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Miniaturized Tables of Bessel Functions. III*

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Abstract. After the manner of our previous studies, coefficients for the expansion of $J_\nu(z)$ and $Y_\nu(z)$ in double series of Chebyshev polynomials are presented. For $J_\nu(z)$, the ranges are (1) $0 < z \leqq 8$, $0 \leqq \nu \leqq 4$, (2) $0 < z \leqq 8$, $4 \leqq \nu \leqq 8$. For $J_\nu(z) + iY_\nu(z)$, the ranges are $z \geqq 5$ and $0 \leqq \nu \leqq 1$. The coefficients are given with sufficient accuracy to enable the evaluation of the Bessel functions to at least 20 decimals.

1. Introduction. In previous studies [1], [2], we considered the expansion of two parameter functions in a double series of Chebyshev polynomials and developed coefficients for the evaluation of $K_\nu(z)$ and $I_\nu(z)$ over a large part of the real z and ν lines. In the present paper, we give similar type coefficients for the evaluation of $J_\nu(z)$ and $Y_\nu(z)$.

2. Chebyshev Expansion for $J_\nu(z)$. From [3, Vol. 1, p. 212 and Vol. 2, p. 35],

$$J_\nu(z) = z^\nu \sum_{k=0}^{\infty} A_k(\nu, \lambda) T_{2k}(z/\lambda), \qquad 0 < z \leqq \lambda, \tag{1}$$

$$A_k(\nu, \lambda) = G_k(\nu, \lambda)/2^\nu\Gamma(\nu + 1), \tag{2}$$

$$G_k(\nu, \lambda) = \frac{\epsilon_k(-)^k\lambda^{2k}\Gamma(\nu + 1)}{2^{4k}k!\,\Gamma(k + \nu + 1)}\, {}_1F_2\left(\begin{matrix} \frac{1}{2} + k \\ 1 + 2k, \nu + 1 + k \end{matrix} \,\middle|\, -\lambda^2/4\right), \tag{3}$$

$$\begin{aligned} \frac{2G_k(\nu, \lambda)}{\epsilon_k} = \frac{(k + 1)}{(k + 2)} \{G_{k+1}(\nu, \lambda) - G_{k+3}(\nu, \lambda)\} - \frac{16(k + 1)(k + \nu + 1)}{\lambda^2} G_{k+1}(\nu, \lambda) \\ + \left\{1 - \frac{16(k + 1)(k + 2 - \nu)}{\lambda^2}\right\} G_{k+2}(\nu, \lambda), \end{aligned} \tag{4}$$

where

$$\epsilon_0 = 1, \qquad \epsilon_k = 2 \quad \text{for } k > 0. \tag{5}$$

It is readily shown that

$$G_k(\nu, \lambda) = \frac{\epsilon_k(-)^k\lambda^{2k}k^{-\nu}}{2^{4k}(k!)^2} [1 + O(k^{-1})], \tag{6}$$

and for ν and λ fixed,

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(7) $$\lim_{k\to\infty} G_k(\nu, \lambda) = 0.$$

Thus, the expansion (1) converges and by letting $z \to 0$, we have the useful normalization relation

(8) $$\sum_{k=0}^{\infty} (-)^k A_k(\nu, \lambda) = 1.$$

Further, after the manner of the discussion presented in [3, Vol. 2, pp. 159-166], we can show that use of the recursion formula (4) in the backward direction is convergent. Thus, for a fixed λ, we can generate the coefficients $A_k(\nu, \lambda)$ for any given value of ν. Then, following the discussion given in [1], we can develop coefficients $D_{r,k}(\lambda)$ such that

(9) $$A_k(\nu, \lambda) = \sum_{r=0}^{\infty} D_{r,k}(\lambda) T_r^*\left(\frac{\nu - s}{t}\right), \qquad s \leqq \nu \leqq s + t.$$

We remark that 20 decimal values of $A_k(\nu, \lambda)$ are given in [3, pp. 331, 332, 352–356] for $\lambda = 8$ and $\nu = 0, \pm\frac{1}{4}, \pm\frac{1}{3}, \pm\frac{1}{2}, \pm\frac{2}{3}, \pm\frac{3}{4}, 1$. Coefficients for the evaluation of $Y_0(z)$ and $Y_1(z)$ for $0 < z \leqq 8$ are also given in [3, pp. 331, 332].

We next present a descending type expansion in series of Chebyshev polynomials for the evaluation of $J_\nu(z)$ and $Y_\nu(z)$ in the vicinity of $z = +\infty$. Now,

(10) $$H_\nu^{(1)}(z) = -\frac{2i}{\pi} e^{-i\nu\pi/2} K_\nu(ze^{-i\pi/2}),$$

and from [1], we have

(11) $$K_\nu(z) = (\pi/2z)^{1/2} e^{-z} \sum_{k=0}^{\infty} G_k(\nu, \lambda) T_k^*(\lambda/z), \; \lambda \text{ fixed}, \; \lambda/z \leqq 1, \; |\arg z| < 3\pi/2.$$

The recursion formula for $G_k(\nu, \lambda)$ and other properties of these coefficients are given in [1]. If we write

(12) $$\begin{aligned} H_\nu^{(1)}(z) &= J_\nu(z) + i Y_\nu(z) \\ &= (2/\pi z)^{1/2} e^{i(z - \nu\pi/2 - \pi/4)} \sum_{k=0}^{\infty} H_k(\nu, \lambda) T_k^*(\lambda/z), \qquad z \geqq \lambda, \end{aligned}$$

then the recurrence formula and other properties of the coefficients $H_k(\nu, \lambda)$ follow from those for $G_k(\nu, \lambda)$ upon replacing λ by $\lambda e^{-i\pi/2}$. We have the normalization relation

(13) $$\sum_{k=0}^{\infty} (-)^k H_k(\nu, \lambda) = 1$$

and from [1], use of the backward recurrence relation for $H_k(\nu, \lambda)$ is convergent provided $|\arg \lambda| < \pi/2$.

3. Numerical Results. From (1) and (9) with a slight change of notation, we have

(14) $$J_\nu(z) = z^\nu \sum_{k=0}^{\infty} A_k(\nu) T_{2k}(z/8), \qquad 0 < z \leqq 8,$$

$$A_k(\nu) = \sum_{r=0}^{\infty} D_{r,k} T_r^*\left(\frac{\nu - s}{t}\right), \qquad s \leqq \nu \leqq s + t. \tag{15}$$

In Tables 1 and 2, in the microfiche section of this issue, we present values of $D_{r,k}$ which were computed by the technique depicted in [1] for $s = 0$, $t = 4$ and $s = t = 4$, respectively. Values of $\Gamma(\nu + 1)$ required in the numerics were obtained by use of the schema of my previous paper [4]. Numerous checks were made on the coefficients. They are of the type previously discussed in [1], [2] and we dispense with further details. The computations were designed so that the coefficients for $0 \leqq \nu \leqq 4$ are accurate to about 25D while those for $4 \leqq \nu \leqq 8$ are accurate to about 27D. To evaluate $J_\nu(z)$, we must incorporate the value of z^ν. As $0 \leqq z \leqq 8$, we see that the coefficients are sufficiently accurate to produce $J_\nu(z)$ to about 20 decimals at least.

Now,

$$Y_\nu(z) = (\csc \nu\pi)[(\cos \nu\pi) J_\nu(z) - J_{-\nu}(z)] \tag{16}$$

and both $J_\nu(z)$ and $Y_\nu(z)$ satisfy the same recurrence formula

$$J_{\nu+1}(z) + J_{\nu-1}(z) = (2\nu/z) J_\nu(z). \tag{17}$$

Further, the recurrence formula for $J_\nu(z)$ is always stable in the backward direction, but only conditionally stable in the forward direction. On the other hand, the recurrence formula for $Y_\nu(z)$ is always stable in the forward direction. Thus, with the aid of the coefficients just described and the recurrence formulas, we can evaluate $Y_\nu(z)$ for all z such that $0 \leqq z \leqq 8$ and all $\nu > 0$, ν an integer excepted. We have already referred to the availability of coefficients to compute $Y_0(z)$ and $Y_1(z)$. These together with the recurrence formula for $Y_\nu(z)$ can be used to generate values of $Y_n(z)$, $n = 2, 3, \cdots$. As use of the recurrence formula in the forward direction for $J_\nu(z)$ is limited, we leave for a future paper the development of Chebyshev coefficients for $0 \leqq z \leqq 8$ and $\nu > 8$.

Using (12) with a slight change of notation, we write

$$J_\nu(z) + i\,Y_\nu(z) = (2/\pi z)^{1/2} e^{i(z-\nu\pi/2-\pi/4)} \sum_{k=0}^{\infty} H_k(\nu) T_k^*(5/z), \qquad z \geqq 5. \tag{18}$$

Let

$$H_k(\nu) = \sum_{r=0}^{\infty} E_{r,k} T_r^*(\nu), \qquad 0 \leqq \nu \leqq 1. \tag{19}$$

Table 3, also in the microfiche section of this issue, lists values of the real and imaginary parts of $E_{r,k}$. These were obtained and checked by the methods previously described and we omit further details. The coefficients are sufficiently accurate to produce values of $J_\nu(z)$ and $Y_\nu(z)$ for ν and z as noted to about 25 decimals. Since

$$Y_{-\nu}(z) = (\cos \nu\pi) Y_\nu(z) + (\sin \nu\pi) J_\nu(z), \tag{20}$$

$$J_{-\nu}(z) = (\cos \nu\pi) J_\nu(z) - (\sin \nu\pi) Y_\nu(z), \tag{21}$$

the coefficients $E_{r,k}$ together with the recurrence formula for $Y_\nu(z)$ enable the evaluation of $Y_\nu(z)$ for all $\nu > 0$ and $z \geqq 5$. A like statement cannot be made for $J_\nu(z)$ as use of the recurrence formula in the forward direction for $J_\nu(z)$ is limited. We defer the development of coefficients to facilitate the evaluation of $J_\nu(z)$ when both ν and z are large to a later paper.

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