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On Jacobi and Jacobi-Like Algorithms for a Parallel Computer

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Abstract. Many existing algorithms for obtaining the eigenvalues and eigenvectors of matrices would make poor use of such a powerful parallel computer as the ILLIAC IV. In this paper, Jacobi's algorithm for real symmetric or complex Hermitian matrices, and a Jacobi-like algorithm for real nonsymmetric matrices developed by P. J. Eberlein, are modified so as to achieve maximum efficiency for the parallel computations.

1. Introduction. With the advent of parallel computers, the study of computationally massive problems became economically possible. Such problems include, for example, solution of sets of partial differential equations over sizable grids, and multiplication, inversion, or determination of eigenvalues and eigenvectors of large matrices.

An example of a parallel computer is the ILLIAC IV.* This computer is essentially an array of coupled arithmetic units driven by instructions from a common control unit. Each of the arithmetic units, called processing elements (PE's), have 2048 words of 64-bit memory with an access time under 420 nanoseconds. Each PE is capable of 64-bit floating-point multiplication in about 550 nanoseconds. Two 32-bit floating-point operations may be performed in each PE in approximately the same times. The PE instruction set is similar to that of conventional machines with two exceptions. First, the PE's are capable of communicating data to four neighboring PE's by means of routing instructions. Second, the PE's are able to set their own mode registers to effectively disable or enable themselves. For a more detailed description of this system, the reader is referred to [2], [8], [9], [12].

The purpose of this paper is to introduce modified Jacobi and Jacobi-like algorithms for the computation of the eigenvalues and eigenvectors of large real symmetric or complex Hermitian matrices, and real nonsymmetric matrices, respectively, that are suitable for a parallel computer.

2. Jacobi's Algorithm. In the classical method of Jacobi (1846), [13], a real symmetric matrix is reduced to the diagonal form by a sequence of plane rotations $A_{k+1} = R_k A_k R_k^t$ $(k = 1, 2, \cdots)$, where $A_1 = A$ is the original matrix and each rotation $R_k \equiv R(p, q, \alpha_{pq}^{(k)})$ in the p, q plane through an angle $\alpha_{pq}^{(k)}$ eliminates the off-diagonal element $a_{pq}^{(k)}$ (and hence $a_{qp}^{(k)}$), and affects only elements in rows and columns p and q. See the Appendix for the appropriate value of $\alpha_{pq}^{(k)}$ to annihilate the element $a_{pq}^{(k)}$. Because of symmetry, only the off-diagonal elements above the main diagonal are considered in what follows.

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* The ILLIAC IV system will soon be in operation.

It is possible, however, to modify the present Jacobi algorithm for a parallel computer so as to eliminate more than one off-diagonal element. For example, for a matrix A of order 4, if the orthogonal transformation R is chosen as,

$$R = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ 0 & c_2 & 0 & s_2 \\ -s_1 & 0 & c_1 & 0 \\ 0 & -s_2 & 0 & c_2 \end{bmatrix}, \tag{2.1}$$

where $c_i = \cos \alpha_i$, $s_i = \sin \alpha_i$ $(i = 1, 2)$, then RAR^t would have zero elements in positions (1, 3) and (2, 4) provided that the angles α_1 and α_2 are properly chosen. α_1 and α_2 are determined by (a_{11}, a_{33}, a_{13}) and (a_{22}, a_{44}, a_{24}), respectively.

Define m by $[(n + 1)/2]$, where n is the order of the matrix A and $[x]$ is the greatest integer less than or equal to x. Let each $(2m - 1)$ orthogonal transformations be denoted by a sweep. Observing that there are $n(n - 1)/2$ off-diagonal elements, and that the maximum number of these elements which can be annihilated by an orthogonal transformation of the type (2.1) is $[n/2]$, then the modified Jacobi algorithm will attain maximum efficiency of parallel computation if the following two conditions are satisfied:

(i) Each orthogonal transformation R_k should be constructed so as to annihilate $[n/2]$ off-diagonal elements.

(ii) Each sweep should annihilate each off-diagonal element only once, i.e., each of the $(2m - 1)$ orthogonal transformations in a sweep should annihilate different $[n/2]$ off-diagonal elements.

Several annihilation regimes that satisfy the above requirements are possible. Two different regimes are discussed below.

First Annihilation Regime. For a given sweep, each of the $(2m - 1)$ orthogonal matrices R_k consists of the elements,

$$\begin{aligned} R_{pp}^{(k)} = R_{qq}^{(k)} = \cos \alpha_{pq}^{(k)}; \qquad R_{pq}^{(k)} &= -R_{qp}^{(k)} = \sin \alpha_{pq}^{(k)}, && p < q, \\ &= -\sin \alpha_{pq}^{(k)}, && p > q, \end{aligned} \tag{2.2}$$

where p and q are sequences defined by

(a) for $k = 1, 2, \cdots, m - 1$,

$$\begin{aligned} q &= m - k + 1, m - k + 2, \cdots, n - k, \\ p &= (2m - 2k + 1) - q, && m - k + 1 \leqq q \leqq 2m - 2k, \\ &= (4m - 2k) - q, && 2m - 2k < q \leqq 2m - k - 1, \\ &= n, && 2m - k - 1 < q, \end{aligned} \tag{2.3}$$

(b) for $k = m, m + 1, \cdots, 2m - 1$,

$$\begin{aligned} q &= 4m - n - k, 4m - n - k + 1, \cdots, 3m - k - 1, \\ p &= n, && q < 2m - k + 1, \\ &= (4m - 2k) - q, && 2m - k + 1 \leqq q \leqq 4m - 2k - 1, \\ &= (6m - 2k - 1) - q, && 4m - 2k - 1 < q. \end{aligned} \tag{2.4}$$

The remaining elements of R_k are zero except for n odd, then $R^{(k)}_{2m-k,2m-k} = 1$. For a given k, the angles $\alpha^{(k)}_{pq}$ are determined for all (p, q) such that $\alpha^{(k)}_{pq}$ eliminates the element $a^{(k)}_{pq}$; see the Appendix.

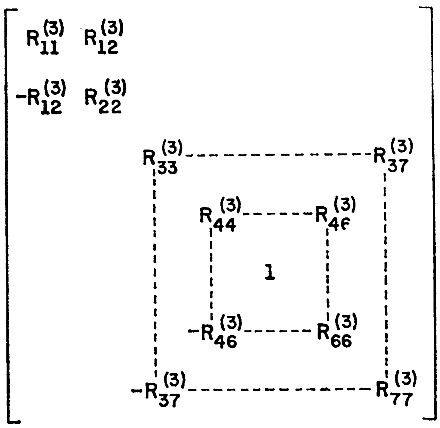
Let $n = 8$ and $k = 2$, then the pairs (p, q) are given by $\{(2, 3); (1, 4); (7, 5); (8, 6)\}$ and R_2 is of the form

$$\begin{bmatrix}
R^{(2)}_{11} & & & R^{(2)}_{14} & & & & \\
& R^{(2)}_{22} & R^{(2)}_{23} & & & & & \\
& -R^{(2)}_{23} & R^{(2)}_{33} & & & & & \\
-R^{(2)}_{14} & & & R^{(2)}_{44} & & & & \\
& & & & R^{(2)}_{55} & & R^{(2)}_{57} & \\
& & & & & R^{(2)}_{66} & & R^{(2)}_{68} \\
& & & & -R^{(2)}_{57} & & R^{(2)}_{77} & \\
& & & & & -R^{(2)}_{68} & & R^{(2)}_{88}
\end{bmatrix}$$

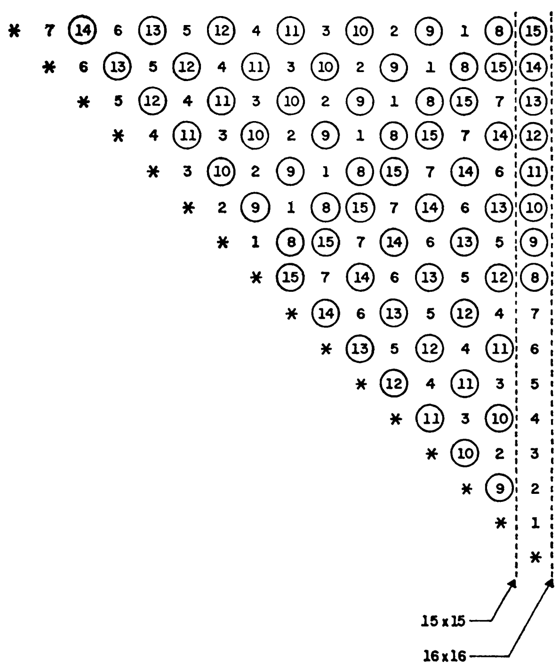
while for $k = 7$ the pairs (p, q) are $\{(8, 1); (7, 2); (6, 3); (5, 4)\}$ and R_7 is of the form

$$\begin{bmatrix}
R^{(7)}_{11} & & & & & & & R^{(7)}_{18} \\
& R^{(7)}_{22} & & & & & R^{(7)}_{27} & \\
& & R^{(7)}_{33} & & & R^{(7)}_{36} & & \\
& & & R^{(7)}_{44} & R^{(7)}_{45} & & & \\
& & & -R^{(7)}_{45} & R^{(7)}_{55} & & & \\
& & -R^{(7)}_{36} & & & R^{(7)}_{66} & & \\
& -R^{(7)}_{27} & & & & & R^{(7)}_{77} & \\
-R^{(7)}_{18} & & & & & & & R^{(7)}_{88}
\end{bmatrix}$$

If the order of the matrix is odd, say $n = 7$, then for $k = 3$ the pairs (p, q) are given by $\{(1, 2); (7, 3); (6, 4)\}$ and R_3 is of the form



For example, in a given sweep, denoting each element eliminated in the kth transformation by the integer k, the patterns of the annihilated elements for matrices of orders 16 and 15 are shown below.



Second Annihilation Regime. This regime satisfies conditions (i) and (ii) for matrices of order $n = 2^\gamma$, where γ is an integer. The elements of each orthogonal transformation, in a given sweep, R_k $(k = 1, 2, \cdots, n-1)$ are given by (2.2). For $k = 1, 2, \cdots, n/2$, the pairs (p, q) are defined by

$$
\begin{aligned}
&q = 2, 4, 6, \cdots, n, \\
(2.5) \qquad &p = q + (n - 2k + 1), \qquad q < 2k, \\
& = q - 2k + 1, \qquad q \geqq 2k.
\end{aligned}
$$

Let $n = 8$ and $k = 3$, then the pairs (p, q) are $\{(5, 2); (7, 4); (1, 6); (3, 8)\}$ and R_3

is of the form

$$\begin{bmatrix}
R_{11}^{(3)} & & & & & R_{16}^{(3)} & & \\
& R_{22}^{(3)} & & & R_{25}^{(3)} & & & \\
& & R_{33}^{(3)} & & & & & R_{38}^{(3)} \\
& & & R_{44}^{(3)} & & & R_{47}^{(3)} & \\
& -R_{25}^{(3)} & & & R_{55}^{(3)} & & & \\
-R_{16}^{(3)} & & & & & R_{66}^{(3)} & & \\
& & & -R_{47}^{(3)} & & & R_{77}^{(3)} & \\
& & -R_{38}^{(3)} & & & & & R_{88}^{(3)}
\end{bmatrix}$$

In order to construct the orthogonal transformations R_k for $k = n/2 + 1, n/2 + 2, \cdots, n - 1$, consider the sequence $L = 1, 2, \cdots, \gamma - 1$. For each value of L, there are $N = 2^{\gamma - L - 1}$ orthogonal matrices R_k given by

$$R_k = \operatorname{diag}\,(H_1^{(k)}, H_2^{(k)}, \cdots, H_t^{(k)}), \tag{2.6}$$

where $t = 2^{L-1}$, $k = n(1 - 2^{-L}) + l$, and $l = 1, 2, \cdots, N$. The sequences p and q for each $H_M^{(k)}$ $(M = 1, 2, \cdots, t)$, are defined by

$$p = i + 4N(M - 1), \qquad i = 1, 2, \cdots, 2N,$$
$$q = p + 2(N + l - 1) - 2N[O(1)],$$

where

$$\begin{aligned} O(1) &= 0, \qquad i + 2(N + l - 1) \leqq 4N, \\ &= 1, \qquad \text{otherwise}. \end{aligned}$$

Let $n = 8$, $L = 2$, and $l = 1$, then $k = 7$, and the pairs (p, q) are given by $\{(1, 3); (2, 4); (5, 7); (6, 8)\}$ and R_7 is of the form

$$\begin{bmatrix}
R_{11}^{(7)} & & R_{13}^{(7)} & & & & & \\
& R_{22}^{(7)} & & R_{24}^{(7)} & & & & \\
-R_{13}^{(7)} & & R_{33}^{(7)} & & & & & \\
& -R_{24}^{(7)} & & R_{44}^{(7)} & & & & \\
& & & & R_{55}^{(7)} & & R_{57}^{(7)} & \\
& & & & & R_{66}^{(7)} & & R_{68}^{(7)} \\
& & & & -R_{57}^{(7)} & & R_{77}^{(7)} & \\
& & & & & -R_{68}^{(7)} & & R_{88}^{(7)}
\end{bmatrix}$$

The pattern of the annihilated elements in one sweep for a matrix of order 16 is shown below, where those elements annihilated in the kth transformation are denoted by the integer k.

*	1	(15)	2	(13)	3	(14)	4	(9)	5	(10)	6	(11)	7	(12)	8
	*	8	(15)	7	(13)	6	(14)	5	(9)	4	(10)	3	(11)	2	(12)
		*	1	(14)	2	(13)	3	(12)	4	(9)	5	(10)	6	(11)	7
			*	8	(14)	7	(13)	6	(12)	5	(9)	4	(10)	3	(11)
				*	1	(15)	2	(11)	3	(12)	4	(9)	5	(10)	6
					*	8	(15)	7	(11)	6	(12)	5	(9)	4	(10)
						*	1	(10)	2	(11)	3	(12)	4	(9)	5
							*	8	(10)	7	(11)	6	(12)	5	(9)
								*	1	(15)	2	(13)	3	(14)	4
									*	8	(15)	7	(13)	6	(14)
										*	1	(14)	2	(13)	3
											*	8	(14)	7	(13)
												*	1	(15)	2
													*	8	(15)
														*	1
															*

Using one quadrant of the ILLIAC IV (64 PE's), then for a 128×128 matrix, the 64 angles of each transformation are determined simultaneously, one angle per PE. Once the transformation matrix R_k is determined, the matrix $A_{k+1} = R_k A_k R_k^t$ is computed in parallel [7]. Assuming that the matrix has converged, (using some criterion [13]), to the diagonal form after u sweeps, or after $r - 1 = u(2m - 1)$ orthogonal transformations, then the diagonal elements of $A_r = WAW^t$ are taken to be the eigenvalues of A. The columns of $W^t = (V_u V_{u-1} \cdots V_1)^t$ are the corresponding eigenvectors, where for the jth sweep $V_j = \prod_{k=1}^{2m-1} (R_k)_j$ $(j = 1, 2, \cdots, u)$.

A similar algorithm as that described above [11] has been programmed in ILLIAC IV assembly language and successfully tested on an ILLIAC IV execution simulator [1].

3. A Jacobi-Like Algorithm for Nonsymmetric Matrices. Eberlein [3], [4] showed that for an $n \times n$ matrix A, complex in general, there exists a matrix $U = \prod_l U_l(k, m)$ generated from a sequence of two-dimensional transformations $U_l(k, m)$, where (k, m) is the pivot pair, such that $A_L = U^{-1}AU$ is arbitrarily close to being normal, i.e., the matrix $(A_L A_L^* - A_L^* A_L)$ is arbitrarily small. At each stage of the iteration, based on the elements of the kth and mth rows and columns, the parameters of U_l were chosen such that the decrement of the Euclidean norm of A_l is given by

$$N^2(A_l) - N^2(U_l^{-1} A_l U_l) \geqq [1/3n(n-1)] \cdot N^2(A_l A_l^* - A_l^* A_l)$$

where $N^2(A) = \sum_{i,j} |a_{ij}|^2$.

In this paper, the above algorithm has been modified for parallel computation. The transformations U_l are n-dimensional, and their parameters are based on all the elements of the matrix A_l. A lower bound on the decrement of the Euclidean norm of A_l is given by

$$N^2(A_l) - N^2(U_l^{-1} A_l U_l) \geqq (1/4n) N^2(A_l A_l^* - A_l^* A_l).$$

Once the matrix is practically normal, one can use the optimal procedure of Goldstine and Horwitz [5] for reducing it to the diagonal form; thus the eigenvalues and eigenvectors of A are obtained.

Since a nondiagonable matrix cannot be similar to a normal matrix, then this procedure yields its best results for diagonable matrices (see Example 7 in [3, p. 84]).

Let the original matrix A be real, diagonable, and of an even order $n = 2r$ (if n is odd A is replaced by diag (A, ν) of order $n + 1$), then it can be partitioned as follows

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ A_{21} & A_{22} & \cdots & A_{2r} \\ \vdots & \vdots & & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rr} \end{bmatrix} \tag{3.1}$$

where each submatrix A_{km}, $(k, m = 1, 2, \cdots, r)$, is given by

$$A_{km} = \begin{bmatrix} a_{2k-1,2m-1} & a_{2k-1,2m} \\ a_{2k,2m-1} & a_{2k,2m} \end{bmatrix}. \tag{3.2}$$

Let

$$\begin{aligned} D_{km} &= (a_{2k-1,2m-1} - a_{2k,2m}), \\ E_{km} &= (a_{2k-1,2m} - a_{2k,2m-1}), \\ B_{km} &= (a_{2k-1,2m} + a_{2k,2m-1}), \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \kappa_1(A) &= \sum_{k,m} (D_{km}^2 + E_{km}^2), \\ \kappa_2(A) &= \sum_{k,m} D_{km}E_{km}. \end{aligned} \tag{3.4}$$

Assume also that A has been scaled such that $N^2(A) \leqq 1$, and denote the matrix $(AA^t - A^tA)$ by C.

LEMMA 1. *Let* $A' = Q^{-1}AQ$, *where* $Q = \text{diag}\,(S_1, S_2, \cdots, S_r)$, *and* $S_1 = S_2 = \cdots = S_r = S$ *is given by*

$$S = \begin{bmatrix} \cosh \psi & \sinh \psi \\ \sinh \psi & \cosh \psi \end{bmatrix}. \tag{3.5}$$

Define ψ *by*

$$\tanh 4\psi = -2\kappa_2(A)/\kappa_1(A). \tag{3.6}$$

Provided that $\kappa_1(A) > 2|\kappa_2(A)|$, *the following relation holds*

$$\Delta N^2(A) \geqq \kappa_2^2(A)/\kappa_1(A), \tag{3.7}$$

where $\Delta N^2(A) = N^2(A) - N^2(A')$ *is the decrement of the Euclidean norm of* A.

Proof. The elements of each submatrix $A'_{km} = S^{-1}A_{km}S$ are given by

$$(3.8)\quad \begin{aligned} a'_{2k-1,2m-1} &= a_{2k-1,2m-1}\cosh^2\psi - a_{2k,2m}\sinh^2\psi + \tfrac{1}{2}E_{km}\sinh 2\psi,\\ a'_{2k,2m} &= -a_{2k-1,2m-1}\sinh^2\psi + a_{2k,2m}\cosh^2\psi - \tfrac{1}{2}E_{km}\sinh 2\psi,\\ a'_{2k-1,2m} &= \tfrac{1}{2}\,D_{km}\sinh 2\psi + a_{2k-1,2m}\cosh^2\psi - a_{2k,2m-1}\sinh^2\psi,\\ a'_{2k,2m-1} &= -\tfrac{1}{2}\,D_{km}\sinh 2\psi - a_{2k-1,2m}\sinh^2\psi + a_{2k,2m-1}\cosh^2\psi. \end{aligned}$$

Therefore,

$$N^2(A'_{km}) = N^2(A_{km}) + (D^2_{km} + E^2_{km})\sinh^2 2\psi + D_{km}E_{km}\sinh 4\psi$$

and consequently,

$$(3.9)\qquad \Delta N^2(A_{km}) = -D_{km}E_{km}\sinh 4\psi - \tfrac{1}{2}(D^2_{km} + E^2_{km})(\cosh 4\psi - 1).$$

Since $N^2(A) = \sum_{k,m} N^2(A_{km})$, then

$$(3.10)\qquad \Delta N^2(A) = -\tfrac{1}{2}(\cosh 4\psi - 1)\kappa_1(A) - (\sinh 4\psi)\kappa_2(A).$$

A necessary condition for $\Delta N^2(A)$ to be an extremum with respect to ψ is $(\partial/\partial\psi)\ \Delta N^2(A) = 0$; this yields relation (3.6),

$$\tanh 4\psi = -2\kappa_2(A)/\kappa_1(A).$$

From the definition (3.4), it is clear that $\kappa_1(A) \geqq 2\ |\kappa_2(A)|$. Excluding for the time being the case $\kappa_1(A) = 2|\kappa_2(A)|$, then the second derivative of $\Delta N^2(A)$ with respect to ψ evaluated for ψ in (3.6) is given by

$$(3.11)\qquad -8\kappa_1(A)[1 - (4\kappa_2^2(A)/\kappa_1^2(A))](\cosh 4\psi)$$

and is less than zero. Thus, for the choice (3.6) of ψ, $\Delta N^2(A)$ achieves its maximum value,

$$(3.12)\qquad \Delta N^2(A) = \tfrac{1}{2}\kappa_1(A)[1 - \{1 - (4\kappa_2^2(A)/\kappa_1^2(A))\}^{1/2}]$$

which vanishes only if $\kappa_2(A) = 0$. Since one is considering the case $\kappa_1(A) > 2\ |\kappa_2(A)|$, then by the binomial theorem,

$$(3.13)\qquad (1 - 4\kappa_2^2(A)/\kappa_1^2(A))^{1/2} = 1 - \tfrac{1}{2}(4\kappa_2^2(A)/\kappa_1^2(A)) - \tfrac{1}{8}(4\kappa_2^2(A)/\kappa_1^2(A))^2 - \cdots$$

and (3.12) yields the relation (3.7). If $\kappa_1(A) = 2\ |\kappa_2(A)|$, then from (3.10), $\Delta N^2(A)$ is given by $\tfrac{1}{2}\kappa_1(A)[1 - \{(1 \pm \tanh 4\psi)/(1 - \tanh^2 4\psi)^{1/2}\}]$. Choosing $\tanh 4\psi = \mp(1 - \epsilon^2)/(1 + \epsilon^2)$, where ϵ is a small number, then $\Delta N^2(A) = \tfrac{1}{2}(1 - \epsilon)\kappa_1(A)$ which is greater than zero.

LEMMA 2. *Let $A' = P^tAP$, where P is the orthogonal transformation,*

$$(3.14)\qquad P = \operatorname{diag}(T_1, T_2, \cdots, T_r),$$

in which

$$(3.15)\qquad T_k = \begin{bmatrix} \cos\varphi_k & \sin\varphi_k \\ -\sin\varphi_k & \cos\varphi_k \end{bmatrix} \qquad (k = 1, 2, \cdots, r).$$

Then, if φ_k is determined by

$$(3.16)\qquad \tan 2\varphi_k = \frac{c_{2k-1,2k-1} - c_{2k,2k}}{2c_{2k-1,2k}},$$

where c_{ij} are the elements of the matrix $C = AA^t - A^tA$,

$$\kappa_2^2(A') \geqq \frac{1}{2n} N^2(C). \tag{3.17}$$

Proof. The 2×2 diagonal submatrices C_{kk} of the matrix C can be expressed as

$$C_{kk} = \sum_{m=1}^{r} [A_{km}A_{km}^t - A_{mk}^t A_{mk}], \qquad k = 1, 2, \cdots, r. \tag{3.18}$$

Therefore,

$$\sum_{k=1}^{r} C_{kk} = \sum_{k,m=1}^{r} [A_{km}A_{km}^t - A_{km}^t A_{km}] \tag{3.19}$$

where

$$(A_{km}A_{km}^t - A_{km}^t A_{km}) = \begin{bmatrix} E_{km}B_{km} & -D_{km}E_{km} \\ -D_{km}E_{km} & -E_{km}B_{km} \end{bmatrix}. \tag{3.20}$$

Equating the off-diagonal elements of the left- and right-hand sides of (3.19),

$$\sum_{k=1}^{r} c_{2k-1,2k} = -\sum_{k,m} D_{km}E_{km} = -\kappa_2(A). \tag{3.21}$$

Consequently, if the orthogonal matrix P is chosen such that the off-diagonal elements $c_{2k-1,2k}$, for all k, attain their maximum positive values, then the inequality (3.17) is achieved. To show that, consider the matrix $C' = A'A'^t - A'^tA'$. Since $A' = P^tAP$, then $C' = P^tCP$, and the elements of the diagonal submatrices $C'_{kk} = T_k^tC_kT_k$ are given by

$$\begin{aligned} c'_{2k-1,2k} &= c_{2k-1,2k} \cos 2\varphi_k + \tfrac{1}{2}(c_{2k-1,2k-1} - c_{2k,2k}) \sin 2\varphi_k, \\ c'_{2k-1,2k-1} &= c_{2k-1,2k-1} \cos^2 \varphi_k + c_{2k,2k} \sin^2 \varphi_k - c_{2k-1,2k} \sin 2\varphi_k, \\ c'_{2k,2k} &= c_{2k-1,2k-1} \sin^2 \varphi_k + c_{2k,2k} \cos^2 \varphi_k + c_{2k-1,2k} \sin 2\varphi_k, \quad \text{and} \\ c'_{2k,2k-1} &= c'_{2k-1,2k}. \end{aligned} \tag{3.22}$$

Hence, for $c'_{2k-1,2k}$ to be an extremum, (3.16) must hold. Also, for the choice (3.16) of φ_k, the second derivative of $c'_{2k-1,2k}$ with respect to φ_k is given by

$$-(h^2/c_{2k-1,2k}) \cos 2\varphi_k, \tag{3.23}$$

where $h = [4c_{2k-1,2k}^2 + (c_{2k-1,2k-1} - c_{2k,2k})^2]^{1/2}$. As a result if $\cos 2\varphi_k$ is of the same sign as $c_{2k-1,2k}$, $c'_{2k-1,2k}$ attains its maximum value. Restricting φ_k to the interval $[0, \pi]$, the elements of T_k are given by

$$\sin^2 \varphi_k = \tfrac{1}{2} - (c_{2k-1,2k}/h), \qquad \cos^2 \varphi_k = \tfrac{1}{2} + (c_{2k-1,2k}/h), \tag{3.24}$$

in which $\sin \varphi_k > 0$ and $\cos \varphi_k$ is of the same sign as $(c_{2k-1,2k-1} - c_{2k,2k})$. The maximum value of $c'_{2k-1,2k}$ turns out to be $\tfrac{1}{2}h$, and

$$c'_{2k-1,2k-1} = c'_{2k,2k} = \tfrac{1}{2}(c_{2k-1,2k-1} + c_{2k,2k}).$$

Excluding the case when $c_{2k-1,2k-1} = c_{2k,2k}$ and $c_{2k-1,2k} = 0$, which results in T_k

being the identity matrix and hence $c'_{2k-1,2k} = 0$, then from (3.21) one obtains the inequality

$$\kappa_2^2(A') > \sum_{k=1}^{r} c'^2_{2k-1,2k}. \tag{3.25}$$

Assuming that $\sum_{k=1}^{r} c'^2_{2k-1,2k} \geqq (1/2n)N^2(C')$, then, from the fact that the Euclidean norm is invariant under orthogonal transformations and from (3.25), one obtains relation (3.17).

From Lemmas 1 and 2, it can be seen that in order to obtain the largest possible value of $\Delta N^2(A)$, the matrix A should be subjected to the orthogonal transformation M^tAM where M is a permutation matrix determined as follows: Let $A'' = M^tAM$ and $C'' = A''A''^t - A''^tA''$, then M is chosen such that each 2×2 diagonal submatrix C''_{kk} has an element $c''_{2k-1,2k}$ of at least average absolute value of all the off-diagonal elements of C'' if any, and/or the difference $(c''_{2k-1,2k-1} - c''_{2k,2k})$ different from zero. For example, in order to bring the off-diagonal element c_{uv}, $(u < v)$, of maximum absolute value in the position (1, 2), M is given by I_{1u}, I_{2v} where $I_{ij} = I - (e_i - e_j)(e_i - e_j)^t$. Essentially, $I^t_{ij}AI_{ij}$ has the ith and jth rows and columns of A exchanged.

After the matrix A is "prepared" by the transformation M, $A' = P^tA''P$ will produce a matrix C' whose off-diagonal elements $c'_{2k-1,2k}$ are of such magnitudes that $\sum_{k=1}^{r} c'^2_{2k-1,2k}$ is at least equal to $(1/2n)N^2(C)$.

THEOREM. *Let $A = A_1$ be a diagonable matrix with an even order $n = 2r$ and $N^2(A) \leqq 1$. Let $A_{l+1} = U_l^{-1}A_lU_l$, where $U_l = M_lP_lQ_l$. If these transformations are defined as follows:*

(i) *M_l is chosen as discussed above.*

(ii) $P_l = \operatorname{diag}(T_1^{(l)}, T_2^{(l)}, \cdots, T_r^{(l)})$ *in which*

$$T_k^{(l)} = \begin{bmatrix} \cos\varphi_k^{(l)} & \sin\varphi_k^{(l)} \\ -\sin\varphi_k^{(l)} & \cos\varphi_k^{(l)} \end{bmatrix}$$

with

$$\tan 2\varphi_k^{(l)} = \frac{c_{2k-1,2k-1}^{(l)} - c_{2k,2k}^{(l)}}{2c_{2k-1,2k}^{(l)}}.$$

(iii) $Q_l = \operatorname{diag}(S_1^{(l)}, S_2^{(l)}, \cdots, S_r^{(l)})$ *in which*

$$S_1^{(l)} = S_2^{(l)} = \cdots = S_r^{(l)} = \begin{bmatrix} \cosh\psi_l & \sinh\psi_l \\ \sinh\psi_l & \cosh\psi_l \end{bmatrix}$$

with

$$\tanh 4\psi_l = -2\kappa_2(A_l')/\kappa_1(A_l')$$

where

$$A_l' = (M_lP_l)^t A_l(M_lP_l).$$

Then, $\lim_{l\to\infty} N^2(C_l) = 0$.

Proof. With no loss of generality, assume that $M_l = I$. By Lemma 2, $\kappa_2^2(A_l') \geqq (1/2n)N^2(C_l)$. From (3.3), $(D_{km}^{(l)})^2 + (E_{km}^{(l)})^2 \leqq 2N^2(A_{km}^{(l)})$, then (3.4) yields, $\kappa_1(A_l) \leqq$

$2N^2(A_l) \leqq 2$. Since the Euclidean norm is invariant under orthogonal transformations, then $\kappa_1(A_l') \leqq 2$, and hence by Lemma 1,

$$\Delta N^2(A_l) \geqq \kappa_2^2(A_l')/\kappa_1(A_l') \geqq \frac{1}{4n} N^2(C_l).$$

But since $N^2(A_l)$ is a decreasing monotone function bounded below by $\sum_i |\lambda_i|^2$, where λ_i are the eigenvalues of A, [10], then $\Delta N^2(A_l) \to 0$ as $l \to \infty$. Hence $N^2(C_l) \to 0$, and A_l is arbitrarily close to being normal.

Let A be a 128×128 matrix. Using one quadrant of the ILLIAC IV (64 PE's), the matrix can be stored in memory such that for a given m the 2×2 submatrices A_{km} $(k = 1, 2, \cdots, 64)$ are assigned to the mth PE. Once the matrix C is determined by parallel multiplication and stored in the same way, i.e., the kth PE contains the submatrix C_{kk}, the 64 angles φ_k can then be determined simultaneously. Also for each k the submatrices $A_{km}' = T_k^t A_{km} T_m$ are computed simultaneously for all m, hence the updated matrix $A' = P^t AP$ is computed with all the PE's working. Similarly the quantities D_{km}', E_{km}', and B_{km}' of the submatrices A_{km}', and consequently the submatrices $S^{-1}A_{km}'S$ are computed with full efficiency. This part of the algorithm has been coded and successfully tested on the ILLIAC IV simulator [1].

Once the matrix A is reduced to a matrix $\tilde{A}$ which is practically normal, then for any diagonal submatrix

$$\begin{bmatrix} \tilde{a}_{pp} & \tilde{a}_{pq} \\ \tilde{a}_{qp} & \tilde{a}_{qq} \end{bmatrix}$$

either $\tilde{a}_{pq} = \tilde{a}_{qp}$; or $\tilde{a}_{pq} = -\tilde{a}_{qp}$ and $\tilde{a}_{pp} = \tilde{a}_{qq}$, to within a reasonable computational error. The matrix $\tilde{A}$ is reduced to the diagonal form by the unitary transformations $V_j^* \tilde{A}_j V_j$ $(j = 1, 2, 3, \cdots)$, where $V_j = \prod_{k=1}^{2m-1} (R_k)_j$, as in Section 2, is the transformation matrix of the jth sweep. For each off-diagonal element $\tilde{a}_{pq}$ or $\tilde{a}_{qp}$ above the diagonal, the elements of the diagonal submatrices of R_k are given by

(a) $\tilde{a}_{pq}^{(k)} = \tilde{a}_{qp}^{(k)}$;

the elements $R_{pp}^{(k)}$, $R_{qq}^{(k)}$, $R_{pq}^{(k)}$, and $R_{qp}^{(k)}$ are determined as in Section 2.

(b) $\tilde{a}_{pq}^{(k)} = -\tilde{a}_{qp}^{(k)}$ and $\tilde{a}_{pp}^{(k)} = \tilde{a}_{qq}^{(k)}$;

$$R_{pp}^{(k)} = R_{qq}^{(k)} = \frac{1}{\sqrt{2}}; \qquad R_{pq}^{(k)} = R_{qp}^{(k)} = \frac{i}{\sqrt{2}}, \quad \text{where} \quad i = (-1)^{1/2} \qquad [5].$$

Denoting the resulting matrix by $\Lambda = Y^{-1}AY$, the diagonal elements of Λ are then the eigenvalues of A, and the columns of the matrix $Y = (\prod_l U_l)(\prod_j V_j)$ are the corresponding eigenvectors.

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Appendix. The orthogonal matrix $R(p, q, \alpha_{pq}^{(k)})$ differs from the identity matrix by a 2×2 diagonal submatrix whose elements are

(A.1) $$R_{pp} = R_{qq} = \cos \alpha_{pq}^{(k)}; \qquad R_{pq} = -R_{qp} = \sin \alpha_{pq}^{(k)}$$

where $p < q$. In order to eliminate the off-diagonal element $a_{pq}^{(k)}$, the angle α_{pq} is chosen such that

(A.2) $$\tan 2\alpha_{pq}^{(k)} = \frac{2a_{pq}^{(k)}}{a_{pp}^{(k)} - a_{qq}^{(k)}}$$

in which $\alpha_{pq}^{(k)}$ is restricted by $|\alpha_{pq}^{(k)}| \leqq \pi/4$, [6]. Let

$$t_k = |2a_{pq}^{(k)}|, \qquad x_k = |a_{pp}^{(k)} - a_{qq}^{(k)}|, \qquad y_k = (t_k^2 + x_k^2)^{1/2};$$

then

(A.3) $$\cos^2 \alpha_{pq}^{(k)} = \frac{1}{2}\left(1 + \frac{x_k}{y_k}\right); \qquad \sin^2 \alpha_{pq}^{(k)} = \frac{1}{2}\left(1 - \frac{x_k}{y_k}\right).$$

Since $|\alpha_{pq}^{(k)}| \leqq \pi/4$, then $\cos \alpha_{pq}^{(k)}$ will always be taken positive and $\sin \alpha_{pq}^{(k)}$ will be of the same sign as $[2a_{pq}^{(k)}/(a_{pp}^{(k)} - a_{qq}^{(k)})]$.

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