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The Orders of Approximation of the First Derivative of Cubic Splines at the Knots

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Abstract. The order of approximation of the first derivative of four types of interpolating cubic splines are found. The splines are defined by a variety of endpoint conditions and include the natural cubic spline and the periodic cubic spline. It is found that for two types there is an increase in the order of approximation when equal intervals are used, and that for a special distribution of knots the same order can be realized for the natural spline.

1. Introduction. The cubic spline is now a well established tool for smooth interpolation in a table of a function defined at a discrete set of points. A useful account of the basic properties of this spline and an algorithm for constructing it can be found in [1], and an analysis of the convergence of the spline to the function it interpolates is given in [4].

The present paper is devoted to an investigation of the problem of finding how well the first derivative, taken at the knots, of the spline approximates the first derivative of the interpolated function there. It was shown in [4] that there is $O(h^3)$ approximation uniformly over the range of the knots, as the maximum interval tends to zero, but as it is often the case that the derivative is taken at the knots, it is felt that the results may be of some value.

2. Notation. The set of real numbers, $t_0, t_1, \cdots, t_N$, will be called *knots* and will satisfy

$$-\infty < t_0 < t_1 < \cdots < t_{N-1} < t_N < \infty, \qquad N \geqq 2.$$

The *interval* $t_i \leqq t \leqq t_{i+1}$ will have *length* $h_i = t_{i+1} - t_i$, $i = 0(1)N - 1$, and the *maximum interval length* will be h, that is,

$$h = \max_{0 \leqq i \leqq N-1} h_i.$$

y will denote a *cubic spline* with the above knots. As stated in Section 1, more than one kind of spline will be considered but they will have the common property that each is a member of $C^2(-\infty, \infty)$ and that in each interval they are polynomials of degree at most three.

x will be a member of $C^5[t_0, t_N]$ and will be the function with which the spline agrees at the knots. For brevity, define

$$x_i^{(r)} = \left(\frac{d}{dt}\right)^r x(t), \quad \text{for } t = t_i,\ i = 0(1)N,\ r = 0(1)5.$$

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Then

$$y_i = x_i, \qquad i = 0(1)N.$$

The norms which will be used are the uniform norms for functions, vectors and matrices, namely,

$$||x|| = \max_{t_0 \leq t \leq t_N} |x(t)|, \qquad ||\mathbf{x}|| = \max_i |x_i|, \qquad ||A|| = \max_i \sum_j |a_{ij}|.$$

The domain of the suffixes in the vector and matrix norms will be clear from the context.

It is convenient to define here

$$M_4 = ||x^{(4)}||, \qquad M_5 = ||x^{(5)}||.$$

The first and last columns of the $(n + 1) \times (n + 1)$ unit matrix will be written respectively as $\mathbf{e}_0$, $\mathbf{e}_n$; the jth element of the vector $\mathbf{x}$ will be denoted by $[\mathbf{x}]_j$.

3. The Cubic Splines. Four types of cubic splines will be described in this section. Cubic splines are usually characterized by the value of their second derivative at each of the knots (see for example [1]), but for the purpose of this note, an alternative method will be used.

Let

$$\lambda_i = y_i^{(1)}, \qquad i = 0(1)N,$$

then, if $y(t)$ takes the same value as $x(t)$ at each of the knots, it follows from Hermite's two point interpolation formula that, for $t_i \leqq t \leqq t_{i+1}$,

$$\begin{aligned} y(t) = {} & \left[3\left(\frac{t_{i+1} - t}{h_i}\right)^2 - 2\left(\frac{t_{i+1} - t}{h_i}\right)^3\right]x_i + \left[3\left(\frac{t - t_i}{h_i}\right)^2 - 2\left(\frac{t - t_i}{h_i}\right)^3\right]x_{i+1} \\ & + h_i\left[\left(\frac{t_{i+1} - t}{h_i}\right)^2 - \left(\frac{t_{i+1} - t}{h_i}\right)^3\right]\lambda_i - h_i\left[\left(\frac{t - t_i}{h_i}\right)^2 - \left(\frac{t - t_i}{h_i}\right)^3\right]\lambda_{i+1}, \\ & \qquad i = 0(1)N - 1. \end{aligned} \tag{1}$$

A simple calculation shows that

$$\begin{aligned} h_i^2 y_i^{(2)} &= 6(x_{i+1} - x_i) - h_i(4\lambda_i + 2\lambda_{i+1}), \\ h_i^2 y_{i+1}^{(2)} &= -6(x_{i+1} - x_i) + h_i(2\lambda_i + 4\lambda_{i+1}). \end{aligned} \tag{2}$$

Now, as $y \in C^2(-\infty, \infty)$, the two expressions for $y_i^{(2)}$ from the equations which arise from the intervals (t_{i-1}, t_i), (t_i, t_{i+1}) must be equal. The identification gives the equations:

$$\frac{\lambda_{i-1} + 2\lambda_i}{h_{i-1}} + \frac{2\lambda_i + \lambda_{i+1}}{h_i} = 3\left[\frac{x_{i+1} - x_i}{h_i^2} + \frac{x_i - x_{i-1}}{h_{i-1}^2}\right], \qquad i = 1(1)N - 1. \tag{3}$$

It is convenient to define

$$\alpha_i = h_{i-1}/(h_{i-1} + h_i),$$

then the equations become

$$(1 - \alpha_i)\lambda_{i-1} + 2\lambda_i + \alpha_i\lambda_{i+1} = 3\left[\alpha_i \frac{(x_{i+1} - x_i)}{h_i} + (1 - \alpha_i)\frac{(x_i - x_{i-1})}{h_{i-1}}\right],$$

$$i = 1(1)N - 1,$$

which can be written as

$$(1 - \alpha_i)(\lambda_{i-1} - x_{i-1}^{(1)}) + 2(\lambda_i - x_i^{(1)}) + \alpha_i(\lambda_{i+1} - x_{i+1}^{(1)})$$
$$= -(1 - \alpha_i)x_{i-1}^{(1)} - 2x_i^{(1)} - \alpha_i x_{i+1}^{(1)} + 3\left[\alpha_i \frac{(x_{i+1} - x_i)}{h_i} + (1 - \alpha_i)\frac{(x_i - x_{i-1})}{h_{i-1}}\right],$$

$$i = 1(1)N - 1.$$

Finally, the use of Peano's method for finding remainders gives the result that

$$(1 - \alpha_i)(\lambda_{i-1} - x_{i-1}^{(1)}) + 2(\lambda_i - x_i^{(1)}) + \alpha_i(\lambda_{i+1} - x_{i+1}^{(1)})$$
$$= \frac{1}{24} h_{i-1}h_i(h_{i-1} - h_i)x_i^{(4)} - \frac{1}{60} h_{i-1}h_i(h_{i-1}^2 + h_i^2 - h_{i-1}h_i)x^{(5)}(\tau_i), \tag{4}$$

where $t_{i-1} \leqq \tau_i \leqq t_{i+1}$, $i = 1(1)N - 1$.

The sets of Eqs. (3), (4) are satisfied by $\lambda_0, \lambda_1, \cdots, \lambda_N$ for each of the splines to be considered. Clearly, two further relations are needed in order that a unique interpolating spline may be found. The equations (3) are the useful ones for the actual calculation of the splines and, for completeness, the two relations to be adjoined to (3) will be given for the different types of splines to be described. For this note, however, (4) are the useful equations and these relations will have to be written in a form similar to (4).

(A) *Natural Cubic Spline.* The relations which help to define this spline are [1]

$$y_0^{(2)} = y_N^{(2)} = 0,$$

whence, from (2), the equations additional to (3) are

$$2\lambda_0 + \lambda_1 = \frac{3}{h_0}(x_1 - x_0),$$
$$\lambda_{N-1} + 2\lambda_N = \frac{3}{h_{N-1}}(x_N - x_{N-1}). \tag{5a}$$

With the aid of Peano's method these can be written

$$2(\lambda_0 - x_0^{(1)}) + (\lambda_1 - x_1^{(1)}) = \frac{1}{2}h_0x_0^{(2)} - \frac{1}{24}h_0^3x_0^{(4)} - \frac{1}{60}h_0^4x^{(5)}(\tau_0),$$

$$t_0 \leqq \tau_0 \leqq t_1, \quad \text{and}$$

$$(\lambda_{N-1} - x_{N-1}^{(1)}) + 2(\lambda_N - x_N^{(1)}) = -\frac{1}{2}h_{N-1}x_N^{(2)} + \frac{1}{24}h_{N-1}^3x_N^{(4)} + \frac{1}{60}h_{N-1}^4x^{(5)}(\tau_N), \tag{5b}$$

$$t_{N-1} \leqq \tau_N \leqq t_N.$$

These equations together with (4) are, in matrix form,

$$\text{(5c)}\quad \begin{bmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1-\alpha_1 & 2 & \alpha_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 2 \end{bmatrix} (\boldsymbol{\lambda} - \mathbf{x}^{(1)})$$

$$= \frac{1}{2} h_0 \left[x_0^{(2)} - \frac{1}{12} h_0^2 x_0^{(4)} \right] \mathbf{e}_0 - \frac{1}{2} h_{N-1} \left[x_N^{(2)} - \frac{1}{12} h_{N-1}^2 x_N^{(4)} \right] \mathbf{e}_N + \mathbf{x}^{(4)} + \mathbf{x}^{(5)}$$

where

$$\boldsymbol{\lambda} - \mathbf{x}^{(1)} = [\lambda_0 - x_0^{(1)}\ \lambda_1 - x_1^{(1)} \cdots \lambda_N - x_N^{(1)}]^T,$$

$$\text{(5d)}\quad \mathbf{x}^{(4)} = \frac{1}{24} [0\ h_0 h_1 (h_0 - h_1) x_1^{(4)} \cdots h_{N-2} h_{N-1} (h_{N-2} - h_{N-1}) x_{N-1}^{(4)}\ 0]^T,$$

$$\text{(5e)}\quad \mathbf{x}^{(5)} = -\frac{1}{60} [h_0^4 x^{(5)}(\tau_0)\ h_0 h_1 (h_0^2 + h_1^2 - h_0 h_1) x^{(5)}(\tau_1) \cdots -h_{N-1}^4 x^{(5)}(\tau_N)]^T.$$

(B) *Cubic Spline D*1. Here, $y_0^{(1)}$ and $y_N^{(1)}$ are fitted exactly, and so

$$\text{(6a)}\quad \lambda_0 = x_0^{(1)}, \qquad \lambda_N = x_N^{(1)}$$

are the equations to be put with (3) for the calculation of this spline. Further, (4) can now be written as

$$\text{(6b)}\quad \begin{bmatrix} 2 & \alpha_1 & 0 & \cdots & 0 & 0 \\ 1-\alpha_2 & 2 & \alpha_2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1-\alpha_{N-1} & 2 \end{bmatrix} (\boldsymbol{\lambda} - \mathbf{x}^{(1)}) = \mathbf{x}^{(4)} + \mathbf{x}^{(5)},$$

where

$$\boldsymbol{\lambda} - \mathbf{x}^{(1)} = [\lambda_1 - x_1^{(1)} \cdots \lambda_{N-1} - x_{N-1}^{(1)}]^T,$$

$$\text{(6c)}\quad \mathbf{x}^{(4)} = \frac{1}{24} [h_0 h_1 (h_0 - h_1) x_1^{(4)} \cdots h_{N-2} h_{-N-1} (h_{N-2} - h_{N-1}) x_{N-1}^{(4)}]^T,$$

$$\text{(6d)}\quad \mathbf{x}^{(5)} = -\frac{1}{60} [h_0 h_1 (h_0^2 + h_1^2 - h_0 h_1) x^{(5)}(\tau_1) \cdots h_{N-2} h_{N-1} (h_{N-2}^2 + h_{N-1}^2 - h_{N-2} h_{N-1}) x^{(5)}(\tau_{N-1})]^T.$$

(C) *Cubic Spline D*2. If

$$y_0^{(2)} = x_0^{(2)}, \qquad y_N^{(2)} = x_N^{(2)},$$

then, from (2), the equations additional to (3) are

$$\text{(7)}\quad \begin{aligned} 2\lambda_0 + \lambda_1 &= \frac{3}{h_0} (x_1 - x_0) - h_0 x_0^{(2)}, \\ \lambda_{N-1} + 2\lambda_N &= \frac{3}{h_{N-1}} (x_N - x_{N-1}) + h_{N-1} x_N^{(2)}. \end{aligned}$$

Peano's theorem gives the results

$$2(\lambda_0 - x_0^{(1)}) + (\lambda_1 - x_1^{(1)}) = -\frac{1}{24} h_0^3 x_0^{(4)} - \frac{1}{60} h_0^4 x^{(5)}(\tau_0),$$

$$(\lambda_{N-1} - x_{N-1}^{(1)}) + 2(\lambda_N - x_N^{(1)}) = \frac{1}{24} h_{N-1}^3 x_N^{(4)} - \frac{1}{60} h_{N-1}^4 x^{(5)}(\tau_N).$$

On comparison with the corresponding ones for the cubic spline, namely (5b), it is seen that the matrix equation for this spline is identical with (5c) except that the terms $x_0^{(2)}$ and $x_N^{(2)}$ are replaced by zero.

(D) *Periodic Cubic Spline.* When x has period $t_N - t_0$ and $x_0^{(r)} = x_N^{(r)}, r = 0, 1, \cdots$, then the spline can be taken to be periodic in the sense that

$$y_0^{(r)} = y_N^{(r)}, \qquad r = 0, 1, 2. \tag{8a}$$

The Eqs. (3) remain valid but in the first λ_0, x_0 can be replaced by λ_N, x_N, respectively. An additional equation arises from the observation that $y_0^{(2)} = y_N^{(2)}$ and is, after simplification,

$$\beta\lambda_1 + (1 - \beta)\lambda_{N-1} + 2\lambda_N = 3\left[\beta\left(\frac{x_1 - x_0}{h_0}\right) + (1 - \beta)\left(\frac{x_N - x_{N-1}}{h_{N-1}}\right)\right]. \tag{8b}$$

In the required form, this is

$$\begin{aligned} &\beta(\lambda_1 - x_1^{(1)}) + (1 - \beta)(\lambda_{N-1} - x_{N-1}^{(1)}) + 2(\lambda_N - x_N^{(1)}) \\ &\quad = \frac{1}{24} h_{N-1}h_0(h_{N-1} - h_0)x_0^{(4)} - \frac{1}{60} h_{N-1}h_0(h_{N-1}^2 + h_0^2 - h_{N-1}h_0)x^{(5)}(\pi) \end{aligned} \tag{8c}$$

where $\beta = h_{N-1}/(h_0 + h_{N-1})$, and $t_0 - h_{N-1} \leqq \pi \leqq t_1$.

Thus, the matrix equation is

$$\begin{bmatrix} 2 & \alpha_1 & 0 & \cdots & 0 & 1 - \alpha_1 \\ 1 - \alpha_2 & 2 & \alpha_2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \beta & 0 & 0 & \cdots & 1 - \beta & 2 \end{bmatrix} (\boldsymbol{\lambda} - \mathbf{x}^{(1)}) = \mathbf{x}^{(4)} + \mathbf{x}^{(5)}, \tag{8d}$$

where

$$\begin{aligned} \boldsymbol{\lambda} - \mathbf{x}^{(1)} &= [\lambda_1 - x_1^{(1)} \cdots \lambda_N - x_N^{(1)}]^T, \\ \mathbf{x}^{(4)} &= \frac{1}{24} [h_0h_1(h_0 - h_1)x_1^{(4)} \cdots h_{N-1}h_0(h_{N-1} - h_0)x_N^{(4)}]^T, \end{aligned} \tag{8e}$$

$$\begin{aligned} \mathbf{x}^{(5)} = -\frac{1}{60} [&h_0h_1(h_0^2 + h_1^2 - h_0h_1)x^{(5)}(\tau_1) \\ &\cdots h_{N-1}h_0(h_{N-1}^2 + h_0^2 - h_{N-1}h_0)x^{(5)}(\pi)]^T. \end{aligned} \tag{8f}$$

4. Error in the First Derivatives of the Splines at the Knots. It will be noticed that the matrices which occurred in Section 3 for each of the splines are strictly diagonally dominant, and so the equations can be solved. Further, if A represents any of them then, with the uniform norm $||A^{-1}|| \leqq 1$. This follows from the observation that if $||A\mathbf{x}|| \geqq 1$ for $||\mathbf{x}|| = 1$, then $||A^{-1}|| \leqq 1$. Now, $A = 2I + B$, where $||B|| \leqq 1$ and so $||A\mathbf{x}|| \geqq 2||\mathbf{x}|| - ||B\mathbf{x}||$ and, as $||B\mathbf{x}|| \leqq ||B|| \leqq 1$, the result is proved.

THEOREM 1. *If y is either a cubic spline $D1$ or a periodic cubic spline, then*

$$||\lambda - \mathbf{x}^{(1)}|| \leqq \frac{1}{24} h^2 \max_i |h_{i-1} - h_i| \cdot M_4 + \frac{1}{60} h^4 M_5. \tag{9}$$

Proof. In (6b) and (8d), multiply by the inverse of the respective matrices, and take the uniform norm of each side. Then,

$$||\lambda - \mathbf{x}^{(1)}|| \leqq ||\mathbf{x}^{(4)}|| + ||\mathbf{x}^{(5)}||,$$

where $\mathbf{x}^{(4)}$, $\mathbf{x}^{(5)}$ are defined by (6c), (6d) for the $D1$ spline and by (8e), (8f) for the periodic spline.

The results now follow on taking the uniform norms of $\mathbf{x}^{(4)}$, $\mathbf{x}^{(5)}$.

COROLLARY. *If $h_i = h$, $i = 0(1)N - 1$, then, if y is either a cubic spline $D1$ or a periodic cubic spline, then*

$$||\lambda - \mathbf{x}^{(1)}|| \leqq \frac{1}{60} h^4 M_5. \tag{10}$$

The remaining types of splines will be taken together as the analysis is common to them both. The equations for the natural cubic spline are given by (5c). Denote by A the matrix. Then, after multiplying (5c) by A^{-1} it will easily be seen that

$$\begin{aligned} |\lambda_j - x_j^{(1)}| \leqq\ & h_0[C_1 + h_0^2 D_1]\, |[A^{-1}\mathbf{e}_0]_j| + h_{N-1}[C_2 + h_{N-1}^2 D_2]\, |[A^{-1}\mathbf{e}_N]_j| \\ & + \frac{1}{24} h^2 \max |h_{i-1} - h_i| \cdot M_4 + \frac{1}{60} h^4 M_5, \qquad j = 0(1)N, \end{aligned} \tag{11}$$

where

$$C_1 = \frac{1}{2} |x_0^{(2)}|, \qquad C_2 = \frac{1}{2} |x_N^{(2)}|, \qquad D_1 = \frac{1}{24} |x_0^{(4)}|, \qquad D_2 = \frac{1}{24} |x_N^{(4)}|.$$

The corresponding inequalities for the cubic spline $D2$ are found by putting $C_1 = C_2 = 0$ in (11) and are

$$\begin{aligned} |\lambda_j - x_j^{(1)}| \leqq\ & h_0^3 D_1\, |[A^{-1}\mathbf{e}_0]_j| \\ & + h_{N-1}^3 D_2\, |[A^{-1}\mathbf{e}_N]_j| + \frac{1}{24} h^2 \max_i |h_{i-1} - h_i|\, M_4 + \frac{1}{60} h^4 M_5. \end{aligned} \tag{12}$$

Clearly, the nonvanishing of the multipliers of $[A^{-1}\mathbf{e}_0]_j$, $[A^{-1}\mathbf{e}_N]_j$ have an adverse effect on the approximations in (12) when the intervals are equal, and for the natural spline this is apparently disastrous, even when the intervals are equal. But, on examination, it is seen that to increase the order of approximation in both cases it is necessary only to make the first and last intervals small enough. The situations can be saved a little in the general case of unequal intervals as shown in the following theorems.

THEOREM 2. *If y is a natural cubic spline, $h < 1$ and if $N \geqq 2 - 2r \log h/\log \alpha$, there exist integers p, q, $0 \leqq p < q \leqq N$, such that, for $p \leqq j \leqq q$,*

$$\begin{aligned} |\lambda_j - x_j^{(1)}| \leqq\ & \tfrac{2}{3} h_0 h^r [C_1 + h_0^2 D_1] + \tfrac{2}{3} h_{N-1} h^r [C_2 + h_{N-1}^2 D_2] \\ & + \frac{1}{24} h^2 \max_i |h_{i-1} - h_i| \cdot M_4 + \frac{1}{60} h^4 M_5, \end{aligned}$$

where the real number α is

(i) $2 + \sqrt{3}$ *if* $h_i = h$,

(ii) 2 *when the intervals are unequal.*

Also

$$t_p - t_0 < h[1 - r \log h/\log \alpha], \qquad t_N - t_q < h[1 - r \log h/\log \alpha].$$

Proof. This depends on results from [2], where it is shown that for equal intervals

$$|[A^{-1}\mathbf{e}_0]_j| = U_{N-j}(2)/U_{N+1}(2), \qquad |[A^{-1}\mathbf{e}_N]_j| = U_j(2)/U_{N+1}(2), \qquad j = 0(1)N,$$

and from [3], where it is shown that when the intervals are not equal

$$|[A^{-1}\mathbf{e}_0]_j| \leqq \tfrac{2}{3}\cdot 2^{j-N}, \qquad |[A^{-1}\mathbf{e}_N]_j| \leqq \tfrac{2}{3}\cdot 2^{-j}.$$

Now,

$$\frac{U_j(2)}{U_{N+1}(2)} = \frac{(2+\sqrt{3})^{j+1} - (2-\sqrt{3})^{j+1}}{(2+\sqrt{3})^{N+2} - (2-\sqrt{3})^{N+2}} < (2+\sqrt{3})^{j-N-1}, \qquad j = 0(1)N,$$

and similarly,

$$U_{N-j}(2)/U_{N+1}(2) < (2+\sqrt{3})^{-j-1}, \qquad j = 0(1)N.$$

Hence, (11) can be replaced by

$$|\lambda_j - x_j^{(1)}| \leqq h_0[C_1 + h_0^2 D_1]\alpha^{-j} + h_{N-1}[C_2 + h_{N-1}^2 D_2]\alpha^{j-N-1}$$

$$+ \frac{1}{24} h^2 \max_i |h_{i-1} - h_i| \cdot M_4 + \frac{1}{60} h^5 M_5, \qquad j = 0(1)N, \tag{13}$$

where $\alpha = 2 + \sqrt{3}$ if $h_i = h$ and $\alpha = 2$ otherwise.

(For simplicity of presentation, the factor $\frac{2}{3}$ which should occur in these inequalities when $\alpha = 2$ and the factor $2 - \sqrt{3}$ when $\alpha = 2 + \sqrt{3}$ have been replaced by unity.)

As $\alpha > 1$, it follows that α^{-j} decreases with increasing j, and so $\alpha^{-j} \leqq h^r$ for all $j \geqq p$ where the integer p satisfies $\alpha^{-p} \leqq h^r < \alpha^{-p+1}$, that is

$$-r \log h/\log \alpha \leqq p < 1 - r \log h/\log \alpha.$$

Similarly, $\alpha^{j-N} \leqq h^r$ for all $j \leqq q$ where the integer q satisfies

$$N - 1 + r \log h/\log \alpha < q \leqq N + r \log h/\log \alpha.$$

In order that $p < q$, it is sufficient that

$$N - 1 - r \log h/\log \alpha - 1 - r \log h/\log \alpha \geqq 0$$

which is equivalent to

$$N \geqq 2 - 2r \log h/\log \alpha.$$

It remains to note that

$$t_p - t_0 \leqq ph < h[1 - r \log h/\log \alpha],$$

$$t_N - t_q \leqq (N - q)h < h[1 - r \log h/\log \alpha].$$

(The inequality $N \geqq 2 - 2r \log h/\log \alpha$ will be satisfied for sufficiently large N as $Nh \geqq t_N - t_0$.)

COROLLARY. *If y is a cubic spline ($D2$), $h < 1$ and if $N \geqq 2 - 2r \log h/\log \alpha$ then there exist integers p, q, $0 \leqq p < q \leqq N$ such that, for $p \leqq j \leqq q$,*

$$|\lambda_j - x_j^{(1)}| \leqq \frac{2}{3} h_0^3 h^r D_1 + \frac{2}{3} h_{N-1}^3 h^r D_2 + \frac{1}{24} h^2 \max_i |h_{i-1} - h_2| \cdot M_4 + \frac{1}{60} h^4 M_5,$$

where α is

(i) $2 + \sqrt{3}$ *if* $h_i = h$,

(ii) 2 *when the intervals are unequal.*

Also

$$t_p - t_0 < h[1 - r \log h/\log \alpha], \qquad t_N - t_q < h[1 - r \log h/\log \alpha].$$

Proof. This follows from Theorem 2 on setting $C_1 = C_2 = 0$.

Conclusions. The approximation of the first derivative at the knots is best when equal intervals are used both for the cubic spline $D1$ and the periodic cubic spline. In each case, the approximation is $O(h^4)$. When unequal intervals are used, it drops to $O(h^3)$. For the cubic spline $D2$, the order is generally $O(h^3)$ whether the intervals are equal or not, but with equal intervals and for a large enough number of points, the order is $O(h^4)$ at a number of internal knots.

The first derivative of the natural cubic spline is only an $O(h)$ approximation to the first derivative of the interpolated function at the knots, although for a sufficiently large number of knots the order can be made $O(h^3)$ or $O(h^4)$ at a range of internal points if the intervals are respectively unequal or equal.

Similar theorems can be proved for other types of cubic splines with mixed end conditions. It is worth remarking that if one end only is 'natural', for example $y_N^{(2)} = 0$, then the effect of this on the approximation will decrease rapidly as this point is left (by a factor of $2 - \sqrt{3}$ for equal intervals and 0.5 for unequal intervals).

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