + \sum_{k=m+1}^{\infty} \gamma_{k,n}^2 \tag{7}$$
$$\leqq \sum_{k=1}^{m} \gamma_{k,n}^2 + \sum_{k=m+1}^{\infty} \Gamma_k^2 .$$

Let $\epsilon > 0$. Choose a positive integer m_ϵ so large that

$$\sum_{k=m_\epsilon+1}^{\infty} \Gamma_k^2 < \frac{\epsilon^2}{2} .$$

There exists a positive integer N_ϵ such that, for all $n \geqq N_\epsilon$, one has

$$\gamma_{k,n}^2 < \frac{\epsilon^2}{2m_\epsilon}, \qquad k = 1, 2, \cdots, m_\epsilon .$$

Together with (7), the last two inequalities yield $(x_n, x_n) < \epsilon^2$, for all $n \geqq N_\epsilon$, as desired.

1. A $\not\equiv 0$, Compact, Selfadjoint, and Positive Semidefinite. The relationship between the various theorems concerning the equation $Ax = y$, and the procedures for solving this same equation, are illustrated schematically by means of the following diagram.

$$\begin{tikzcd}
\text{Existence of a Solution} \arrow[r, leftrightarrow, "(a_1)"] \arrow[d, leftrightarrow, "(d_1)"] & \text{Formula for the Inverse, plus } y \perp \eta(A) \arrow[d, leftrightarrow, "(b_1)"] \\
\text{Picard's Condition} \arrow[r, leftrightarrow, "(c_1)"] & \text{Iteration Scheme Converges, plus } y \perp \eta(A)
\end{tikzcd}$$

Throughout this section, the operator $A \not\equiv 0$ will be assumed to be compact, selfadjoint, and positive semidefinite. The real number μ will be supposed to be such that $0 < \mu < 2/\lambda_1$. The equivalence of some of the various items in the diagram will now be explained in detail.

(a_1) *Existence of a Solution* $\rightleftarrows$ *Formula for the Inverse, plus* $y \perp \eta(A)$.

THEOREM a_1. *Let* $y \in H$. *Then there exists an* $x \in H$, *with* $Ax = y$, *if and only if*
(α) *the sequence of partial sums*

$$\left\{ \sum_{k=0}^{n} (I - \mu A)^k y \right\}_{n=0}^{\infty}$$

converges strongly;

(β) $(y, u) = 0$ *for every* u *such that* $Au = 0$ (*in other words*, $y \perp \eta(A)$).

Proof. The following identity, which actually holds for any real number μ and any operator A, is the basis for this equivalence:

$$\mu A \sum_{k=0}^{n} (I - \mu A)^k = I - (I - \mu A)^{n+1}, \tag{8}$$

where I is the identity operator and n is any nonnegative integer. This identity follows at once from

$$\begin{aligned} (I - \mu A)^{n+1} &= I - \sum_{k=0}^{n} (I - \mu A)^k + \sum_{k=1}^{n+1} (I - \mu A)^k \\ &= I - \sum_{k=0}^{n} (I - \mu A)^k + (I - \mu A) \sum_{k=0}^{n} (I - \mu A)^k \\ &= I - \mu A \sum_{k=0}^{n} (I - \mu A)^k. \end{aligned}$$

First, suppose that a solution of $Ax = y$ exists, that is, there is an $x \in H$ such that $Ax = y$. Then (β) holds, because $y \in R(A)$ and $H = \mathrm{cl}\,(R(A)) \oplus \eta(A)$. Also, by (8),

$$\mu A \sum_{k=0}^{n} (I - \mu A)^k x = x - (I - \mu A)^{n+1} x;$$

but,

$$\begin{aligned} \mu A \sum_{k=0}^{n} (I - \mu A)^k x &= \mu \sum_{k=0}^{n} (I - \mu A)^k Ax \\ &= \mu \sum_{k=0}^{n} (I - \mu A)^k y; \end{aligned}$$

hence,

$$\mu \sum_{k=0}^{n} (I - \mu A)^k y = x - (I - \mu A)^{n+1} x.$$

(If it were true that $\|I - \mu A\| < 1$, then it would follow at once that $\sum_{k=0}^{n} (I - \mu A)^k y$ converges strongly, as $n \to +\infty$. However, it may be verified that $\|I - \mu A\| = 1$.)

By Lemma 1, one has

$$x = x_N + \sum_{k=1}^{\infty} c_k u_k,$$

where $Ax_N = 0$ and $\sum_{k=1}^{\infty} c_k^2 < \infty$. Hence,

$$\begin{aligned} (I - \mu A)^{n+1} x &= (I - \mu A)^{n+1} \left(x_N + \sum_{k=1}^{\infty} c_k u_k \right) \\ &= x_N + (I - \mu A)^{n+1} \sum_{k=1}^{\infty} c_k u_k \\ &= x_N + \sum_{k=1}^{\infty} c_k (1 - \mu \lambda_k)^{n+1} u_k, \end{aligned}$$

which, by Lemma 2 (putting $\gamma_{k,n} = c_k(1 - \mu\lambda_k)^n$ and $\Gamma_k = |c_k|$ for $k = 1, 2, \cdots$; and using $0 < \mu < 2/\lambda_1$), implies that the strong limit of $(I - \mu A)^{n+1}x$, as $n \to +\infty$, is x_N. Thus, in the sense of strong convergence,

$$\mu \lim_{n\to\infty} \sum_{k=0}^{n} (I - \mu A)^k y = \mu \sum_{k=0}^{\infty} (I - \mu A)^k y$$

$$= x - x_N = \sum_{k=1}^{\infty} c_k u_k,$$

so that (α) also holds. This equation is, in a sense, "a formula for finding an inverse to A at the vector y". Actually, this is a formula for the "principal part" of the inverse of A at y. Formally, this formula for the inverse follows from

$$A^{-1}y = \frac{\mu I}{I - (I - \mu A)} y = \mu \sum_{k=0}^{\infty} (I - \mu A)^k y.$$

Secondly, suppose that

$$\mu \sum_{k=0}^{\infty} (I - \mu A)^k y$$

converges strongly, and that also $y_N = 0$. Then, from (8), one has

$$\mu A \sum_{k=0}^{n} (I - \mu A)^k y = y - (I - \mu A)^{n+1} y.$$

The right-hand side in the last equation tends strongly, as $n \to +\infty$, to $y - y_N = y$. On the other hand, the left-hand side tends strongly, as $n \to +\infty$, to

$$A\left(\mu \sum_{k=0}^{\infty} (I - \mu A)^k y\right);$$

so that, finally,

$$A\left(\mu \sum_{k=0}^{\infty} (I - \mu A)^k y\right) = y.$$

(b_1) *Formula for the Inverse, plus $y \perp \eta(A) \rightleftarrows$ Iteration Scheme Converges, plus $y \perp \eta(A)$.*

THEOREM b_1. *Let $y \in H$ be such that $(y, u) = 0$ for every u such that $Au = 0$ (in other words, $y \perp \eta(A)$). Then the sequence of partial sums*

$$\left\{\sum_{k=0}^{n} (I - \mu A)^k y\right\}_{n=0}^{\infty}$$

converges strongly if and only if the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly, for every $x_0 \in H$, where

$$x_{n+1} = x_n + \mu(y - Ax_n) = (I - \mu A)x_n + \mu y, \qquad n = 0, 1, \cdots.$$

Proof. The following formula, which gives x_n explicitly in terms of x_0 and y, is the basis for this equivalence:

$$x_n = (I - \mu A)^n x_0 + \mu \sum_{k=0}^{n-1} (I - \mu A)^k y, \qquad n = 1, 2, \cdots. \tag{9}$$

This identity may be proved by mathematical induction. As in part (a_1), one also has that $(I - \mu A)^n x_0$ converges strongly to x_{0N}, the component of x_0 in the null space of A. Using this information, one then sees from the identity (9) that the sequence $\{x_n\}$ converges strongly if and only if the sequence of partial sums

$$\left\{\mu \sum_{k=0}^{n-1} (I - \mu A)^k y\right\}_{n=1}^{\infty}$$

converges strongly, that is, if and only if the infinite series

$$\mu \sum_{k=0}^{\infty} (I - \mu A)^k y$$

converges strongly.

(c_1) *Iteration Scheme Converges, plus* $y \perp \eta(A) \rightleftarrows$ *Picard's Condition.*

THEOREM c_1. *Let $y \in H$ be such that $(y, u) = 0$ for every u such that $Au = 0$ (in other words, $y \perp \eta(A)$). Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly, for every $x_0 \in H$, where*

$$x_{n+1} = x_n + \mu(y - Ax_n) = (I - \mu A)x_n + \mu y, \qquad n = 0, 1, \cdots, \tag{10}$$

if and only if

$$\sum_{k=1}^{\infty} \frac{(y, u_k)^2}{\lambda_k^2} < \infty. \tag{11}$$

Proof. Let $x_0 \in H$. Then, from Lemma 1, one has

$$x_0 = \sum_{k=1}^{\infty} c_{0,k} u_k + x_{0N},$$

where $Ax_{0N} = 0$. Hence, one has that $(x_{0N}, u_k) = 0$ for $k = 1, 2, \cdots$, since

$$(x_{0N}, u_k) = \left(x_{0N}, \frac{1}{\lambda_k} Au_k\right) = \frac{1}{\lambda_k}(Ax_{0N}, u_k) = 0,$$

for $k = 1, 2, \cdots$ (this last may be seen, without computation, from (4), since $x_{0N} \in \eta(A)$, while $u_k \in R(A)$). Likewise, x_n may be written as

$$x_n = \sum_{k=1}^{\infty} c_{n,k} u_k + x_{nN}, \qquad n = 1, 2, \cdots,$$

where $(x_{nN}, u_k) = 0$ for $k = 1, 2, \cdots$, and $Ax_{nN} = 0$. By the hypothesis that $y \perp \eta(A)$, one has that

$$y = \sum_{k=1}^{\infty} (y, u_k) u_k.$$

Use of (9), from part (b_1), now gives

$$\begin{aligned} x_n &= (I - \mu A)^n \left(x_{0N} + \sum_{k=1}^{\infty} c_{0,k} u_k\right) + \mu \sum_{j=0}^{n-1} (I - \mu A)^j \left(\sum_{k=1}^{\infty} (y, u_k) u_k\right) \\ &= x_{0N} + \sum_{k=1}^{\infty} c_{0,k} (I - \mu A)^n u_k + \sum_{k=1}^{\infty} \frac{(y, u_k)}{\lambda_k} \left[\mu A \sum_{j=0}^{n-1} (I - \mu A)^j\right] u_k. \end{aligned}$$

But, the identity (8) of part (a_1), and the fact that $Au_k = \lambda_k u_k$, allows one to rewrite this in the form

$$
\begin{aligned}
x_n &= x_{0N} + \sum_{k=1}^{\infty} c_{0,k}(1 - \mu\lambda_k)^n u_k \\
&\quad + \sum_{k=1}^{\infty} \frac{(y, u_k)}{\lambda_k} [I - (I - \mu A)^n] u_k \\
&= x_{0N} + \sum_{k=1}^{\infty} \left[\frac{(y, u_k)}{\lambda_k} + (1 - \mu\lambda_k)^n \left(c_{0,k} - \frac{(y, u_k)}{\lambda_k} \right) \right] u_k .
\end{aligned}
\tag{12}
$$

Hence, equating coefficients of u_k,

$$
c_{n,k} = \frac{(y, u_k)}{\lambda_k} + (1 - \mu\lambda_k)^n \left(c_{0,k} - \frac{(y, u_k)}{\lambda_k} \right), \qquad k = 1, 2, \cdots ; \tag{13}
$$

and, equating components in $\eta(A)$,

$$
x_{nN} = x_{0N}, \quad \text{for } n = 1, 2, \cdots .
$$

These preliminaries over, the proof proceeds as follows. Suppose, in the first place, that (11) holds. Then, by (11), the sum

$$
\sum_{k=1}^{\infty} \frac{(y, u_k)}{\lambda_k} u_k
$$

is an element of H (that is, the sequence of partial sums,

$$
\left\{ \sum_{k=1}^{m} \frac{(y, u_k)}{\lambda_k} u_k \right\}_{m=1}^{\infty},
$$

converges strongly in H). Hence, from (12),

$$
x_n = \sum_{k=1}^{\infty} \frac{(y, u_k)}{\lambda_k} u_k - x_{0N} = \sum_{k=1}^{\infty} (1 - \mu\lambda_k)^n \left(c_{0,k} - \frac{(y, u_k)}{\lambda_k} \right) u_k,
$$

which by Lemma 2 (putting

$$
\gamma_{k,n} = (1 - \mu\lambda_k)^n \left(c_{0,k} - \frac{(y, u_k)}{\lambda_k} \right)
$$

and

$$
\Gamma_k = \left| c_{0,k} = \frac{(y, u_k)}{\lambda_k} \right|
$$

for $k = 1, 2, \cdots$; and using $0 < \mu < 2/\lambda_1$), implies that the strong limit of x_n, as $n \to +\infty$, is

$$
x_{0N} + \sum_{k=1}^{\infty} \frac{(y, u_k)}{\lambda_k} u_k .
$$

Suppose, in the second place, that, for every $x_0 \in H$, the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly. Since

$$
x_n = \sum_{k=1}^{\infty} c_{n,k} u_k + x_{nN},
$$

where $(x_{nN}, u_k) = 0$ for $k = 1, 2, \cdots$ (from (4), since $x_{0N} \in \eta(A)$, while $u_k \in R(A)$), it follows that both of the sequences

$$\left\{\sum_{k=1}^{\infty} c_{n,k} u_k\right\}_{n=1}^{\infty} \quad \text{and} \quad \{x_{nN}\}_{n=1}^{\infty}$$

also converge strongly. Let

$$z = \sum_{k=1}^{\infty} (z, u_k) u_k$$

denote the strong limit of the sequence

$$\left\{\sum_{k=1}^{\infty} c_{n,k} u_k\right\}_{n=1}^{\infty}.$$

This implies that

$$\lim_{n\to\infty} c_{n,k} = (z, u_k)$$

for each $k = 1, 2, \cdots$. But, from (13), one has also

$$\lim_{n\to\infty} c_{n,k} = \frac{(y, u_k)}{\lambda_k}$$

for each $k = 1, 2, \cdots$. Consequently,

$$\sum_{k=1}^{\infty} \frac{(y, u_k)^2}{\lambda_k^2} = \sum_{k=1}^{\infty} (z, u_k)^2 < \infty,$$

which is (11).

(d_1) *Picard's Condition* $\rightleftarrows$ *Existence of a Solution.*

THEOREM d_1. *Let* $y \in H$. *Then*

$$\sum_{k=1}^{\infty} \frac{(y, u_k)^2}{\lambda_k^2} < \infty$$

and $(y, u) = 0$ *for every* u *such that* $Au = 0$ (*in other words*, $y \perp \eta(A)$), *if and only if there exists an* $x \in H$, *with* $Ax = y$.

This is a classical result of Picard [1] (see also Smithies [2, p. 164]) who deals with integral equations; and a detailed proof need not be reproduced here. However, if desired, a proof follows (see the diagram) merely by combining the proofs of Theorems a_1, b_1, and c_1, just by "following the implication arrows in the diagram in the right direction", although this may not be the most direct way of arriving at the result.

2. $A \not\equiv 0$ and Compact. The considerations of Section 1 may be extended to the equation $Ax = y$, where A ($\not\equiv 0$) is now a compact operator. As a matter of fact, all that one needs to suppose about A is that both AA^* and A^*A are compact. (However, as kindly pointed out to us by Professor Jerry Eisenfeld, this already implies that A itself is compact, by Exercise 18, p. 1260 of N. Dunford and J. T. Schwartz, *Linear Operators*, Part II, Interscience, New York, 1963.) The following diagram is applicable here. This diagram, of course, reduces to that of Section 1 when the operator is also selfadjoint and positive semidefinite.

$$\begin{array}{ccc} \text{Existence of a Solution} & \overset{(a_2)}{\longleftrightarrow} & \text{Formula for the Inverse, plus } y \perp \eta(A^*) \\ \updownarrow (d_2) & & \updownarrow (b_2) \\ \text{Picard's Condition} & \overset{(c_2)}{\longleftrightarrow} & \text{Iteration Scheme Converges, plus } y \perp \eta(A^*) \end{array}$$

Throughout this section, the operator $A \not\equiv 0$ will be assumed to be compact. The real number μ will be assumed to be such that $0 < \mu < 2/\lambda_1$, where λ_1^2 denotes the largest eigenvalue of the compact, selfadjoint, and positive semidefinite operator A^*A (that is, $\lambda_1^2 = ||A^*A||$). The equivalence of the various items in the diagram will now be explained in detail for this case. The basic idea behind all the proofs of the present section is to reduce all arguments relative to the compact operator A to arguments involving the operator A^*A, to which the results of Section 1 are then directly applicable.

(a_2) *Existence of Solution* $\rightleftarrows$ *Formula for the Inverse, plus* $y \perp \eta(A^*)$.

THEOREM a_2. *Let* $y \in H$. *Then there exists an* $x \in H$, *with* $Ax = y$, *if and only if*

(α) *the sequence of partial sums*

$$\left\{\sum_{k=0}^{n} (I - \mu A^* A)^k A^* y\right\}_{n=0}^{\infty} = \left\{A^* \sum_{k=0}^{n} (I - \mu A A^*)^k y\right\}_{n=0}^{\infty}$$

converges strongly;

(β) $(y, u) = 0$ *for every* u *such that* $A^*u = 0$ (*in other words*, $y \perp \eta(A^*)$).

Proof. It is to be noticed that the equality of the two sequences of partial sums which appear in (α) follows from

$$(A^* A)^k A^* = A^*(A A^*)^k, \qquad k = 0, 1, \cdots,$$

which may be verified by mathematical induction.

Suppose, first, that there exists an $x \in H$ such that $Ax = y$. Then $A^*Ax = A^*y$. From this equation, as a result of applying Theorem a_1 to the operator A^*A and the vector A^*y, one has that the sequence of partial sums

$$\left\{\sum_{k=0}^{n} (I - \mu A^* A)^k A^* y\right\}_{n=0}^{\infty}$$

converges strongly, which is condition (α). Also $(y, u) = (Ax, u) = (x, A^*u) = 0$, whenever $A^*u = 0$, which is condition (β).

Suppose, now, that conditions (α) and (β) of the present theorem hold. From this it follows that conditions (α) and (β) of Theorem a_1 are valid for the operator A^*A and the vector A^*y, namely:

(α') the sequence of partial sums

$$\left\{\sum_{k=0}^{n} (I - \mu A^* A)^k A^* y\right\}_{n=0}^{\infty}$$

converges strongly;

(β') $(A^*y, u) = 0$ for every u such that $A^*Au = 0$. Notice that (β') follows from $(A^*y, u) = (y, Au)$ and the fact that if $A^*Au = 0$, then the vector Au, being both in the range of A and in the null space of A^*, must be zero (since $H = \mathrm{cl}\,(R(A)) \oplus \eta(A^*)$, which is to be compared with (4). Hence, Theorem a_1 yields the existence of an x in H such that

$$A^*Ax = A^*y,$$

i.e., such that $A^*(Ax - y) = 0$; that is, $Ax - y$ is in the null space of A^*. But, Ax is in the range of A; while y, in view of hypothesis (β) of the present theorem, is in the closure of the range of A. Thus, the vector $Ax - y$, besides being in the null space of A^*, is also in the closure of the range of A. Hence, $Ax - y$ must be zero; in other words, $Ax = y$.

(b_2) *Formula for the Inverse, plus* $y \perp \eta(A^*) \rightleftarrows$ *Iteration Scheme Converges, plus* $y \perp \eta(A^*)$.

THEOREM b_2. *Let $y \in H$ be such that $(y, u) = 0$ for every u such that $A^*u = 0$ (in other words, $y \perp \eta(A^*)$). Then, the sequence of partial sums*

$$\left\{\sum_{k=0}^{n} (I - \mu A^*A)^k A^*y\right\}_{n=0}^{\infty} = \left\{A^* \sum_{k=0}^{n} (I - \mu AA^*)^k y\right\}_{n=0}^{\infty}$$

converges strongly if and only if the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly, for every $x_0 \in H$, where

$$x_{n+1} = x_n + \mu A^*(y - Ax_n), \qquad n = 0, 1, \cdots .$$

Proof. This result follows immediately from Theorem b_1, upon application of that result to the operator A^*A and the vector A^*y.

(c_2) *Iteration Scheme Converges, plus* $y \perp \eta(A^*) \rightleftarrows$ *Picard's Condition.*

THEOREM c_2. *Let $y \in H$ be such that $(y, u) = 0$ for every u such that $A^*u = 0$ (in other words, $y \perp \eta(A^*)$). Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly, for every $x_0 \in H$, where*

$$x_{n+1} = x_n + \mu A^*(y - Ax_n), \qquad n = 0, 1, \cdots ,$$

if and only if

$$\sum_{k=1}^{\infty} \frac{(y, u_k)^2}{\lambda_k^2} < \infty,$$

where $\lambda_1^2 > \lambda_2^2 > \cdots$ are the eigenvalues of A^*A and $u_1, u_2, \cdots$ are the eigenvectors of AA^*.

Proof. Theorem c_1, applied to the operator A^*A and the vector A^*y, gives: $\{x_n\}_{n=0}^{\infty}$ converges strongly, for every $x_0 \in H$, if and only if

$$\sum_{k=1}^{\infty} \frac{(A^*y, v_k)^2}{\lambda_k^4} < \infty,$$

where $v_1, v_2, \cdots$ are eigenvectors of the operator A^*A, with corresponding eigenvalues $\lambda_1^2, \lambda_2^2, \cdots$ (which are all positive). In particular, one can choose

$$v_k = \frac{1}{\lambda_k} A^*u_k, \qquad k = 1, 2, \cdots ,$$

since

$$A^*A\left(\frac{1}{\lambda_k}A^*u_k\right) = \frac{1}{\lambda_k}A^*(AA^*u_k) = \frac{1}{\lambda_k}A^*(\lambda_k^2 u_k) = \lambda_k^2\left(\frac{1}{\lambda_k}A^*u_k\right),$$

and

$$\begin{aligned}\left(\frac{1}{\lambda_j}A^*u_j, \frac{1}{\lambda_k}A^*u_k\right) &= \frac{1}{\lambda_j\lambda_k}(AA^*u_j, u_k)\\ &= \frac{\lambda_k}{\lambda_j}(u_j, u_k) = 0, \quad \text{for } j \neq k,\\ &= 1, \quad \text{for } j = k,\end{aligned}$$

for $j, k = 1, 2, \cdots$. With this particular choice of the v_k, one has

$$\frac{(A^*y, v_k)^2}{\lambda_k^4} = \frac{\left(y, \frac{1}{\lambda_k}Av_k\right)^2}{\lambda_k^2} = \frac{\left(y, \frac{1}{\lambda_k^2}AA^*u_k\right)^2}{\lambda_k^2} = \frac{(y, u_k)^2}{\lambda_k^2}, \qquad k = 1, 2, \cdots,$$

which leads to the desired result.

(d_2) *Picard's Condition* $\rightleftarrows$ *Existence of a Solution.*

THEOREM d_2. *Let* $y \in H$. *Then*

$$\sum_{k=1}^{\infty}\frac{(y, u_k)^2}{\lambda_k^2} < \infty$$

and $(y, u) = 0$ *for every* u *such that* $A^*u = 0$ (*in other words,* $y \perp \eta(A^*)$), *if and only if there exists an* $x \in H$ *with* $Ax = y$. (*Here,* $\lambda_1^2 > \lambda_2^2 > \cdots$ *are the eigenvalues of* A^*A *and* $u_1, u_2, \cdots$ *are the eigenvectors of* AA^*.)

A proof follows (see the diagram) merely by combining the proofs of Theorems a_2, b_2, and c_2, just by "following the implication arrows in the diagram in the right direction", although this may not be the most direct way of arriving at the result.

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