REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

The numbers in brackets are assigned according to the indexing system printed in Volume 22, Number 101, January 1968, page 212.

28[2.10, 7].—A. GHIZZETTI & A. OSSICINI, *Quadrature Formulae*, Academic Press' New York, 1970, 192 pp., 24 cm. Price $10.00.

During the last fifteen years the authors have developed an approach to the construction of quadrature formulae based on earlier work of Von Mises and Radon in 1935. Briefly, given a set of abscissas $\{x_i\}$, a weight function $w(x)$ and a linear differential operator

$$Ef = \sum_{k=0}^{n} a_k(x)f^{(n-k)}(x),$$

one may construct a quadrature formula

$$\int_a^b w(x)f(x)\,dx = \sum_{h=0}^{n-1}\sum_{i=1}^{m} A_{h,i}f^{(h)}(x_i) + Rf$$

having the property that it is exact, i.e., $Rf = 0$ for all functions $f(x)$ satisfying $Ef = 0$. The method of construction involves determining a set of analytic solutions $\phi_j(x)$, $j = 0, 1, \cdots, m$, to the differential equation

$$E^*\phi = w(x),$$

where E^* is the adjoint operator of E. Using the Green-Lagrange identity, a relatively straightforward formula for the weights $A_{h,i}$ ensues which involves linear combinations of the functions $\phi_i(x)$ and their derivatives evaluated at $x = x_i$.

This approach is naturally of theoretical interest and was entirely new to this reviewer. Chapters 1, 2 and 5 describe this theory. These consist essentially of an edited translation of the authors' research papers. They are compactly written, provide little in the way of motivation and are just as difficult to follow as the average research paper. But they are in English (and not Italian) and, in providing the translation, the authors have carried out a very useful service.

The book contains six chapters and a largely disjoint bibliography. Over half the book is contained in Chapters 3 and 4. These are essentially devoted to the individual derivation of virtually every quadrature formula of specified polynomial degree of which this reviewer has ever heard. In Chapter 3, a concise derivation of the required properties of the special functions required for this purpose is given. This includes topics such as the Bernoulli and Jacobi polynomials and the Gamma function. In Chapter 4, a concise derivation of each set of quadrature rules is carried out, one by one. These chapters are also written in the 'research journal' style but are almost devoid of references.

It is only human for a reader who is familiar with this topic to compare this approach with the standard approach. One difficulty which is present in this approach but not present in standard treatments arises in cases in which one is seeking a derivative free quadrature formula. The $m(n - 1)$ arbitrary parameters which occur in the choice of $\phi_i(x)$ have to be chosen to make the weights $A_{h,i} = O_i h \neq 0$. This introduces a linear constraint problem. Once this is solved or circumvented, the derivation of a quadrature formula follows the standard familiar pattern, with occasionally some minor variant.

The only 'new' formulae which I noticed consisted of a set due to Rebolia and Varna which were analogues of up to five-point Gaussian type formulae involving, besides function values, up to four derivatives at each point. Apart from this short collection, the authors apparently have not used their theory to provide any new quadrature formulae or to derive any hitherto undiscovered properties of known quadrature formulae.

To summarize, the part of this book which deals with the theory should be of interest to research workers in the field of numerical quadrature. The remainder of the book consists of a repository of special derivations of quadrature formulae using this new method, and may be of interest to research workers.

J. N. L.

29[2.20, 2.35, 3, 13.35].—E. Polak, *Computational Methods in Optimization: A Unified Approach*, Academic Press, New York, 1971, xvii + 329 pp., 24 cm. Price $17.50.

The intent of this detailed book is to present in a unified manner almost all of the important algorithms invented to date for solving nonlinear programming, optimal control, root finding, and boundary value problems. The first chapter contains the statements of the problems to be solved, the John, Kuhn-Tucker, and Pontryagin conditions which characterize solutions to these problems, and an exposition of simple prototype models of algorithms for solving these problems. The second chapter deals with methods for finding points in R^n which minimize continuously differentiable functions and then applies these to the problem of finding solutions to unconstrained discrete optimal control and unconstrained continuous optimal control problems. Included are the methods of steepest descent, golden section search (for unconstrained problems in one variable), Newton-Raphson, local variations, conjugate gradients, variable metric (Davidon-Fletcher-Powell), and a modified quasi-Newton method of the author's. Several modifications of the above methods are described. In Chapter 3, the Newton-Raphson method for solving nonlinear equations is used as a basis for algorithms to solve equality constrained optimization problems in R^n, boundary value and discrete optimal control problems, and boundary value and continuous optimal control problems.

Algorithms for solving the general nonlinear programming problem with inequality and equality constraints are described in Chapter 4. Interior and exterior penalty function methods, methods of centers, methods of feasible directions, and gradient projection methods are covered. The use of some of these methods for solving optimal control problems is explained. Chapter 5 deals with discrete optimal control problems which can be cast as convex nonlinear programming problems. Some of these are shown to be solvable by standard linear and quadratic programming techniques. A dual decomposition algorithm is described for solving a restricted class of convex optimal control problems which can be transcribed into a 'primal problem.' A more general class of problems, 'geometric problems,' is defined, and a primal decomposition algorithm for solving these is explained. The final chapter gives theorems on the rates of convergence of the steepest descent, Newton-Raphson, conjugate gradient, and variable metric methods for minimizing an unconstrained function. The rates of convergence of the conjugate gradient and variable metric methods have been only recently proved (by A. Cohen and M. J. D. Powell, respectively) and their inclusion makes this chapter very current.

Three concluding appendices contain generalizations of the author's ideas on the implementation of conceptual algorithms, background material on properties of continuous and convex functions, and a summary of the author's preferences for implementable versions of many of the algorithms presented in the book.

Most students using this book as a text would face several difficulties. The writing is generally good, and some explanations (such as that on the reason why the method of steepest descent works poorly) are illuminating, but the author uses hardly any examples (there are no small illustrative examples with numbers in the book). The exercises, some of which are intriguing for advanced researchers who already know the material, are no aid to learning since they consist almost entirely of requests to complete uncompleted proofs or generalize already extended concepts or algorithms. The exposition contains much material which is not explained but could easily have been explained in Appendix B. The appendix covers convex functions but makes no mention of convex sets. The text contains such terms as 'strictly convex set' and uses the fundamental separation theorem for convex sets. Topological concepts such as Banach spaces and Hilbert spaces appear without definition, usually when such generality is unwarranted.

The exposition is uneven and often emphasizes and complicates small points while ignoring or failing to make clear significant ones. For example, in Section 1.2, he discusses optimality conditions. Because the concept of a local minimizer is not brought up, the reader is not informed that the John conditions apply to other than global minimizers. He omits the second order necessary and second order sufficiency conditions, thereby failing to inform the reader of an important subject matter of optimization theory (and practice). In the statement of the John theorem (without proof), the feasibility requirement on the optimal point is omitted, the notation does not show the association of the multipliers with the particular optimal point, and it is not at all clear what the phrase "not all zero" modifies. Then, an interesting but minor proposition is proved using terms 'ray,' 'cone,' and 'separated' when it could have been proved directly from the John theorem.

There is a serious question whether the author's material on 'simple' prototype models makes it easier for the student to learn the variety of algorithms presented later.

The instructor, in addition to explaining the material in the book to the student, supplying examples and exercises, must also supply additional material to ensure full coverage of the growing field of optimization theory. Two key omissions are the generalized reduced gradient method and the combined gradient projection-variable metric method. These are two of the most successful algorithms implemented to date for solving nonlinear programming problems. Methods of feasible directions, on which the author spends much time, have been quite unsuccessful for nonlinear problems.

The instructor will also have to unteach much of the vocabulary learned from the book. The author uses 'quasi-Newton' to mean Newton-Raphson type algorithms. The term 'quasi-Newton' has come to apply to first order iteration schemes for approximation of the inverse Hessian matrix. Most people use 'methods of conjugate directions' for the class of algorithms the author calls 'conjugate gradient methods.' The conjugate gradient method is a particular member of this class. The author also includes, without discussion, the variable metric method as a member of this class although it should be obvious from the proof by Powell that it owes much of the superlinear rate of convergence characteristics to its 'quasi-Newton' character. The author never explains the principles behind the methods of conjugate directions, using as a prototype algorithm a useless general statement.

Finally, the instructor will have to disabuse the student of almost all the opinions about what constitutes a 'good' 'implementable' algorithm. Suggestions abound for parameter selection. These selections seem to be drawn from extensive computer efforts on problems with two variables. Each implementable algorithm has at least two parameters so one can probably obtain very good performance indeed on such problems. In statements like "Generally, exterior penalty function methods are considered to perform better than interior penalty function methods . . ." the author strays from his personal opinions and implies some concensus on this point. The general concensus is just the opposite. In fact, the author's earlier reasoned discussion tends to contradict this.

Another way of looking at this book is for its value as a reference or research monograph. Generally speaking, the author has been meticulous in his proofs of theorems and the book is almost error free. Where the author falls down is in the theorem statements, the hypotheses and conclusions. For one example, consider the convergence proofs for exterior penalty function methods. For the problem: minimize $f^0(z)$ subject to $z \in C \subset R^n$, he assumes that the set $\{z | f^0(z) \leqq f^0(z^1)\}$ is compact for some $z^1 \in C$. This makes the proving of convergence very easy but avoids all the hard questions such as what happens when the exterior penalty function method is applied to a problem when this assumption does not hold (and it usually does not). There are interesting answers to this question involving convergence to compact sets of local minimizers and, for the convex case, proofs of the boundedness of penalty function contours deriving from compact set solutions of the original problem. In short, the author leaves out much interesting fundamental material.

For another example, in the proof of the convergence of the variable metric method and its rate of convergence by Powell, the author has confused to a high degree the hypotheses of both portions of the paper. On p. 46, he presents a confused statement about the uniform bounds required on the eigenvalues of the Hessian matrix which is required for Powell's convergence proof. On p. 58, he states a Lipschitz condition which should be required for the rate of convergence proof and later on p. 269 he changes that to the assumption that $f^0 \in C^3$. Since one of the fascinating aspects of Powell's paper was the level of assumptions needed, it is important to reproduce these correctly.

The author makes much of his use of algorithm models as an aid in unifying the presentation of algorithms and their proofs. It would have been most interesting to see if his approach is more general than that of Zangwill who did the first extensive work in this area. Instead, we are given the sentence, "The following set of assumptions are due to Zangwill [Z1] and can be shown, though not very easily, to be stronger than [my assumptions] . . .". Such distinctions are the meat of research and it is very important not to omit proofs of such statements.

Finally, it is very amusing after the author has written so much about the importance of proposing 'implementable' as opposed to 'conceptual' algorithms to read the following step in at least seven of his 'implementable' algorithms for minimizing an unconstrained function $f^0(z)$. "Step 0. Select a $z_0 \in R^n$ such that the set $C(z_0) = \{z | f^0(z) \leqq f^0(z_0)\}$ is bounded."

This book is important because of the breadth of material it contains. The chapter on the rate of convergence of unconstrained minimization techniques is very up-to-date. For these reasons, it is a useful addition to anyone's library.

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30[2.20, 2.40].—CHIH-BING LING & JUNG LIN, *Values of Coefficients in Problems of Rotational Symmetry*, Virginia Polytechnic Institute and State University, Blacksburg, Virginia, and Tennessee Technological University, Cookeville, Tennessee, February 1972, ms. of 23 typewritten pages deposited in the UMT file.

The finite difference $\Delta^s \sigma_n$ arises frequently in problems of rotational symmetry, where σ_n is the sum of the nth powers of the roots of the equation $u^k - (u - 1)^k = 0$, $k \geqq 2$. In general, σ_n is real.

The authors tabulate $\Delta^s \sigma_n$ to 11S for $k = 3(1)8$, with $n = -4(1)65$ and $s = 0(1)k - 1$. For $s \geqq k$, values of the differences can be found from the tabulated values by the relation $\Delta^s \sigma_n = \Delta^{s-mk} \sigma_{n+mk}$, where m is a positive integer such that $0 \leqq s - mk \leqq k - 1$. In particular, for $k = 2$ we have $\Delta^s \sigma_n = (-1)^s / 2^{n+s}$.

AUTHORS' SUMMARY

31[3].—STEFAN FENYÖ, *Moderne Mathematische Methoden in der Technik*, Vol. II, Birkhäuser Verlag, Basel, 1971, 336 pp., 25 cm. Price 62—Fr.

This second volume, in contrast to the first, may be described as dealing with finite methods in applied mathematics. In three chapters, it covers linear algebra, linear and convex programming, and graph theory. While the first two chapters would offer ample opportunity for including computational considerations, the author deliberately omits such topics. He feels that their inclusion would lead beyond the scope of this book and would be contrary to the principal aim of the book, which is to develop basic mathematical ideas in as simple a manner as possible. More bluntly, the author expresses his view that "anybody who has mastered the underlying mathematical principles will have no difficulty in learning the numerical methods very quickly from the vast relevant literature."

The reviewer was astonished to discover that the second chapter (on optimization) is essentially a reproduction of the first two chapters of the book by Collatz and Wetterling [1]. Moreover, he is appalled at the incredible amount of typographical and grammatical errors, revealing little if any editorial supervision.

W. G.

1. L. Collatz & W. Wetterling, *Optimierungsaufgaben*, Springer, Berlin and New York, 1966.

32[5].—Garrett Birkhoff, *The Numerical Solution of Elliptic Equations*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa., 1971, xi + 82 pp. Price $4.20.

This book consists of the revised notes of a series of lectures given at an NSF sponsored Regional Conference in Applied Mathematics. It gives a concise, readable and up to date survey of most available methods for the numerical solution of elliptic equations. It contains many well-chosen references and should therefore also be quite useful as a guide to further studies.

There are nine lectures. The first describes typical elliptic problems, and, in the last, the author discusses some of his experiences with complicated practical problems. Lectures two and three are on classical analysis and finite difference methods, while the following two lectures survey the well-known successive overrelaxation, semi-iterative and alternating direction methods. The sixth lecture discusses the use of the classical integral equation approach, a topic often neglected in surveys of this kind. In addition, there are two sections on approximation theory and closely related variational methods.

A discussion of special methods for problems which can be solved by separation of variables such as Hockney's and Buneman's methods, of great importance in specialized applications, is missing. Cf. Hockney, *Methods in Computational Physics*, vol. 9, 1970.

The finite element method which is now rapidly being developed (to perhaps the most important numerical method for elliptic problems) is discussed only briefly. (It should be noted, however, that the following regional NSF conference dealt exclusively with variational methods. The notes by R. S. Varga are going to appear in the same series.)

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33[6].—PHILIP M. ANSELONE, *Collectively Compact Operator Approximation Theory and Applications to Integral Equations*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1971, xiii + 138 pp., 24 cm. Price $12.50.

One widely used numerical method for solving a linear Fredholm integral equation of the second kind,

$$x(s) - \int_0^1 k(s, t)x(t)\,dt = y(s) \qquad (0 \leqq s \leqq 1), \tag{1}$$

is to approximate the integral by an appropriate quadrature formula. This leads to a system of linear equations by collocation. An approximation to the exact solution is obtained in the form

$$x_n(s) = K_n x_n(s) + y(s), \tag{2}$$

where K_n is the operator defined by

$$K_n x(s) = \sum_{j=1}^{n} w_j k(s, t_j) x(t_j), \tag{3}$$

where the w_j's and the t_j's are weights and nodes for the quadrature rule used for the integral in (1). The operator K_n may be regarded as an approximation to the exact integral operator K.

In this way, one obtains a collection of operators $\{K_n\}_{n=1}^{\infty}$. In a typical situation, where $k(s, t)$ in (1) is continuous, the K_n's may be regarded as operators on $C[0, 1]$ into itself. Moreover, in such cases, the collection $\{K_n\}$ has an additional property called *collective compactness*, that is, the unit ball of $C[0, 1]$ is mapped by the K_n's into *a fixed compact* set.

This book presents a systematic study of collectively compact operators by an author who is directly responsible for the development of this subject matter. The Galerkin method or the projection method for Eq. (1) is a special case of collectively compact operator approximations. Also, this method may be applicable to not only Eq. (1) but also to eigenvalue problems of linear integral operators. The reviewer feels that this book is a "must" for those who wish to engage in a serious study of numerical techniques for linear and nonlinear integral equations.

The organization of the book is excellent. Theorems are clearly stated and proofs are carried out in full. The book is essentially self-contained.

Chapter 1 of this book is devoted to that part of the general theory of linear operators needed in the subsequent development and to a quick development of approximation theory for collectively compact operators. In Chapter 2 and Chapter 3, applications to integral equations are discussed in length. As an example of an application, the approximate solution of the transport equations is included. A brief section in Chapter 3 directs the reader to references on applications of this method to boundary value problems of partial differential equations.

Chapter 4 of this book discusses spectral approximation by collectively compact operators. This chapter should be very valuable because many useful theorems (e.g., Theorems 4.7 and 4.8) are proved in full.

Chapter 5 may be regarded as one in functional analysis in which collectively compact sets of operators are compared with bounded sets and with totally bounded sets.

The last chapter (Chapter 6) is mainly devoted to applications to nonlinear integral equations via linearization by Newton's method.

There are two appendices. The first one by the author summarizes basic properties of compact operators and provides a convenient reference for reading this book. The second appendix, by Joel Davis, discusses practical implementation of some of the methods in this book and provides numerical examples (Table 1 to Table 6). There is a bibliography at the end of the book which cites 99 references.

As a whole, this book gives, in a relatively few pages (136 pages), a first-hand account of essential parts of collectively compact operator theory which have been developed in the past decade, mainly by the author and his colleagues.

This book may be used in an advanced graduate course or in a research seminar. In the former, the instructor may feel the need to supplement his lecture with appropriate exercises since no exercises are provided in this book.

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34[7].—P. F. BYRD & M. D. FRIEDMAN, *Handbook of Elliptic Integrals for Engineers and Scientists*, Springer-Verlag, New York, 1971, xvi + 358 pp., 24 cm. Price $18.50.

The first edition of this volume appeared in 1954 and, to my surprise, I find that it was never reviewed in these annals. This valuable and useful tome I am sure is quite well known. Nonetheless, some descriptive remarks are in order. Its title is a good summary of its contents. It contains over 3000 important formulas to facilitate the reduction and evaluation of elliptic integrals. Also included are short tables of the elliptic integrals of the first and second kind, Jacobi's q-function, Heuman's Λ_0-function and Jacobi's (K-multiplied) zeta-function.

The second edition is unfortunately not much different from the first. The authors recognize that since 1954 numerous computational methods for efficient calculation of elliptic integrals and Jacobian elliptic functions have been published. Further, an abundance of new tabular material has appeared. To improve the first edition by augmenting its contents to include this material would, in the view of the authors, necessitate another volume and must be deferred. Thus, the first edition is reproduced essentially without change, except for a supplementary bibliography and corrections. Unfortunately, the bibliography is incomplete, especially with regard to tabular material, and not all errors noted in the first edition have been corrected.

Errata to the first edition have been recorded in these annals [1], [2], [3], [4]. We have examined these data against the new edition and find some errors still persist. For example, consider the evaluation of

$$\Pi(\phi, \alpha^2, 1) = \int_0^{\varphi} \frac{\alpha\theta}{(1 - \alpha^2 \sin^2 \theta) \cos \theta}.$$

In the first edition, on p. 10, the third line of formula 111.04 reads

$$(1)\ \Pi(\phi, \alpha^2, 1) = \frac{1}{1-\alpha^2}\left[\ln(\tan\phi + \sec\phi) - \alpha\ln\left(\frac{1+\alpha\sin\phi}{1-\alpha\sin\phi}\right)^{1/2}\right], \quad \alpha^2 \neq 1.$$

In [2, p. 533], H. E. Fettis remarked that for the validity of (1), one should have $\alpha^2 > 0$ and $\alpha \sin\varphi < 1$. He suggested that (1) be replaced by two equations. Thus,

$$(2)\quad \Pi(\phi, \alpha^2, 1) = \frac{1}{1-\alpha^2}\left[\ln(\tan\phi + \sec\phi) - \alpha\ln\left(\frac{1+\alpha\sin\phi}{1-\alpha\sin\phi}\right)^{1/2}\right], \quad \alpha^2 > 0,$$

$$(3)\quad \Pi(\phi, \alpha^2, 1) = \frac{1}{1-\alpha^2}\left[\ln(\tan\phi + \sec\phi) + |\alpha| \arctan(|\alpha|\sin\phi)\right], \quad \alpha^2 < 0.$$

In the second edition, (3) is given, but, in place of (2), we find (1) with the added restriction, $\alpha^2 \neq 1$. Clearly, if φ is not an odd multiple of $\pi/2$, (1) can be evaluated for $\alpha^2 = 1$ with the aid of L'Hospital's theorem. Actually, for the validity of (2) and (3), we should add the restrictions φ real and $\alpha \sin\varphi \neq 1$ with the further understanding that the integral is a Cauchy principal value integral if $\alpha\sin\varphi > 1$. The conditions are sufficient, since we can easily continue the integral into the complex plane for complex values of α and φ. Note that for φ real, we can use the restriction $0 < \varphi < \pi/2$ in view of periodicity and the fact that the integral has the same value for $\varphi = \pi/2 \pm \omega$, $0 < \omega \leqq \pi/2$.

In the tables for $KZ(\beta, k)$, the following corrections should be made in both editions:

Page	$\sin^{-1} k$	β	for	read
339	15°	44°	.027204	.027203
340	40°	73°	.124059	.124061

These as well as others were reported in [3, p. 639], and all have been corrected in the second edition except those just noted.

An error noted in [4] has not been corrected in the second edition. On p. 289, Formula 800.07 (same in both editions), the third term in the third line should be $-\pi K'/2$ instead of $+\pi K'/2$. In the first edition, the upper limit of the first integral in this formula was given incorrectly as K. It should read 1. This error was noted in [4] and has been corrected in the second edition.

Some further errors have been noticed by H. E. Fettis and appear in the errata section of this issue.

Y. L. L.

1. *MTAC*, v. 13, 1959, p. 141.
2. *Math. Comp.*, v. 18, 1964, pp. 533, 687.
3. *Math. Comp.*, v. 20, 1966, pp. 344, 639.
4. *Math. Comp.*, v. 23, 1969, p. 468.

35[9].—EDGAR KARST, *The Second* 2500 *Reciprocals and their Partial Sums of all Twin Primes* ($p, p + 2$) *between* (102911, 102913) *and* (239387, 239389), Department of Mathematics, University of Arizona, Tucson, Arizona, February 1972. Ms. of 253 computer sheets deposited in the UMT file.

This manuscript table is a direct continuation of one [1] by the author giving 20D reciprocals of the first 2500 twin primes, together with 20D cumulative sums of these reciprocals. As in the previous table a useful supplementary table of two computer sheets here lists the first member of each of the subject prime pairs.

The author states that comparison of his list of twin primes with the tables of Selmer & Nesheim [2] and of Tietze [3] has revealed no discrepancies.

The announced motivation for the present tables is the testing of the author's conjecture that the sum of the reciprocals of the twin primes (counting 1 as a prime) closely approximates π. However, the calculation of Fröberg [4] implies that this sum to 4D is 3.0352, which does not appear to substantiate this conjecture to any reasonable degree.

J. W. W.

1. EDGAR KARST, *The First* 2500 *Reciprocals and their Partial Sums of all Twin Primes* ($p, p + 2$) *between* (3, 5) *and* (102761, 102763), Department of Mathematics, University of Arizona, Tucson, Arizona, January 1969. (See *Math. Comp.*, v. 23, 1969, p. 686, RMT **52.**)

2. E. S. SELMER & G. NESHEIM, "Tafel der Zwillingsprimzahlen bis 200000," *Norske Vid. Selsk. Forh. Trondheim*, v. 15, 1942, pp. 95–98.

3. H. TIETZE, "Tafel der Primzahl-Zwillinge unter 300000," *Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B.*, 1947, pp. 57–72.

4. CARL-ERIK FRÖBERG, "On the sum of inverses of primes and of twin primes," *Nordisk Mat. Tidskr. Informationsbehandling*, v. 1, 1961, pp. 15–20.

36[9].—ELVIN J. LEE & JOSEPH S. MADACHY, "The history and discovery of amicable numbers—Part 1," *J. Recreational Math.*, v. 5, April 1972.

This is the text of the published version of our previously reviewed [1]. The table of amicable numbers has been increased from the previous 977 pairs to 1095 pairs and includes all pairs known to the authors through the end of 1971. The table is not given here but will follow in "succeeding issues" of the *Journal of Recreational Mathematics.*

For more detail of the contents of this paper see our previous review [1]. The main change in the present edition, besides a second author (the editor of JRM) and a slightly changed title, is the inclusion of brief reports on the subsequent work of Henri Cohen, Walter Borho, A. Wolf, Richard David and Harry Nelson. These authors account for the extra 118 pairs in the table. No doubt there will be a supplement at the end of the table since new pairs are still coming in.

The paper lists a new "aliquot 4-cycle" due to R. David, and subsequently David found three others. Adding these to Cohen's eight cycles and Borho's one gives a present total of thirteen 4-cycles. Counting Tuckerman's new perfect number, which is also listed here, the number of cycles satisfying $s^{(k)}(n) = n$ is now 24 for $k = 1$, 1095 for $k = 2$, 13 for $k = 4$, and 1 each for $k = 5$ and 28. There still are none for

$k = 3$ (called "crowds" in England) but Borho's analytical work may assist in settling this.

It is stated here that no amicable pair is known that does not terminate an aliquot chain. A priori, the reviewer sees no compelling reason to doubt the existence of one since, analogously, the perfect number 28 does not terminate a chain.

D. S.

1. ELVIN J. LEE, *The Discovery of Amicable Numbers, Math. Comp.*, v. 24, 1970, pp. 493–494, RMT **40.**

37[9].—RUDOLF ONDREJKA, *Mersenne Primes and Perfect Numbers*, ms. of 91 computer sheets (undated) deposited in the UMT file.

Herein are listed in decimal, octal, and binary form, respectively, the exact values of the first 23 Mersenne primes and the corresponding perfect numbers. Also presented are such relevant statistics as the number of decimal digits in each number, the corresponding digital sum, the frequency distribution of these digits and the associated cumulative frequency distribution.

The author includes explicit expressions of the perfect numbers as sums of cubes of successive odd numbers, sums of successive powers of 2, and sums of arithmetic progressions.

Appropriate entries in these tables were compared by the author with corresponding results of Uhler [1]. Also, the eighteenth Mersenne prime was checked against the value of Riesel [2], and the last three Mersenne primes listed here were checked against the corresponding results of Gillies [3].

Furthermore, this reviewer has successfully compared the statistics herein with corresponding data found by Lal [4].

It seems appropriate to note here that an additional Mersenne prime has been recently announced by Tuckerman [5].

J. W. W.

1. H. S. UHLER, "Full values of the first seventeen perfect numbers," *Scripta Math.*, v. 20, 1954, p. 240, where references to Professor Uhler's previous related calculations are given.
2. H. RIESEL, "A new Mersenne prime," *MTAC*, v. 12, 1958, p. 60.
3. D. B. GILLIES, *Three New Mersenne Primes and a Conjecture*, Report No. 138, Digital Computer Laboratory, University of Illinois, Urbana, Illinois, 1964.
4. M. LAL, *Decimal Expansion of Mersenne Primes*, Department of Mathematics, Memorial University of Newfoundland, St. John's, Newfoundland, 1967. (See *Math. Comp.*, v. 22, 1968, p. 232, RMT **20.**)
5. B. TUCKERMAN, "The 24th Mersenne prime," *Proc. Nat. Acad. Sci. U.S.A.*, v. 68, 1971, pp. 2319–2320.

38[9].—G. AARON PAXSON, *Table of Aliquot Sequences*, Standard Oil Co. of California, 225 Bush Street, San Francisco, California 94120, computer output, 134 sheets filed in stiff covers and deposited in the UMT file in 1966.

Let $s(n)$ be the sum of the *aliquot parts* of n, i.e. divisors of n other than n itself. According as $s(n) = n$, $<n$ or $>n$, n is *perfect*, *deficient* or *abundant*. Define $s^0(n) = n$, $s^{k+1}(n) = s(s^k(n))$, $k \geqq 0$. The author tabulates $s^k(n)$ for $k = 0, 1, 2, \cdots$, and each abundant n through $n = 3040$. Each sequence is pursued until either (i) a prime p occurs, whereafter $s(p) = 1$, or (ii) a member of an earlier sequence occurs, or (iii) the next term would exceed 10^{10}. No instructions or explanation are given; here are six consecutive entries from the table, with an interpretation:

552	9	43	6526189068	2	1	2	1	2	3	7	31	358031
558			558	1	2	1		2	3	31		
558	8	1	690					462	1			
560			560	4	1	1		2	5	7		
560		1	928	5	1			2	29			
560		2	962	1	1	1		2	13	37		

Column 1 contains n, column 3 contains k (unless it is zero) and the 9 (resp. 8) in column 2 is a flag indicating that case (iii) (resp. (i) or (ii)) has occurred. The last columns contain the distinct prime factors of $s^k(n)$, and the preceding ones give the exponents to which these primes occur (except that the last exponent is missing when flag 9 appears, and there is no factorization when flag 8 appears). These entries may be written, with the factorizations in an obvious notation, as

$$\begin{aligned}
s^{43}(552) &= 6526189068 = 2(2)3.7(2)31.358031,\\
s^{0}\,(558) &= 558 = 2.3(2)31,\\
s^{1}\,(558) &= 690 = s^{1}(462),\\
s^{0}\,(560) &= 560 = 2(4)5.7,\\
s^{1}\,(560) &= 928 = 2(5)29,\\
s^{2}\,(560) &= 962 = 2.13.37.
\end{aligned}$$

Additional information can be adduced: (a) $s^{44}(552) > 10^{10}$, (b) 558 is the next abundant number after 552 and occurs in no earlier sequence, (c) the sequence with leader 462 is the first in which the term 690 occurs, etc.

The reviewer has checked the table against his work with Selfridge [1]; no errors were found, but the following entry is missing (presumably a card was dropped).

132	1	204	2	1	1	2	3	17

There are 40 instances of flag 9 (case (iii)): $n = 138$, 276, 552, 564, 660, 702, 720, 840, 858, 936, 966, 1074, 1134, 1248, 1316, 1464, 1476, 1488, 1512, 1560, 1578, 1632, 1734, 1848, 1920, 1992, 2058, 2136, 2190, 2232, 2340, 2360, 2484, 2514, 2580, 2664, 2712, 2850, 2880, 2982. The author has written (June, 1966) "As one might infer from [2], I started . . . to handle the 40 unfinished sequences . . . six were finished and a seventh joined one of the remaining. The remainder were pushed to . . . 4×10^{15} to 8×10^{20}." His intention was to cover the range $n < 10^4$; this has now been done in [1]. It is known (D. H. and Emma Lehmer, written communication) that

$$s^{177}(138) = s^{300}(702) = s^{194}(720) = s^{167}(858) = s^{184}(936) = 1,$$

and ([1], and continuing calculations) that $s^{164}(1316) = s^{74}(2136) = s^{189}(2190) = s^{209}(3192) = s^{203}(4500) = s^{161}(5760) = s^{350}(5970) = s^{172}(6450) = s^{125}(8496) = s^{283}(8658) = s^{162}(9576) = s^{198}(9840) = 1$. These sequences all contain terms $> 10^{10}$.

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1. Richard K. Guy & J. L. Selfridge, "Interim report on aliquot series," *Proc. Winnipeg Conf. on Numerical Math.*, October 1971.
2. G. Aaron Paxson, "Aliquot sequences" (preliminary report), *Amer. Math. Monthly*, v. 63, 1956, p. 614.

39[12].—B. A. Galler & A. J. Perlis, *A View of Programming Languages*, Addison-Wesley Publishing Co., Reading, Mass., 1970, vi + 282 pp., 24 cm. Price $12.95.

Alan Perlis and Bernard Galler have been major contributors towards the design of programming languages. In this book, they present their views on the structure of programming, and describe a programming language, Algol D, that reflects these views. The book would be better entitled *A View of a Programming Language*, since they make little attempt to describe or deal with realistic programming languages other than Algol D. Thus, problems such as input-output and parallel processing are barely mentioned, and the reader may have a hard time seeing the connection between the ideas of this book and his favorite programming language. Nevertheless, the book must be regarded as a significant contribution to its field.

The expository quality is excellent. There are many exercises, and these are not only related to the text but are actually referenced within it. Thus, the reader who does the exercises will have extended the material covered by the book. The presentation is quite well-organized and ought to be suitable for a variety of readers, though some acquaintance with programming is surely necessary. Although the book is intended for classroom use as well as self-study, an instructor with well-formulated views on programming languages would probably not feel comfortable teaching from it because of its strong and rather personal biases.

There are four chapters. The first chapter is an elegant development of a higher-level (but not really practical) programming language starting with Markov algorithms. With the simplest type of algorithm as a base, the concepts of concatenation of algorithms, subroutining, operation on part of the data space, labelling, and storage addressing are introduced. The result of these successive extensions is the Addressed Labelled Markov Algorithm (ALMA). Emphasis is placed on the use of conventions in order to represent different data structures with the same set of characters. This chapter also introduces the concept of interpretation.

The second chapter is concerned with language. It begins with a discussion of flowcharting, and then moves on to present a version of Algol without arrays. Arrays are then presented in the context of a more general development of data structures, including strings as well. Unfortunately, the section on flowcharts contains a misleading explanation of an algorithm for finding a zero of a continuous function in an interval $[p, q]$. Although the description of the algorithm specifies that the interval must contain an odd number of zeros, the algorithm itself does not assume this. Under the stated assumption, the first half of the algorithm is superfluous.

The third chapter is concerned with data structures. In this chapter, three basic structure-forming operations are introduced, one for strings, one for arrays, and one for nonhomogeneous n-tuples. Since names (i.e., locations) of data are also allowed as data, it becomes possible to build list structures of various kinds. A number of examples of different data structures are given, including multidimensional arrays, complex numbers, ring structures such as those used in graphics, and threaded lists. The notion that complex structures may be built up from simple ones through a hierarchy of definitions is emphasized.

In the final chapter, concerned with extensible languages, the authors discuss the incorporation of definitions into Algol in order to produce Algol D. The definitional method deals not only with the specification of the structures themselves but also with the specification of operators on the structures. Since the same operator may have different meanings when applied to different structures, the process of determining the meaning of an operator is context-dependent. Given a context, it is then possible to invoke an appropriate substitution of text in order to replace an occurrence of a defined operator by its definition. The process of context determination and replacement is discussed in some detail. The book concludes with a discussion of macro schemes suggested by other people.

In summary, this is a worthwhile and meaty book, though one that presents a highly personal and limited view of its field.

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40[12].—Ronald Blum, Editor, *Computers in Undergraduate Science Education*, Commission on College Physics, College Park, Maryland, 1971, x + 499 pp., 23 cm. Available from American Institute of Physics, 335 East 45th Street, New York, New York 10017.

These conference proceedings contain quite a number of papers dealing with the use of computers in college physics instruction, as well as some dealing with broader aspects of the use of computers in universities. None of the material here is of lasting significance, though the book has a certain pragmatic value for those actively engaged in educational projects using the computer as a tool. The numerous reports on active projects and their results do provide a standard of comparison for workers in the field. For other readers, the book may be worth skimming as a matter of interest but does not merit detailed study.

Fortunately, the authors of most of the papers seem to have tried to present their results honestly without exaggerating their successes. I was rather charmed by an article by Edwin F. Taylor entitled "History of a failure in computer interactive instruction," which begins with the sentence "This paper deals with an elegant and technically successful computer interactive display that has not influenced many students."

The first groups of papers discuss applications where the computer is used by students as an experimental tool. Some of these applications require the student to know how to program, while others do not. The simulation of the behavior of physical systems does seem to aid in developing physical intuition in a simpler way than through the use of elaborate laboratory experiments. Later sections of the book discuss more general systems of computer-aided instruction, and these sections are much less physics-dependent than the earlier ones. However, most of the interesting CAI systems have been discussed elsewhere in more detail. I did get the distinct impression that CAI systems have managed to become cost-effective, and that their use will be spreading in colleges and universities. A final section of the book deals with the political problems of computing at universities, and with the prognoses for the future.

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41[13.35].—I. H. MUFTI, *Computational Methods in Optimal Control Problems*, Springer-Verlag, New York, 1970, 45 pp., 26 cm. Price $1.70.

This short monograph is a member of the series, *Lecture Notes in Operations Research and Mathematical Systems*, edited by M. Beckmann and H. P. Kunzi. The author accomplishes his stated purpose of presenting in a concise manner the major points of several computational methods for solving certain optimal control problems. Each method discussed is iterative and uses successive linearization to obtain functions that are solutions of a set of necessary conditions of optimality associated with the given problem. These conditions consist of a set of algebraic and differential equations and inequalities (called the minimum principle of Pontryagin) that any optimal solution of the given problem must satisfy.

The reader is assumed to possess more than a passing acquaintance with variational calculus; no general explanatory comments are included. Algorithms for the solution of problems with no state or control constraints are presented first, progressing to problems with control constraints and 'terminal' state constraints. Problems in which the state trajectories must remain in a set not equal to the entire space are not considered.

This report contains no statements regarding the advantages or disadvantages of any of the methods discussed, and there are no examples or exercises. A good reference for such comments and examples is *Applied Optimal Control* by A. E. Bryson, Jr. and Y. C. Ho, Blaisdell Publishing Co., 1969, Chapter 7.

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