MATHEMATICS OF COMPUTATION, VOLUME 25, NUMBER 114, APRIL, 1971

A Fourth-Order Finite-Difference Approximation for the Fixed Membrane Eigenproblem*

By J. R. Kuttler

Abstract. The fixed membrane problem $\Delta u + \lambda u = 0$ in Ω, $u = 0$ on $\partial\Omega$, for a bounded region Ω of the plane, is approximated by a finite-difference scheme whose matrix is monotone. By an extension of previous methods for schemes with matrices of positive type, $O(h^4)$ convergence is shown for the approximating eigenvalues and eigenfunctions, where h is the mesh width. An application to an approximation of the forced vibration problem $\Delta u + qu = f$ in Ω, $u = 0$ in $\partial\Omega$, is also given.

1. Introduction. Let Ω be a bounded region of the plane with smooth boundary $\partial\Omega$. We consider the fixed membrane problem

$$\Delta u(x) + \lambda u(x) = 0, \quad x \in \Omega, \qquad u(x) = 0, \quad x \in \partial\Omega, \tag{1.1}$$

where Δ is the Laplacian. In [6], this problem was approximated by difference schemes which were of positive type in the interior of the region. Here, we consider a difference scheme for (1.1) which is only monotone. However, by appropriate modifications of the techniques of [6], we can prove that this scheme yields $O(h^4)$ approximations to the eigenvalues and eigenvectors of (1.1). The principal result is Theorem 8.1. An application to a forced vibration problem is also given in Section 9.

2. The Difference Scheme. Let $h > 0$ be given and define the mesh S_h by

$$\{(ih, jh) : i, j \text{ are integers}\}.$$

Points $x, y \in S_h$ will be called nearest neighbors if $|x - y| = h$, where we write

$$|x - y| = ((x_1 - y_1)^2 + (x_2 - y_2)^2)^{1/2}.$$

Let $\Omega_h^{(3)}$ be the set of points in $S_h \cap \Omega$ having at least one nearest neighbor not in Ω. One such point might be $x = (x_1, x_2)$ with $(x_1 - \alpha h, x_2), (x_1, x_2 - \beta h) \in \partial\Omega$ for $0 < \alpha, \beta \leqq 2$. If $(x_1 + h, x_2), (x_1 + 2h, x_2), (x_1, x_2 + h), (x_1, x_2 + 2h) \in \Omega$, we define

Received September 11, 1969, revised September 9, 1970.

AMS 1970 *subject classifications.* Primary 35P15, 35J05, 65N10, 65N15, 65N25; Secondary 15A48, 15A42.

Key words and phrases. Finite-differences, membrane, fixed membrane, eigenvalues, elliptic partial differential equations, monotone matrices, forced vibration problem, discrete Green's function.

* This work was supported by the Department of the Navy, Bureau of Naval Weapons, under Contract No. w-62-0604-c.

$$
(2.1)\qquad
\begin{aligned}
h^2 l_h(x, y) &= \frac{3-\alpha}{\alpha} + \frac{3-\beta}{\beta}, && y = x,\\
&= -\frac{2(2-\alpha)}{1+\alpha}, && y = (x_1 + h, x_2),\\
&= -\frac{2(2-\beta)}{1+\beta}, && y = (x_1, x_2 + h),\\
&= \frac{1-\alpha}{2+\alpha}, && y = (x_1 + 2h, x_2),\\
&= \frac{1-\beta}{2+\beta}, && y = (x_1, x_2 + 2h),\\
&= 0, && \text{otherwise.}
\end{aligned}
$$

Similar formulas apply at other points of $\Omega_h^{(3)}$. One special case may arise, as shown in Fig. 1, where $(x_1, x_2 + h)$, $(x_1, x_2 + 2h)$ do not lie in Ω.

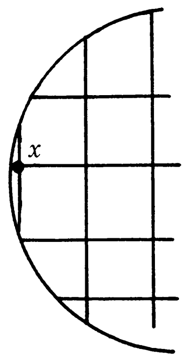

FIGURE 1

In such a case x would be excluded from the difference scheme altogether and the point $(x_1 + h, x_2)$ would be added to $\Omega_h^{(3)}$. For the new point, formula (2.1) would be used with $1 < \alpha \leqq 2$. If $\partial\Omega$ has bounded curvature and h is sufficiently small, there will be no difficulty with the new point.

Next, let $\Omega_h^{(2)}$ be those points of $S_h \cap \Omega$, not in $\Omega_h^{(3)}$ or excluded, which have a nearest neighbor in $\Omega_h^{(3)}$. For $x \in \Omega_h^{(2)}$ define

$$
(2.2)\qquad
\begin{aligned}
h^2 l_h(x, y) &= 4, && y = x,\\
&= -1, && |x - y| = h, \quad y \in S_h,\\
&= 0, && \text{otherwise.}
\end{aligned}
$$

Finally, let Ω_h' be those points of $S_h \cap \Omega$ not in $\Omega_h^{(2)} \cup \Omega_h^{(3)}$ or excluded. For $x \in \Omega_h'$ define

$$
(2.3)\qquad
\begin{aligned}
h^2 l_h(x, y) &= 5, && y = x,\\
&= -\tfrac{4}{3}, && |x - y| = h, \quad y \in S_h,\\
&= \tfrac{1}{12}, && |x - y| = 2h, \quad y \in S_h,\\
&= 0, && \text{otherwise.}
\end{aligned}
$$

Let $\Omega_h = \Omega_h' \cup \Omega_h^{(2)} \cup \Omega_h^{(3)}$. We approximate the Laplacian of a function u vanishing on $\partial\Omega$ by

$$-\Delta_h u(x) = \sum_{y \in \Omega_h} l_h(x, y)u(y), \qquad x \in \Omega_h. \tag{2.4}$$

Let us agree to use C as a generic constant, whose value may change at each usage, but which is always independent of h. Then, if also $u \in C^6(\bar{\Omega})$ (u has continuous sixth derivatives on the closure of Ω), it can be seen from Taylor series expansions that

$$\begin{aligned} |\Delta u(x) - \Delta_h u(x)| &\leqq Ch^4, \qquad x \in \Omega_h', \\ &\leqq Ch^2, \qquad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}. \end{aligned} \tag{2.5}$$

Bramble and Hubbard used Δ_h in [2] in approximating the Dirichlet problem for Poisson's equation.

Our difference scheme approximating (1.1) is

$$\Delta_h U_h(x) + \lambda_h U_h(x) = 0, \qquad x \in \Omega_h. \tag{2.6}$$

Problem (2.6) is equivalent to finding the eigenvalues and eigenvectors of the matrix $[l_h(x, y)]_{x, y \in \Omega_h}$. In the next section, we develop some tools to use in studying this matrix which, however, have some independent interest.

3. Monotone Matrices. Let $A = (a_{ij})$ be an $n \times n$ matrix. We say $A \geqq 0$ if each $a_{ij} \geqq 0$ and $A \leqq B$ if $B - A \geqq 0$. The matrix A is *monotone* if $Ax \geqq 0$ implies $x \geqq 0$ for all x. Thus, A is monotone if and only if A^{-1} exists and $A^{-1} \geqq 0$. An easily recognized type of monotone matrix is a matrix of *positive type*. The matrix A is of positive type if A is indecomposable, the diagonal of A is positive, the off-diagonal elements negative, and the row sums are nonnegative with at least one strictly positive. The following theorem is due to Price [8]:

THEOREM 3.1. *A is monotone if and only if there exists M monotone such that*

(i) $M^{-1}(M - A) \geqq 0$,

(ii) $\rho(M^{-1}(M - A)) < 1$.

Here ρ denotes spectral radius, the maximum of the moduli of the eigenvalues. Here and in the corollaries, the "only if" part is trivial: take $M = A$. This theorem generalizes Theorem 2.7 of Bramble and Hubbard [2]. There are a number of important corollaries:

COROLLARY 3.2. *A is monotone if and only if there exists M monotone such that*

(i) $M \geqq A$,

(ii) $\rho(M^{-1}(M - A)) < 1$.

COROLLARY 3.3. *A is monotone if and only if there exists M monotone and $x > 0$ such that*

(i) $M \geqq A$,

(ii) $Ax > 0$.

Proof. By the Gerschgorin circle theorem (see [7, p. 152]),

$$\rho(M^{-1}(M - A)) \leqq \max_i \, [M^{-1}(M - A)x]_i/x_i < 1,$$

since

$$0 \leqq [M^{-1}(M - A)x]_i = x_i - [M^{-1}Ax]_i < x_i,$$

because $Ax > 0$, $M^{-1} \geqq 0$ and no row of M^{-1} can be all zero.

COROLLARY 3.4. *A is monotone if and only if there exists M monotone and* $x \geqq 0$ *such that*

(i) $M \geqq A$,

(ii) $Ax > 0$.

Proof. Let $\delta = \min_i [Ax]_i > 0$ and let $\epsilon = \delta/(2 \max_i \sum_j |a_{ij}|)$. Then $x + \epsilon > 0$ and $A(x + \epsilon) > 0$, so the hypotheses of Corollary 3.4 are satisfied.

COROLLARY 3.5. *A is monotone if and only if there exist* M_1, M_2 *monotone such that*

$$M_1 \leqq A \leqq M_2.$$

Proof. Let x be such that $M_1 x$ is the vector with all components 1. Since M_1 is monotone, x exists and $x \geqq 0$. Also, $Ax \geqq M_1 x > 0$, so the hypotheses of Corollary 3.4 are satisfied.

COROLLARY 3.6. *A is monotone if there is* $\alpha > 0$ *such that* $A + \alpha I$ *is monotone and every eigenvalue* λ *of A has positive real part.*

Proof. Apply Corollary 3.2. We need only show $\rho((A + \alpha I)^{-1}) < \alpha^{-1}$. But $\rho((A + \alpha I)^{-1}) = 1/\min_\lambda |\alpha + \lambda|$, where λ runs over the eigenvalues of A.

At this time, we also note the following:

LEMMA 3.7. *If the partitioned matrix*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

with A invertible has inverse

$$\begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$$

then $W - A^{-1} = -XCA^{-1}$. *In particular, if* $X \geqq 0$, $A^{-1} \geqq 0$, $C \leqq 0$, *then* $A^{-1} \leqq W$.

Proof. Since

$$\begin{bmatrix} W & X \\ Y & Z \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

we have $WA + XC = I$. Multiply on the right by A^{-1}.

4. Discrete Green's Functions. The main tools in our investigations will be discrete analogues of Green's function. These are inverses of matrices related to $[h^2 l_h(x, y)]_{x,y \in \Omega_h}$ and their nonnegativity is crucial to the investigation. This will be established, using results of the previous section.

We define then

$$(4.1) \quad -\Delta_{h,x} g_h(x, y) = h^{-2}\, \delta(x, y),\ x \in \Omega_h' \cup \Omega_h^{(2)}, \quad g_h(x, y) = \delta(x, y),\ x \in \Omega_h^{(3)},$$

for all $y \in \Omega_h$. This is the discrete Green's function considered by Bramble and Hubbard in [2, Eq. (4.5)]. From (4.1), we see that the matrix $[g_h(x, y)]_{x,y \in \Omega_h}$ is the inverse of the partitioned matrix

$$\mathfrak{M} \equiv \begin{bmatrix} A & B \\ 0 & I \end{bmatrix},$$

where $A = [h^2 l_h(x, y)]_{x,y\in\Omega_h'\cup\Omega_h^{(2)}}$, $B = [h^2 l_h(x, y)]_{x\in\Omega_h'\cup\Omega_h^{(2)},y\in\Omega_h^{(3)}}$, and I is the identity on $\Omega_h^{(3)} \times \Omega_h^{(3)}$. It also follows from Lemma 3.7 that the matrix $[g_h(x, y)]_{x,y\in\Omega_h'\cup\Omega_h^{(2)}}$ is the inverse of A. In [2], it was shown that

$$g_h(x, y) \geqq 0, \qquad x, y \in \Omega_h, \tag{4.2}$$

i.e., $\mathfrak{M}$ is monotone. Since g_h is the inverse, it follows that, for any function W defined on Ω_h, all $x \in \Omega_h$,

$$W(x) = h^2 \sum_{y\in\Omega_h'\cup\Omega_h^{(2)}} g_h(x, y)[-\Delta_h W(y)] + \sum_{y\in\Omega_h^{(3)}} g_h(x, y) W(y). \tag{4.3}$$

This is analogous to Poisson's formula. In [2], the following properties were proved of g_h:

$$\sum_{y\in\Omega_h^{(3)}} g_h(x, y) \leqq 1, \tag{4.4}$$

$$\sum_{y\in\Omega_h^{(2)}} g_h(x, y) \leqq C, \tag{4.5}$$

$$h^2 \sum_{y\in\Omega_h} g_h(x, y) \leqq C, \tag{4.6}$$

for all $x \in \Omega_h$. Using these in (4.3), we have the inequality

$$\max_{\Omega_h} |W| \leqq C\left[\max_{\Omega_h'} |\Delta_h W| + h^2 \max_{\Omega_h^{(2)}} |\Delta_h W|\right] + \max_{\Omega_h^{(3)}} |W|. \tag{4.7}$$

Now, on $\Omega_h^{(3)}$, we have

$$W(x) = \left[-h^2\Delta_h W(x) - h^2 \sum_{y\in\Omega_h;y\neq x} l_h(x, y) W(y)\right] \Big/ h^2 l_h(x, x),$$

and from this and (2.1), we see that

$$\max_{\Omega_h^{(3)}} |W| \leqq Ch^2 \max_{\Omega_h^{(3)}} |\Delta_h W| + \theta \max_{\Omega_h} |W|, \tag{4.8}$$

where

$$\theta = \max_{x\in\Omega_h^{(3)}} \sum_{y\in\Omega_h;y\neq x} |l_h(x, y)|/l_h(x, x) < 1.$$

Putting (4.8) into (4.7) and rearranging, we have

$$\max_{\Omega_h} |W| \leqq C\left[\max_{\Omega_h'} |\Delta_h W| + h^2 \max_{\Omega_h^{(2)}\cup\Omega_h^{(3)}} |\Delta_h W|\right]. \tag{4.9}$$

Let us now use (4.7) to estimate $W = \Phi_h - \varphi$ where φ is the torsion function defined by $\Delta\varphi = -1$ on Ω, $\varphi = 0$ on $\partial\Omega$ and $\Phi_h(x) = h^2 \sum_{y\in\Omega_h} g_h(x, y)$, which satisfies $\Delta_h \Phi_h = -1$ on $\Omega_h' \cup \Omega_h^{(2)}$. If $\partial\Omega$ is sufficiently smooth, φ satisfies (2.5) and we see from (4.7) that

$$\max_{\Omega_h} |\Phi_h - \varphi| \leqq Ch^4 + \max_{\Omega_h^{(3)}} |\Phi_h - \varphi| \leqq Ch^4 + \max_{\Omega_h^{(3)}} |\Phi_h| + \max_{\Omega_h^{(3)}} |\varphi|.$$

Now, $\varphi = 0$ on $\partial\Omega$, so $|\varphi(x)| \leqq Ch$ for $|x - \partial\Omega| = \min_{y\in\partial\Omega} |x - y| \leqq Ch$. Also, $\Phi_h = h^2$ on $\Omega_h^{(3)}$ by definition. Hence,

$$|\Phi_h(x)| \leqq |\varphi(x)| + \max_{\Omega_h} |\Phi_h - \varphi| \leqq Ch$$

for $|x - \partial\Omega| \leqq Ch$, i.e.,

$$h^2 \sum_{y \in \Omega_h} g_h(x, y) \leqq Ch. \tag{4.10}$$

Next, we consider the function

$$f_h(x, y) = C_1 - C_2 \log(|x - y|^2 + h^2).$$

It is easily verified that

$$-\Delta_{h,x} f_h(x, y) \geqq 0, \qquad x \in \Omega_h' \cup \Omega_h^{(2)}, \quad y \neq x,$$
$$-\Delta_{h,x} f_h(x, y) \geqq h^{-2}, \qquad x \in \Omega_h' \cup \Omega_h^{(2)}, \quad y = x,$$

provided $C_2 \geqq \frac{1}{4} \log 2$. If we choose

$$C_1 = C_2 \max_{x, y \in \Omega_h} \log(|x - y|^2 + h^2),$$

then $f_h(x, y) \geqq 0$ for $x, y \in \Omega$. Thus, we see that

$$\mathfrak{M}(f_h - g_h) \geqq 0,$$

and, since $\mathfrak{M}$ is monotone,

$$0 \leqq g_h(x, y) \leqq C_1 - C_2 \log(|x - y|^2 + h^2). \tag{4.11}$$

Analogous inequalities to (4.11) are proved by Bramble and Thomée in [3] for discrete Green's functions of positive-type operators. Here, we see monotonicity was sufficient.

An easy consequence of (4.11) is

$$h^2 \sum_{y \in \Omega_h} [g_h(x, y)]^2 \leqq C. \tag{4.12}$$

5. More Inequalities for Green's Functions. This section will be devoted to derivations of some inequalities of more difficulty than those of the previous section.

Recall that $\mathfrak{G} = [g_h(x, y)]_{x, y \in \Omega_h}$ is the inverse of $[h^2 l_h(x, y)]_{x, y \in \Omega_h}$.

The inequality which we next wish to derive is

$$\sum_{y \in \Omega_h''} g_h(x, y) \leqq C \tag{5.1}$$

for all $x \in \Omega_h$, where $\Omega_h'' = \{x \in \Omega_h' : l_h(x, y) \neq 0 \text{ for some } y \in \Omega_h^{(2)} \cup \Omega_h^{(3)}\}$. The method of proof is the matrix splitting technique employed by Bramble and Hubbard in [2]. The analysis which follows is regrettably detailed.

Let us write

$$\mathfrak{G} = [I - H_1 - H_2]^{-1} \tilde{D}^{-1}, \tag{5.2}$$

where $\tilde{D}$ is the diagonal matrix with

$$\begin{aligned} \tilde{d}_{xx}^{-1} &= 1, && x \in \Omega_h^{(3)}, \\ &= \tfrac{1}{4}, && x \in \Omega_h^{(2)}, \\ &= \tfrac{1}{5}, && x \in \Omega_h', \end{aligned}$$

and

$$\begin{aligned}[H_1]_{xy} &= \tfrac{2}{15}, & x &\in \Omega_h', & |x-y| &= h,\\ &= \tfrac{1}{8}, & x &\in \Omega_h^{(2)}, & |x-y| &= h,\\ &= 0, & &\text{otherwise},\\ [H_2]_{xy} &= \tfrac{2}{15}, & x &\in \Omega_h', & |x-y| &= h,\\ &= -\tfrac{1}{60}, & x &\in \Omega_h', & |x-y| &= 2h,\\ &= \tfrac{1}{8}, & x &\in \Omega_h^{(2)}, & |x-y| &= h,\\ &= 0, & &\text{otherwise}.\end{aligned}$$

Let us define the diagonal matrix D by

$$\begin{aligned}(d_{xx})^{-1} = \sum_{y\in\Omega_h} (I - H_1)_{xy} &= \tfrac{7}{15}, & x &\in \Omega_h',\\ &= \tfrac{1}{2}, & x &\in \Omega_h^{(2)},\\ &= 1, & x &\in \Omega_h^{(3)},\end{aligned}$$

so that $D(I - H_1)$ has row sums one, i.e.,

$$\sum_{y\in\Omega_h} [D(I - H_1)]_{xy} = \sum_{y\in\Omega_h} [(I - H_1)^{-1}D^{-1}]_{xy} = 1. \tag{5.3}$$

We write $[I - H_1 - H_2] = [D^{-1}(I - H)][D(I - H_1)]$, where $H = DH_2(I - H_1)^{-1}D^{-1}$. Thus, by (5.3),

$$\begin{aligned}\sum_{y\in\Omega_h} [D^{-1}(I - H)]_{xy} &= \sum_{y,z\in\Omega_h} [D^{-1}(I - H)]_{xy}[D(I - H_1)]_{yz}\\ &= \sum_{z\in\Omega_h} [I - H_1 - H_2]_{xz} = 0, \qquad x \in \Omega_h' \cup \Omega_h^{(2)},\\ &= 1, \qquad x \in \Omega_h^{(3)}.\end{aligned} \tag{5.4}$$

Now, we consider the characteristic function of Ω_h':

$$\begin{aligned}\chi(x) &= 1, & x &\in \Omega_h',\\ &= 0, & x &\in \Omega_h^{(2)} \cup \Omega_h^{(3)}.\end{aligned}$$

Then

$$\begin{aligned}1 \geqq \chi(x) &= \{[(I - H)^{-1}D][D^{-1}(I - H)\chi]\}_x\\ &= \sum_{y\in\Omega_h'} [(I - H)^{-1}D]_{xy}[D^{-1}(I - H)\chi]_y\\ &\quad + \sum_{y\in\Omega_h^{(2)}\cup\Omega_h^{(3)}} [(I - H)^{-1}D]_{xy}[D^{-1}(I - H)\chi]_y\\ &= \sum_{y\in\Omega_h'} [(I - H)^{-1}D]_{xy} \sum_{z\in\Omega_h} [D^{-1}(I - H)]_{yz}\\ &\quad - \sum_{y\in\Omega_h'} [(I - H)^{-1}D]_{xy}[D^{-1}(I - H)(1 - \chi)]_y\\ &\quad + \sum_{y\in\Omega_h^{(2)}\cup\Omega_h^{(3)}} [(I - H)^{-1}D]_{xy}[D^{-1}(I - H)\chi]_y.\end{aligned}$$

By (5.4), the first term vanishes. Using the definitions of H and χ, this can be written as

$$\begin{aligned}\sum_{y\in\Omega_h'}[(I-H)^{-1}D]_{xy}\sum_{y\in\Omega_h^{(2)}\cup\Omega_h^{(3)}}[H_2(I-H_1)^{-1}D^{-1}]_{yz}\\ -\sum_{y\in\Omega_h^{(2)}\cup\Omega_h^{(3)}}[(I-H)^{-1}D]_{xy}\sum_{y\in\Omega_h'}[H_2(I-H_1)^{-1}D^{-1}]_{yz}\leqq 1.\end{aligned}\tag{5.5}$$

Now, we estimate the factors in each term. First, note that $(I-H)^{-1}\geqq 0$. This is not obvious, but follows from $H\geqq 0$ and $\rho(H)<1$. That $H\geqq 0$, is due to $0\leqq H_2(I-H_1)^{-1}=H_2+H_2H_1+\cdots$, since the negative terms in H_2 are cancelled by positive terms in H_2H_1 as in [2]. That $\rho(H)<1$ is due to $\rho(H)=\rho((I-H_1)^{-1}H_2)<1$, since the row sums of

$$\begin{aligned}(I-(I-H_1)^{-1}H_2)&=(I-H_1)^{-1}(I-H_1-H_2)\\&=(I-H_1-H_2)+H_1(I-H_1-H_2)+\cdots\end{aligned}$$

are positive. Again negative row sums of $(I-H_1-H_2)$ are cancelled by corresponding positive row sums of $H_1(I-H_1-H_2)$.

Next, for $y\in\Omega_h^{(2)}\cup\Omega_h^{(3)}$,

$$\begin{aligned}&\sum_{z\in\Omega_h'}[H_2(I-H_1)^{-1}D^{-1}]_{yz}\\&\qquad\leqq\sum_{z\in\Omega_h}[H_2(I-H_1)^{-1}D^{-1}]_{yz}=\sum_{z\in\Omega_h}[D^{-1}-D^{-1}(I-H)]_{yz}\\&\qquad\leqq 1-\sum_{z\in\Omega_h}[D^{-1}(I-H)]_{yz}=1-\sum_{z\in\Omega_h}[I-H_1-H_2]_{yz}\leqq 1.\end{aligned}$$

Now, we consider, for $y\in\Omega_h''$, the term

$$\sum_{z\in\Omega_h^{(2)}\cup\Omega_h^{(3)}}[H_2(I-H_1)^{-1}D^{-1}]_{yz}.\tag{5.6}$$

Expanding the summand in a Neumann series, it becomes

$$[(H_2+H_2H_1+H_2H_1^2+\cdots)D^{-1}]_{yz}.$$

If $y\in\Omega_h''$, $z\in\Omega_h^{(2)}\cup\Omega_h^{(3)}$ is such that $|y-z|=2h$, then $[H_2]_{yz}=-1/60$. However, let x be the point such that $|y-x|=|x-z|=h$. Then $[H_2H_1]_{yz}$ contains the term $[H_2]_{yx}[H_1]_{xz}=4/225$. Similarly, each negative term in $H_2H_1^k$ is compensated for by a positive term in $H_2H_1^{k+1}$. Thus, for $y\in\Omega_h''$,

$$\sum_{z\in\Omega_h^{(2)}\cup\Omega_h^{(3)}}[H_2(I-H_1)^{-1}D^{-1}]_{yz}\geqq\left[-\frac{1}{60}+\frac{4}{225}\right]\cdot\frac{1}{2}=\frac{1}{1800}.$$

It follows from (5.5) and the above that

$$\sum_{y\in\Omega_h''}[(I-H)^{-1}D]_{xy}\leqq 1800\left\{1+\sum_{y\in\Omega_h^{(2)}\cup\Omega_h^{(3)}}[(I-H)^{-1}D]_{xy}\right\}.\tag{5.7}$$

By similar reasoning, using the function

$$\begin{aligned}\chi(x)&=1,\qquad x\in\Omega_h'\cup\Omega_h^{(2)},\\&=0,\qquad x\in\Omega_h^{(3)},\end{aligned}$$

it can be shown that $\sum_{y\in\Omega_h^{(3)}}[(I-H)^{-1}D]_{xy}\leqq C$. The argument is carried out in [2, Lemma 3.3]. Finally, we note from (5.4) that

$$(5.8) \qquad 1 = \sum_{y\in\Omega_h} [(I - H)^{-1}D]_{xy} \sum_{s\in\Omega_h} [D^{-1}(I - H)]_{ys} = \sum_{y\in\Omega_h^{(s)}} [(I - H)^{-1}D]_{xy}.$$

Combining the above with (5.7), we see that

$$(5.9) \qquad \sum_{y\in\Omega_h''} [(I - H)^{-1}D]_{xy} \leqq C.$$

From (5.2) and (5.3), we finally have

$$\begin{aligned}\sum_{y\in\Omega_h''} g_h(x, y) &= \sum_{y\in\Omega_h''} [(I - H_1 - H_2)^{-1}\tilde{D}^{-1}]_{xy} = \tfrac{1}{5} \sum_{y\in\Omega_h''} [(I - H_1 - H_2)^{-1}]_{xy} \\ &= \tfrac{1}{5} \sum_{y\in\Omega_h''} \sum_{z\in\Omega_h'\cup\Omega_h^{(2)}} \{[D(I - H_1)]^{-1}\}_{xz}[(I - H)^{-1}D]_{zy} \\ &\leqq \tfrac{1}{5} \max_{z\in\Omega_h'\cup\Omega_h^{(2)}} \sum_{y\in\Omega_h''} [(I - H)^{-1}D]_{zy},\end{aligned}$$

or, from (5.9),

$$(5.10) \qquad \sum_{y\in\Omega_h''} g_h(x, y) \leqq C,$$

the desired estimate.

We next define another Green's function G_h by

$$(5.11) \qquad -\Delta_h G_h(x, y) = h^{-2}\, \delta(x, y), \qquad x, y \in \Omega_h.$$

Although G_h may not be nonnegative, it is a perturbation of g_h. We have

THEOREM 5.1. *For any mesh function S,*

$$(5.12) \qquad \begin{aligned}&\max_{x\in\Omega_h} \sum_{y\in\Omega_h} |[G_h(x, y) - g_h(x, y)]S(y)| \\ &\leqq C\left[\max_{\Omega_h^{(s)}} |S| + \max_{x\in\Omega_h''\cup\Omega_h^{(2)}\cup\Omega_h^{(s)}} \sum_{y\in\Omega_h} g_h(x, y)\, |S(y)|\right].\end{aligned}$$

Proof. Let $x_0 \in \Omega$ be the point where $\sum_{y\in\Omega_h} |[G_h(x, y) - g_h(x, y)]S(y)|$ attains its maximum and let

$$W(x) = \sum_{y\in\Omega_h} [G_h(x, y) - g_h(x, y)]S^*(y),$$

where $S^*(y) = |S(y)| \operatorname{sgn} [G_h(x_0, y) - g_h(x_0, y)]$. Employing (4.9), we have

$$\begin{aligned}\max_{\Omega_h} |W| &\leqq C \max_{\Omega_h^{(s)}} |h^2\Delta_h W| \\ &\leqq C\left[\max_{\Omega_h^{(s)}} |S| + \max_{x\in\Omega_h''\cup\Omega_h^{(2)}\cup\Omega_h^{(s)}} \left|\sum g_h(x, y)S^*(y)\right|\right],\end{aligned}$$

and (5.12) follows.

COROLLARY 5.2. *For all $x, z \in \Omega_h$,*

$$(5.13) \qquad \sum_{y\in\Omega_h''\cup\Omega_h^{(2)}\cup\Omega_h^{(s)}} |G_h(x, y)| \leqq C,$$

$$(5.14) \qquad h^2 \sum_{y\in\Omega_h} |G_h(x, y)| \leqq C,$$

$$(5.15) \qquad |G_h(x, z)| \leqq C\, |\log h|,$$

$$(5.16) \qquad h^2 \sum_{y\in\Omega_h} |G_h(x, y)|^2 \leqq C,$$

and for $|x - \partial\Omega| \leqq Ch$,

$$h^2 \sum_{y \in \Omega_h} |G_h(x, y)| \leqq Ch. \tag{5.17}$$

Proof. For (5.13), employ the characteristic function of $\Omega_h'' \cup \Omega_h^{(2)} \cup \Omega_h^{(3)}$ as S in (5.13). Then apply the triangle inequality and (4.4), (4.5), and (5.10). For (5.14), let $S = h^2$ and use (4.6) and (4.10), respectively. For (5.15), let $S(y) = \delta(y, z)$ in (5.12), apply the triangle inequality and (4.11). For (5.16), let x_0 be the point where $\max_{x \in \Omega_h} h^2 \sum_{y \in \Omega_h} |G_h(x, y)|^2$ is attained, and let $S(y) = h^2 G_h(x_0, y)$ in (5.12), from which it follows that

$$h^2 \sum_{y \in \Omega_h} |G_h(x, y)|^2 \leqq Ch^2 \max_{y \in \Omega_h^{(*)}} |G_h(x_0, y)| + \max_{x \in \Omega_h} h^2 \sum_{y \in \Omega_h} g_h(x, y) G_h(x_0, y).$$

Again, using (5.12) with $S(y) = h^2 g_h(x, y)$ for x fixed,

$$h^2 \sum_{y \in \Omega_h} G_h(x_0, y) g_h(x, y) \leqq Ch^2 \max_{y \in \Omega_h^{(*)}} |g_h(x, y)| + \max_{x_0 \in \Omega_h} h^2 \sum_{y \in \Omega_h} g_h(x_0, y) g_h(x, y).$$

By (4.11), this term can be seen to be bounded. Finally, letting $S(y) = h^2\, \delta(y_0, y)$ in (5.12), we have, for any $y_0 \in \Omega_h$,

$$|h^2 G_h(x_0, y_0)| \leqq C\left[h^2 + \max_{x \in \Omega_h} h^2 g_h(x, y_0)\right],$$

which indeed tends to zero as h does, by (4.11), and (5.16) follows. For (5.17) use $S = h^2$ and (4.10).

We require yet one more Green's function G_h' defined by

$$-\Delta_h G_h'(x, y) = h^{-2}\, \delta(x, y), \quad x \in \Omega_h', \qquad G_h'(x, y) = 0, \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}, \tag{5.18}$$

for all $y \in \Omega_h$. Thus, the matrix $[G_h'(x, y)]_{x, y \in \Omega_h'}$ is the inverse of the symmetric matrix $\mathfrak{L} = [h^2 l_h(x, y)]_{x, y \in \Omega_h'}$. We show $\mathfrak{L}$ is monotone by applying Corollary 3.6. First, we show $\mathfrak{L} + \frac{1}{3}I$ monotone from Corollary 3.5: we define M_1 by

$$\begin{aligned} [M_1]_{xy} &= \frac{16}{3}, && x = y, \\ &= -\frac{4}{3}, && |x - y| = h, \\ &= 0, && \text{otherwise}, \end{aligned}$$

for $x, y \in \Omega_h'$, and we define

$$\begin{aligned} [M_2]_{xy} &= \frac{8}{\sqrt{12}}, && x = y, \\ &= -\frac{1}{\sqrt{12}}, && |x - y| = h, \\ &= 0, && \text{otherwise}. \end{aligned}$$

Since M_1 and M_2 are of positive type, they are monotone, hence, so is M_2^2, and it is easy to see that

$$M_1 \leqq \mathfrak{L} + \tfrac{1}{3}I \leqq M_2^2.$$

Thus, $\mathfrak{L}$ is monotone if its eigenvalues, necessarily real by symmetry, are positive. But these are $h^2\mu_h^{(i)}$, where $\mu_h^{(i)}$ is the ith eigenvalue satisfying

$$(5.19) \qquad \Delta_h V_h^{(i)}(x) + \mu_h^{(i)} V_h^{(i)}(x) = 0, \quad x \in \Omega_h', \qquad V_h^{(i)}(x) = 0, \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}.$$

In the next section, we shall show that indeed $|\mu_h^{(i)} - \lambda^{(i)}| \to 0$ as $h \to 0$, for $\lambda^{(i)}$ the ith eigenvalue of (1.1), which is strictly positive. Thus, for h sufficiently small, $\mathfrak{L}$ is monotone and G_h' nonnegative. Thus, as a consequence of Lemma 3.7,

$$(5.20) \qquad 0 \leqq G_h'(x, y) \leqq g_h(x, y), \qquad x, y \in \Omega_h.$$

From (5.20), we see that all of the inequalities proved for g_h hold for G_h'. In particular, the difficult inequality (5.10) does, from which we prove the key inequality

$$(5.21) \qquad \max_{\Omega_h} |W| \leqq C\left[\max_{\Omega_h'} |\Delta_h W| + \max_{\Omega_h^{(2)} \cup \Omega_h^{(3)}} |W|\right],$$

for all W defined on Ω_h. To prove this, let

$$\begin{aligned} W^*(x) &= W(x), \qquad x \in \Omega_h', \\ &= 0, \qquad\quad\; x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}. \end{aligned}$$

Then, by (5.18),

$$\begin{aligned} W^*(x) &= h^2 \sum_{y \in \Omega_h'} G_h'(x, y)[-\Delta_h W^*(y)] \\ &= h^2 \sum_{y \in \Omega_h'} G_h'(x, y)[-\Delta_h W(y)] + h^2 \sum_{y \in \Omega_h''} G_h'(x, y)[\Delta_h W(y) - \Delta_h W^*(y)], \end{aligned}$$

and (5.21) follows from (4.6), (5.10), and (5.20).

6. Convergence of $\mu_h^{(n)}$ to $\lambda^{(n)}$. In this section, we show that the eigenvalue $\mu_h^{(n)}$ of

$$(6.1) \qquad \Delta_h V_h^{(n)}(x) + \mu_h^{(n)} V_h^{(n)}(x) = 0, \quad x \in \Omega_h', \qquad V_h^{(n)}(x) = 0, \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)},$$

converges to $\lambda^{(n)}$ of (1.1) for each n. We will use the variational principles associated with (1.1) and (6.1), and a technique of Weinberger [9].

The nth eigenvalue of (1.1) can be characterized by

$$(6.2) \qquad \lambda^{(n)} = \min \max D(u) \Big/ \int_\Omega u^2 \, dx,$$

where $u = \alpha_1 u_1 + \cdots + \alpha_n u_n$, the max is with respect to the scalars $\alpha_1, \cdots, \alpha_n$, the min is with respect to choices of linearly independent $u_1, \cdots, u_n$, continuous, piecewise differentiable functions vanishing on $\partial\Omega$, and $D(u)$ is the Dirichlet integral.

Similarly, the nth eigenvalue of (6.1) can be characterized by

$$(6.3) \qquad \mu_h^{(n)} = \min \max \frac{h^2 \sum \left[U_{x_1}^2 + U_{x_2}^2 + \frac{h^2}{12} U_{x_1\bar{x}_1}^2 + \frac{h^2}{12} U_{x_2\bar{x}_2}^2 \right]}{h^2 \sum U^2},$$

where $U = \alpha_1 U_1 + \cdots + \alpha_n U_n$, the max is with respect to the scalars $\alpha_1, \cdots, \alpha_n$, the min is with respect to choices of linearly independent mesh functions $U_1, \cdots, U_n$ vanishing on $\Omega_h^{(2)} \cup \Omega_h^{(3)}$, the sum is over the mesh points of Ω_h', and subscript x_i ($\bar{x}_i$)

denotes forward (backward) difference quotient in the x_i direction, $i = 1, 2$, i.e., $U_{x_1}(y_1, y_2) = [U(y_1 + h, y_2) - U(y_1, y_2)]/h$, etc.

First, we show

$$\mu_h^{(n)} \leqq \lambda^{(n)} + O(h). \tag{6.4}$$

Let $u^{(1)}, \cdots, u^{(n)}$ be eigenfunctions associated with $\lambda^{(1)}, \cdots, \lambda^{(n)}$ in (1.1), $u = \alpha_1 u^{(1)} + \cdots + \alpha_n u^{(n)}$, and define

$$\begin{aligned} u(x) &= h^{-1} \int_{Q_h(x)} u(y)\, dy, && x \in \Omega_h', \\ &= 0, && x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}, \end{aligned}$$

where $Q_h(x) = \{(y_1, y_2): |x_1 - y_1| \leqq \frac{1}{2}h,\ |x_2 - y_2| \leqq \frac{1}{2}h\}$ is the square of side h centered at x. Put this U in (6.3). Employing inequalities (2.14), (2.22) and (8.6) of Weinberger [9], we see that

$$\mu_h^{(n)} \leqq \max_\alpha \frac{D(u) + \dfrac{h^2}{12} \displaystyle\int_\Omega \left\{ \left(\frac{\partial^2 u}{\partial x_1^2} \right)^2 + \left(\frac{\partial^2 u}{\partial x_2^2} \right)^2 \right\} dx}{\displaystyle\int_\Omega u^2\, dx - (h^2/\pi^2) D(u)},$$

and Hubbard [5, pp. 568–569], has shown

$$\int_\Omega \left\{ \left(\frac{\partial^2 u}{\partial x_1^2} \right)^2 + \left(\frac{\partial^2 u}{\partial x_2^2} \right)^2 \right\} dx \leqq C(\lambda^{(n)})^2.$$

From these, (6.4) follows.

Next, we show

$$\lambda^{(n)} \leqq \mu_h^{(n)} + O(h). \tag{6.5}$$

Let $V_h^{(1)}, \cdots, V_h^{(n)}$ be eigenvectors associated with $\mu_h^{(1)}, \cdots, \mu_h^{(n)}$ in (6.1), $U = \alpha_1 V_h^{(1)} + \cdots + \alpha_n V_h^{(n)}$, and define u to be the continuous, piecewise linear function interpolating U (see [9, Section 6]). Then, by (6.4), (6.7) of [9] we see that

$$\begin{aligned} \lambda^{(n)} &\leqq \max_\alpha \frac{h^2 \sum (U_{x_1}^2 + U_{x_2}^2)}{h^2 \sum U^2 - \frac{1}{4}h^4 \sum (U_{x_1}^2 + U_{x_2}^2)} \\ &\leqq \max_\alpha \frac{h^2 \sum \left[U_{x_1}^2 + U_{x_2}^2 + \dfrac{h^2}{12} U_{x_1\bar{x}_1}^2 + \dfrac{h^2}{12} U_{x_2\bar{x}_2}^2 \right]}{h^2 \sum U^2 - \frac{1}{4}h^2 \sum \left[U_{x_1}^2 + U_{x_2}^2 + \dfrac{h^2}{12} U_{x_1\bar{x}_1}^2 + \dfrac{h^2}{12} U_{x_2\bar{x}_2}^2 \right]} \\ &= \frac{\mu_h^{(n)}}{1 - \frac{1}{4}h^2 \mu_h^{(n)}} \end{aligned}$$

and we obtain (6.5). Combining (6.4) and (6.5), we have

$$|\mu_h^{(n)} - \lambda^{(n)}| \to 0 \quad \text{as } h \to 0, \tag{6.6}$$

for each $n = 1, 2, \cdots$.

7. Convergence of $\lambda_h^{(n)}$ to $\lambda^{(n)}$ by Perturbation. Next, we will show that the $\lambda_h^{(n)}$ are a perturbation of the $\mu_h^{(n)}$, and that as h tends to zero, $\lambda_h^{(n)}$ tends to $\mu_h^{(n)}$, hence to $\lambda^{(n)}$, by Section 6. We employ the following theorem of Wielandt:

THEOREM 7.1. *If A, B are $\nu \times \nu$ matrices and A has an orthonormal basis of eigenvectors, then the eigenvalues of B lie in the union of the ν discs $|\mu^{(i)} - z| \leqq \|A - B\|_2$, where the $\mu^{(i)}$ are the eigenvalues of A. If k discs are disjoint from the others, they contain exactly k eigenvalues of B.*

In the theorem, $\|\cdot\|_2$ is the spectral norm of a matrix, defined by

$$\|M\|_2 = \sup_{\xi} \|M\xi\|_2/\|\xi\|_2, \quad \text{where } \|\xi\|_2 = \left(\sum_{i=1}^{\nu} |\xi_i|^2\right)^{1/2}$$

for a ν-vector $\xi = (\xi_1, \cdots, \xi_\nu)$. For a proof of the theorem, see [6].

We apply the theorem as follows. For A, we take the matrix $[h^2G_h'(x, y)]_{x,y\in\Omega_h}$. Note that the minor $[h^2G_h'(x, y)]_{x,y\in\Omega_h'}$ is symmetric, while $h^2G_h'(x, y) = 0$ for $x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}$, so that A has an orthonormal basis of eigenvectors, and the eigenvalues are simply $[\mu_h^{(i)}]^{-1}$ plus some zeros. For B, we take the matrix $[h^2G_h(x, y)]$ whose eigenvalues are $[\lambda_h^{(i)}]^{-1}$. Thus, we must estimate $\|h^2(G_h - G_h')\|_2$. However, for any matrix,

$$\|M\|_2 \leqq [\rho(MM^T)]^{1/2} \leqq \|MM^T\|_1^{1/2},$$

where $\|\cdot\|_1$ is the maximum of the absolute row sums of the matrix. This is a consequence of the Gerschgorin circle theorem (see, e.g., [7, p. 146]). Thus, we need to estimate

$$h^4 \max_{x\in\Omega_h} \sum_{y\in\Omega_h} \left| \sum_{z\in\Omega_h} [G_h(x, z) - G_h'(x, z)][G_h(y, z) - G_h'(y, z)] \right|. \tag{7.1}$$

Let x_0 be the point where the max is attained and put

$$\sigma(y) = \operatorname{sgn} \sum_{z\in\Omega_h} [G_h(x_0, z) - G_h'(x_0, z)][G_h(y, z) - G_h'(y, z)].$$

Then, let

$$W(x) = h^4 \sum_{y,z\in\Omega_h} [G_h(x, z) - G_h'(x, z)][G_h(y, z) - G_h'(y, z)]\sigma(y)$$

in (4.9). Then, (7.1) is bounded by

$$\begin{aligned} &Ch^4 \max_{z\in\Omega_h^{(3)}} \sum_{y\in\Omega_h} |G_h(y, z) - G_h'(y, z)| \\ &\qquad + Ch^4 \max_{x\in\Omega_h''} \sum_{z\in\Omega_h'} G_h'(x, z) \sum_{y\in\Omega_h} |G_h(y, z) - G_h'(y, z)|. \end{aligned} \tag{7.2}$$

Now,

$$h^2 \sum_{y\in\Omega_h} |G_h(y, z) - G_h'(y, z)| \leqq C \max_{y,z\in\Omega_h} [|G_h(y, z)| + G_h'(y, z)] \leqq C|\log h|,$$

by (4.11), (5.15) and (5.20). Using this in (7.2) and also (4.10) and (5.20), we have (7.2) bounded by $Ch|\log h|$, which tends to zero as h tends to zero. Thus, the radii of the discs in Theorem 7.1 tend to zero as h does. Since the $\mu_h^{(n)}$ tend to the $\lambda^{(n)}$, which have no finite accumulation point, the disc associated with $[\mu_h^{(n)}]^{-1}$ for any fixed n eventually becomes disjoint from the remaining discs. Consequently, for any fixed n and $\epsilon > 0$, there is h sufficiently small that

$$|\lambda_h^{(n)} - \lambda^{(n)}| < \epsilon. \tag{7.3}$$

8. Main Theorem. We are now ready to state and prove our main theorem:

THEOREM 8.1. *Let $\lambda^{(n)}$ be the nth eigenvalue of* (1.1), *let $\lambda_h^{(n)}$ be the nth eigenvalue of* (2.6) *with associated eigenvector $U_h^{(n)}$. For each $n = 1, 2, \cdots$, there are constants C_n, h_n such that for $h < h_n$*

$$|\lambda_h^{(n)} - \lambda^{(n)}| < C_n h^4, \tag{8.1}$$

and there is an eigenfunction $u^{(n)}$ associated with $\lambda^{(n)}$ such that

$$\max_{\Omega_h} |U_h^{(n)} - u^{(n)}| < C_n h^4. \tag{8.2}$$

Proof. With the machinery generated in the previous sections, our proof will have exactly the form of the proof of the corresponding Theorem 5.1 of [6]. For this reason, we only sketch the proof.

By (7.3)

$$|\lambda_h^{(n)}| \leqq C_n. \tag{8.3}$$

By (5.11), (2.6) is equivalent to

$$U_h^{(n)}(x) = \lambda_h^{(n)} h^2 \sum_{y \in \Omega_h} G_h(x, y) U_h^{(n)}(y), \qquad x \in \Omega_h. \tag{8.4}$$

Let us use the notations

$$\langle U, V \rangle_h \equiv h^2 \sum_{y \in \Omega_h} U(y)\overline{V(y)}, \qquad ||U||_h \equiv \langle U, U \rangle_h^{1/2},$$

$$\langle U, V \rangle_h' \equiv h^2 \sum_{y \in \Omega_h'} U(y)\overline{V(y)}, \qquad ||U||_h' \equiv \langle U, U \rangle_h'^{1/2}.$$

If $U_h^{(n)}$ is normalized by requiring $||U_h^{(n)}||_h = 1$, then (8.4), (8.3), the Schwarz inequality, and (5.16) show

$$\max_{\Omega_h} |U_h^{(n)}| \leqq C_n. \tag{8.5}$$

From (8.4), (8.5) and (5.17), we see that for $|x - \partial\Omega| \leqq Ch$

$$|U_h^{(n)}(x)| \leqq C_n h. \tag{8.6}$$

Let us suppose that $\lambda^{(n)} = \lambda^{(n+1)} = \cdots = \lambda^{(n+m)}$ is an eigenvalue of multiplicity $m + 1$. Since Δ_h restricted to Ω_h' is symmetric, the eigenvectors $V_h^{(n)}$ of (6.1) are a complete orthonormal basis on Ω_h':

$$\langle V_h^{(i)}, V_h^{(j)} \rangle_h' = \delta(i, j).$$

If we set

$$\tilde{V}_h^{(i)} = \sum_{j=n}^{n+m} \langle U_h^{(i)}, V_h^{(j)} \rangle_h' V_h^{(j)}, \qquad i = n, \cdots, n + m,$$

then

$$(8.7) \qquad \| U_h^{(i)} - \tilde{V}_h^{(i)} \|_h' \leqq C_n h, \qquad i = n, \cdots, n + m.$$

This follows from Parseval's identity:

$$\| U_h^{(i)} \|_h'^2 = \langle U_h^{(i)}, \tilde{V}_h^{(i)} \rangle_h' + \sum_{j \neq n, \cdots, n+m} |\langle U_h^{(i)}, V_h^{(j)} \rangle_h'|^2$$
$$= \langle U_h^{(i)}, \tilde{V}_h^{(i)} \rangle_h' + \sum_{j \neq n, \cdots, n+m} \left| \frac{\mu_h^{(j)}}{\mu_h^{(j)} - \lambda_h^{(i)}} \langle H_h^{(i)}, V_h^{(j)} \rangle_h' \right|^2,$$

where $H_h^{(i)}$ is uniquely defined by

$$\Delta_h H_h^{(i)}(x) = 0, \quad x \in \Omega_h', \qquad H_h^{(i)}(x) = U_h^{(i)}(x), \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}.$$

It follows from our hard-won inequality (5.21) that

$$\max_{\Omega_h} |H_h^{(i)}| \leqq \max_{\Omega_h^{(2)} \cup \Omega_h^{(3)}} |U_h^{(i)}| \leqq C_i h,$$

by (8.6), and so

$$\| U_h^{(i)} - \tilde{V}_h^{(i)} \|_h'^2 \equiv \| U_h^{(i)} \|_h'^2 - \langle U_h^{(i)}, \tilde{V}_h^{(i)} \rangle_h' \leqq C_i h^2.$$

In a very similar manner, we show that if

$$\tilde{\tilde{V}}_h^{(i)} = \sum_{j=n}^{n+m} \langle U^{(i)}, V_h^{(j)} \rangle_h' V_h^{(j)}, \qquad i = n, \cdots, n + m,$$

then

$$(8.8) \qquad \| u^{(i)} - \tilde{\tilde{V}}_h^{(i)} \|_h' \leqq C_n h, \qquad i = n, \cdots, n + m.$$

From (8.8), we can conclude that the $(m + 1) \times (m + 1)$ matrix $[\langle u^{(i)}, V_h^{(j)} \rangle_h']$, $i, j = n, \cdots, n + m$, is nonsingular. In particular then, there are eigenvectors

$$u_h^{(i)} = \sum_{j=n}^{n+m} a_{ij}(h) u^{(j)}, \qquad i = n, \cdots, n + m,$$

in the eigenmanifold associated with $\lambda^{(n)}$ such that

$$(8.9) \qquad \langle u_h^{(i)}, V_h^{(j)} \rangle_h' = \langle U_h^{(i)}, V_h^{(j)} \rangle_h', \qquad i, j = n, \cdots, n + m.$$

Moreover, the coefficients $a_{ij}(h)$ are bounded independently of h.

Then, it follows from (8.9) and Parseval's identity that

$$\| U_h^{(i)} - u_h^{(i)} \|_h^2 = h^2 \sum_{\Omega_h^{(2)} \cup \Omega_h^{(3)}} | U_h^{(i)} - u_h^{(i)} |^2 + \sum_{j \neq n, \cdots, n+m} |\langle U_h^{(i)} - u_h^{(i)}, V_h^{(j)} \rangle_h'|^2$$
$$= h^2 \sum_{\Omega_h^{(2)} \cup \Omega_h^{(3)}} | U_h^{(i)} - u_h^{(i)} |^2$$
$$+ \sum_{j \neq n, \cdots, n+m} \left| \frac{\mu_h^{(j)}}{\mu_h^{(j)} - \lambda_h^{(i)}} \langle H_h^{(i)}, V_h^{(j)} \rangle_h' - \frac{\mu_h^{(j)}}{\mu_h^{(j)} - \lambda^{(i)}} \langle \tilde{H}_h^{(i)}, V_j^{(j)} \rangle_h' \right|^2,$$

where $\tilde{H}_h^{(i)}$ is defined by

$$\Delta_h \tilde{H}_h^{(i)}(x) = 0, \quad x \in \Omega_h', \qquad \tilde{H}_h^{(i)}(x) = u_h^{(i)}(x), \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}.$$

Since $|u_h^{(i)}(x)| \leqq C_i h$ for $|x - \partial\Omega| \leqq Ch$, we see that

$$(8.10) \qquad \| U_h^{(i)} - u_h^{(i)} \|_h \leqq C_i h.$$

From (8.10), we also have

$$|\langle U_h^{(i)}, u_h^{(i)}\rangle_h| \geqq 1 - C_i h^2. \tag{8.11}$$

Inequality (8.11) is the key inequality needed to prove the first half of Theorem 8.1, for now

$$\begin{aligned}(\lambda_h^{(i)} - \lambda^{(i)})\langle U_h^{(i)}, u_h^{(i)}\rangle &= \langle U_h^{(i)}, \Delta u_h^{(i)} - \Delta_h^* u_h^{(i)}\rangle_h \\ &= \langle U_h^{(i)}, \tau_h u_h^{(i)}\rangle_h - \langle u_h^{(i)}, \tau_h u_h^{(i)}\rangle_h + \langle \tau_h u_h^{(i)}, u_h^{(i)}\rangle_h \\ &\quad + \langle U_h^{(i)} - u_h^{(i)}, \Delta_h u_h^{(i)} - \Delta_h^* u_h^{(i)}\rangle_h,\end{aligned} \tag{8.12}$$

obtained by adding and subtracting terms. We have used the notations

$$\tau_h u_h^{(i)} \equiv \Delta u_h^{(i)} - \Delta_h u_h^{(i)}$$

for the truncation error, and Δ_h^* for the adjoint of Δ_h defined by

$$\Delta_h^* V(x) = \sum_{y \in \Omega_h} l_h(y, x) V(y).$$

Recall by (2.6) and our smoothness assumption on $u^{(i)}$ that

$$\begin{aligned}|\tau_h u_h^{(i)}| &\leqq C_i h^4, &&\text{on } \Omega_h', \\ &\leqq C_i h^2, &&\text{on } \Omega_h^{(2)} \cup \Omega_h^{(3)}.\end{aligned}$$

However, on $\Omega_h^{(2)} \cup \Omega_h^{(3)}$ both $U_h^{(i)}$ and $u_h^{(i)}$ are bounded by $C_i h$, while the number of points in $\Omega_h^{(2)} \cup \Omega_h^{(3)}$ is only proportional to h^{-1}. From these considerations, we see that the first three terms on the right side of (8.12) are bounded by $C_i h^4$. As for the remaining term,

$$\Delta_h u_h^{(i)}(x) - \Delta_h^* u_h^{(i)}(x)$$

vanishes for $x \notin \Omega_h'' \cup \Omega_h^{(2)} \cup \Omega_h^{(3)}$, and is bounded by

$$Ch^{-2} \max_{\Omega_h'' \cup \Omega_h^{(2)} \cup \Omega_h^{(3)}} |u_h^{(i)}| \leqq C_i h^{-1}$$

for $x \in \Omega_h'' \cup \Omega_h^{(2)} \cup \Omega_h^{(3)}$. Again noting that the number of points in $\Omega_h'' \cup \Omega_h^{(2)} \cup \Omega_h^{(3)}$ is only proportional to h^{-1}, the last term on the right of (8.12) is bounded by

$$C_i \max_{\Omega_h'' \cup \Omega_h^{(2)} \cup \Omega_h^{(3)}} |U_h^{(i)} - u_h^{(i)}| .$$

Thus, using (8.11) we have the inequality

$$|\lambda_h^{(i)} - \lambda^{(i)}| \leqq C_i\left[\max_{\Omega_h'' \cup \Omega_h^{(2)} \cup \Omega_h^{(3)}} |U_h^{(i)} - u_h^{(i)}| + h^4\right]. \tag{8.13}$$

We next employ the discrete Green's function to write

$$\begin{aligned}U_h^{(i)}(x) - u_h^{(i)}(x) &= h^2 \sum_{y \in \Omega_h} G_h(x, y)\, \Delta_h[u_h^{(i)}(y) - U_h^{(i)}(y)] \\ &= -h^2 \sum_{y \in \Omega_h} G_h(x, y)\tau_h u_h^{(i)}(y) + \lambda^{(i)} h^2 \sum_{y \in \Omega_h} G_h(x, y)[U_h^{(i)}(y) - u_h^{(i)}(y)] \\ &\quad + (\lambda_h^{(i)} - \lambda^{(i)}) h^2 \sum_{y \in \Omega_h} G_h(x, y)\, U_h^{(i)}(y).\end{aligned} \tag{8.14}$$

Using inequalities (5.13) and (5.14), we see that the first term on the right of (8.14) is bounded by $C_i h^4$. By (5.14) and (8.5) the last term on the right is bounded by

$C_i|\lambda_h^{(i)} - \lambda^{(i)}|$, or if $|x - \partial\Omega| \leqq Ch$, (5.17) shows the last term bounded by $C_i h$ $|\lambda_h^{(i)} - \lambda^{(i)}|$. Using (8.3), (5.16) and Schwarz's inequality bound the middle term on the right by $||U_h^{(i)} - u_h^{(i)}||_h$, or, if $|x - \partial\Omega| \leqq Ch$, (5.17) bounds it by $C_i h \max_{\Omega_h} |U_h^{(i)} - u_h^{(i)}|$. In summary,

$$(8.15) \qquad \max_{\Omega_h} |U_h^{(i)} - u_h^{(i)}| \leqq C_i[||U_h^{(i)} - u_h^{(i)}||_h + |\lambda_h^{(i)} - \lambda^{(i)}| + h^4],$$

$$(8.16) \qquad \max_{\Omega_h''\cup\Omega_h^{(2)}\cup\Omega_h^{(3)}} |U_h^{(i)} - u_h^{(i)}| \leqq C_i\left[h \max_{\Omega_h} |U_h^{(i)} - u_h^{(i)}| + h\,|\lambda_h^{(i)} - \lambda^{(i)}| + h^4\right].$$

Finally, we use Parseval's identity and (8.9) to conclude that

$$||U_h^{(i)} - u_h^{(i)}||_h^2 = h^2 \sum_{\Omega_h^{(2)}\cup\Omega_h^{(3)}} |U_h^{(i)} - u_h^{(i)}|^2 + \sum_{j\neq n,\cdots,n+m} |\langle U_h^{(i)} - u_h^{(i)}, V_h^{(j)}\rangle_h'|^2,$$

and by a straightforward computation

$$(\mu_h^{(j)} - \lambda^{(i)})\langle U_h^{(i)} - u^{(i)}, V_h^{(j)}\rangle_h' = \langle(\lambda_h^{(i)} - \lambda^{(i)})U_h^{(i)} - \tau_h u_h^{(i)} + \tilde{\tilde{H}}_h^{(i)}, V_h^{(j)}\rangle_h',$$

where $\tilde{\tilde{H}}_h^{(i)}$ is defined by

$$\Delta_h \tilde{\tilde{H}}_h^{(i)}(x) = 0, \quad x \in \Omega_h', \qquad \tilde{\tilde{H}}_h^{(i)}(x) = U_h^{(i)}(x) - u_n^{(i)}(x), \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}.$$

It follows that

$$(8.17) \qquad ||U_h^{(i)} - u_h^{(i)}||_h \leqq C_i\left[\max_{\Omega_h''\cup\Omega_h^{(2)}\cup\Omega_h^{(3)}} |U_h^{(i)} - u_h^{(i)}| + |\lambda_h^{(i)} - \lambda^{(i)}| + h^4\right].$$

Combining (8.13), (8.15), (8.16), and (8.17) yields the proof of Theorem 8.1.

Let us observe some simple consequences of Theorem 8.1. Since the $\lambda^{(i)}$ are real, we have

$$(8.18) \qquad |\mathrm{Re}\, \lambda_h^{(i)} - \lambda^{(i)}| \leqq Ch^4.$$

Also, when $\lambda^{(i)}$ is simple, $\lambda_h^{(i)}$ will be real for h sufficiently small. This is because the matrix $[l_h(x, y)]_{x,y\in\Omega_h}$ is real. Thus, if $\lambda_h^{(i)}$ were complex, its conjugate $[\lambda_h^{(i)}]^-$ would also be a distinct eigenvalue of Δ_h converging to $\lambda_h^{(i)}$. But this is impossible, since $[\lambda_h^{(i)}]^-$ must converge to some $\lambda^{(j)} \neq \lambda^{(i)}$.

We normalized $U_h^{(i)}$ by requiring $||U_h^{(i)}||_h = 1$. This determines $U_h^{(i)}$ only up to a multiplicative constant of modulus 1. If we specify this constant by requiring that $\langle U_h^{(i)}, V_h^{(i)}\rangle_h' \geqq 0$, then when $\lambda^{(i)}$ is simple, $u_h^{(i)}$ is a real multiple of $u^{(i)}$, as can be seen from (8.9).

Theorem 8.1 shows that $U_h^{(i)}$ approximates to $O(h^4)$ an eigenfunction $u_h^{(i)}$ which depends on h. Properly normalized, however, $U_h^{(i)}$ will approximate to $O(h^4)$ an eigenfunction $u_h^{(i)}$ such that $\int_\Omega |u_h^{(i)}|^2\,dx = 1$, independently of h. In particular, when $\lambda^{(i)}$ is simple, $U_h^{(i)}$ will approximate the unique normalized eigenfunction $u^{(i)}$. This normalization is

$$h^2 \sum_{y\in\Omega_h} \alpha_h(y)\,|U_h^{(i)}(y)|^2 = 1,$$

where α_h is given in the appendix of [6]. For a proof, see [6, Corollary 6.2].

9. Forced Vibration Problems. Let us remark that all of the results of the previous sections hold for the problem

$$(9.1) \qquad \Delta u(x) + (q(x) + \lambda)u(x) = 0, \quad x \in \Omega, \qquad u(x) = 0, \quad x \in \partial\Omega,$$

where q is nonpositive and smooth on Ω, and for the discrete Green's function G_h defined by

$$(9.2) \qquad (\Delta_{h,x} + q(x))G_h(x, y) = -h^{-2}\delta(x, y), \qquad x, y \in \Omega_h.$$

The proofs require only that the additional term q be carried along throughout. We make this remark because we next wish to consider the problem

$$(9.3) \qquad \Delta u(x) + r(x)u(x) = F(x), \quad x \in \Omega, \qquad u(x) = 0, \quad x \in \partial\Omega,$$

for F and r given smooth functions on Ω. Problem (9.3) is a forced vibration problem and an $O(h^2)$ analogue of it was studied by Bramble in [1].

Let us rewrite (9.3) in the form

$$(9.4) \qquad \Delta u(x) + q(x)u(x) + \left(\sup_{\Omega} r\right)u(x) = F(x), \quad x \in \Omega, \qquad u(x) = 0, \quad x \in \partial\Omega,$$

where $q(x) \equiv r(x) - \sup_\Omega r \leqq 0$ on Ω. A unique solution u of (9.3) or (9.4) exists if and only if $\sup r$ is not an eigenvalue of the operator $\Delta + q$. Now, we consider the difference approximation

$$(9.5) \qquad \Delta_h U_h(x) + r(x)U_h(x) = F(x), \qquad x \in \Omega_h,$$

where Δ_h is the difference operator defined in Section 2. We prove:

THEOREM 9.1. *If* (9.3) *has a unique solution* $u \in C^6(\bar{\Omega})$, *there are constants* C, h_0 *such that for* $h < h_0$, (9.5) *has a unique solution* U_h *for which*

$$\max_{\Omega_h} |U_h - u| < Ch^4.$$

Proof. Let G_h be the discrete Green's function defined in (9.2). Then, for $x \in \Omega_h$,

$$|U_h(x) - u(x)| = \left| h^2 \sum_{y \in \Omega_h} G_h(x, y)[\Delta_h u(y) + q(y)u(y) - \Delta_h U_h(y) - q(y)U_h(y)] \right|$$

$$\leqq \sup_{\Omega} |q| \, h^2 \sum_{y \in \Omega_h} |G_h(x, y)| \, |U_h(y) - u(y)| + h^2 \sum_{y \in \Omega_h} |G_h(x, y)| \, |\tau_h u(y)|.$$

Therefore, using (5.13) and (5.14) for G_h of (9.2) and (2.5),

$$(9.6) \qquad |U_h(x) - u(x)| \leqq C\left[h^2 \sum_{y \in \Omega_h} |G_h(x, y)| \, |U_h(y) - u(y)| + h^4\right].$$

Employing (5.17), this yields

$$(9.7) \qquad \max_{\Omega_h^{(s)} \cup \Omega_h^{(z)}} |U_h - u| \leqq C\left[h \max_{\Omega_h} |U_h - u| + h^4\right],$$

while (5.16) and Schwarz's inequality yield

$$(9.8) \qquad \max_{\Omega_h} |U_h - u| \leqq C[\|U_h - u\|_h + h^4].$$

From (9.7) and (9.8), we see

$$\|U_h - u\|_h \leqq \|U_h - u\|_h' + Ch^{1/2} \max_{\Omega_h^{(s)} \cup \Omega_h^{(z)}} |U_h - u|$$

$$\leqq \|U_h - u\|_h' + Ch^{1/2}[h \|U_h - u\|_h + h^4].$$

which implies

$$\|U_h - u\|_h \leqq C[\|U_h - u\|_h' + h^4]. \tag{9.9}$$

Finally, we complete the proof by using Parseval's identity to estimate

$$\|U_h - u\|_h' = \left[\sum_i |\langle U_h - u, V_h^{(i)}\rangle_h'|^2\right]^{1/2}, \tag{9.10}$$

where $V_h^{(i)}$ is the eigenvector associated with $\mu_h^{(i)}$ in the symmetric problem

$$\Delta_h V_h^{(i)}(x) + (q(x) + \mu_h^{(i)}) V_h^{(i)}(x) = 0, \quad x \in \Omega_h', \qquad V_h^{(i)}(x) = 0, \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}.$$

Define H_h by

$$\Delta_h H_h(x) + q(x) H_h(x) = 0, \quad x \in \Omega_h', \qquad H_h(x) = U_h(x) - u(x), \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}.$$

From (5.21), we have

$$\max_{\Omega_h'} |H_h| \leqq C \max_{\Omega_h^{(2)} \cup \Omega_h^{(3)}} |U_h - u| ,$$

or, employing (9.7), (9.8), (9.9),

$$\max_{\Omega_h'} |H_h| \leqq C[h \|U_h - u\|_h' + h^4]. \tag{9.11}$$

Then, we have

$$\begin{aligned} \mu_h^{(i)} \langle U_h - u, V_h^{(i)}\rangle_h' &= \langle H_h + u - U_h, (\Delta_h + q) V_h^{(i)}\rangle_h' + \mu_h^{(i)} \langle H_h, V_h^{(i)}\rangle_h' \\ &= \langle (\Delta_h + q)(H_h + u - U_h), V_h^{(i)}\rangle_h' + \mu_h^{(i)} \langle H_h, V_h^{(i)}\rangle_h' \\ &= (\sup r)\langle U_h - u, V_h^{(i)}\rangle_h' - \langle \tau_h u, V_h^{(i)}\rangle_h' + \mu_h^{(i)} \langle H_h, V_h^{(i)}\rangle_h'. \end{aligned}$$

Now, since $\sup r$ is not an eigenvalue $\lambda^{(i)}$ of $\Delta + q$ and $\mu_h^{(i)} \to \lambda^{(i)}$ as $h \to 0$, there are constants C, h_0 such that for $h < h_0$,

$$\max_i |\mu_h^{(i)} - \sup r|^{-1} < C, \qquad \max_i \mu_h^{(i)} / |\mu_h^{(i)} - \sup r| < C,$$

and so

$$|\langle U_h - u, V_h^{(i)}\rangle_h'| \leqq C[|\langle \tau_h u, V_h^{(i)}\rangle_h'| + |\langle H_h, V_h^{(i)}\rangle_h'|].$$

Using this in (9.10), we see that

$$\|U_h - u\|_h' \leqq C[\|\tau_h u\|_h' + \|H_h\|_h'] \leqq C[h^4 + h \|U_h - u\|_h'],$$

by (9.11), from which it follows that

$$\|U_h - u\|_h' \leqq C h^4,$$

completing the proof.

Let us remark that by employing the results of [6], the above technique of proof will show that a unique solution of the forced vibration problem

$$\begin{aligned} \sum_{i=1}^{r} \frac{\partial}{\partial x_i}\left(a_i(x) \frac{\partial u(x)}{\partial x_i}\right) + r(x)u(x) &= f(x), \qquad x \in \Omega, \\ u(x) &= 0, \qquad x \in \partial\Omega, \end{aligned} \tag{9.12}$$

can be approximated to $O(h^2)$ by using the symmetric difference scheme given in [6] at the beginning of Section 7.

Applied Physics Laboratory
The Johns Hopkins University
Silver Spring, Maryland 20910

1. J. H. BRAMBLE, "Error estimates for difference methods in forced vibration problems," *SIAM J. Numer. Anal.*, v. 3, 1966, no. 1, pp. 1–12. MR **34** #969.

2. J. H. BRAMBLE & B. E. HUBBARD, "New monotone type approximations for elliptic problems," *Math. Comp.*, v. 18, 1964, pp. 349–367. MR **29** #2982.

3. J. H. BRAMBLE & V. C. THOMÉE, "Point-wise bounds for discrete Green's functions," *SIAM J. Numer. Anal.*, v. 6, 1969, pp. 583–590.

4. R. COURANT & D. HILBERT, *Methoden der Mathematischen Physik*. Vol. I, Springer, Berlin, 1931; English transl., Interscience, New York, 1953. MR **16,** 426.

5. B. E. HUBBARD, "Bounds for eigenvalues of the free and fixed membrane by finite difference methods," *Pacific J. Math.*, v. 11, 1961, pp. 559–590. MR **25** #4633.

6. J. R. KUTTLER, "Finite difference approximations for eigenvalues of uniformly elliptic operators," *SIAM J. Numer. Anal.*, v. 7, 1970.

7. M. MARCUS & H. MINC, *A Survey of Matrix Theory and Matrix Inequalities*, Allyn and Bacon, Boston, Mass., 1964. MR **29** #112.

8. H. S. PRICE, "Monotone and oscillation matrices applied to finite difference approximations," *Math. Comp.*, v. 22, 1968, pp. 489–516. MR **38** #875.

9. H. F. WEINBERGER, "Lower bounds for higher eigenvalues by finite difference methods," *Pacific J. Math.*, v. 8, 1958, pp. 339–368; erratum, 941. MR **21** #6097.