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How Slowly Can Quadrature Formulas Converge?

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Abstract. Let $\{Q_n\}_{n=1}^{\infty}$ denote a sequence of quadrature formulas, $Q_n(f) \equiv \sum_{j=1}^{k_n} w_j^{(n)} f(x_j^{(n)})$, such that $Q_n(f) \to \int_0^1 f(x)\,dx$ for all $f \in C[0, 1]$. Let $0 < \epsilon < \frac{1}{4}$ and a sequence $\{a_n\}_{n=1}^{\infty}$ be given, where $a_1 \geqq a_2 \geqq a_3 \geqq \cdots$, and where $a_n \to 0$ as $n \to \infty$. Then there exists a function $f \in C[0, 1]$ and a sequence $\{n_k\}_{k=1}^{\infty}$ such that $|f(x)| \leqq 2a_1/|(1 - 4\epsilon)|$, and such that $\int_0^1 f(x)\,dx - Q_{n_k}(f) = a_k$, $k = 1, 2, 3, \cdots$.

1. Introduction and Statement of Results. We consider a sequence of quadrature formulas $\{Q_n\}_{n=1}^{\infty}$ defined by

$$Q_n f = \sum_{j=1}^{k_n} w_j^{(n)} f(x_j^{(n)}) \tag{1.1}$$

where $\{k_n\}_{n=1}^{\infty}$ is a sequence of (increasing) positive integers and $0 \leqq x_j^{(n)} \leqq 1$ for all n and j. The quadrature formulas, we assume, are such that

$$\lim_{n\to\infty} Q_n f = If \equiv \int_0^1 f(x)\,dx \tag{1.2}$$

for all functions f that are continuous on the closed interval [0, 1]; that is, for all f in $C[0, 1]$. For example, the Gaussian quadrature formulas and the well-known trapezoidal formulas have these properties.

In this paper, we show that no matter what the sequence $\{Q_n\}_{n=1}^{\infty}$ defined by (1.1) and (1.2) is, there is a function f in $C[0, 1]$ for which $\{Q_n f\}_{n=1}^{\infty}$ converges to If very slowly. That is, the assumption of continuity is not enough to insure the rapid convergence of any quadrature scheme. More precisely, our main result is the following:

THEOREM 1. *Let a sequence of quadrature formulas $\{Q_n\}_{n=1}^{\infty}$ defined by* (1.1) *and satisfying* (1.2) *for all f in $C[0, 1]$ be given, and let $\{a_n\}$ be any sequence of numbers such that*

$$\lim_{n\to\infty} a_n = 0, \tag{1.3}$$

and

$$S \equiv \sum_{n=1}^{\infty} |a_n - a_{n+1}| < \infty. \tag{1.4}$$

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Corresponding to any ϵ *such that* $0 < \epsilon < \frac{1}{4}$, *there exists a function* f *that is continuous and bounded by* $2S/(1 - 4\epsilon)$ *on the interval* $[0, 1]$, *and an increasing sequence of positive integers* $\{n_k\}_{k=1}^{\infty}$ *such that*

$$(1.5) \qquad If - Q_{n_k}f = a_k \quad \text{for } k = 1, 2, \cdots .$$

We also have

COROLLARY 2. *For every integer* $N > 0$, *there exists a polynomial* P *bounded on the interval* $[0, 1]$ *by* $2S/(1 - 4\epsilon)$ *such that*

$$(1.6) \qquad IP - Q_{n_k}P = a_k \quad \text{for } k = 1, 2, \cdots, N.$$

Remark 3. *We remark that if the sequence* $\{a_k\}_{k=1}^{\infty}$ *is a monotonically increasing or decreasing sequence of real numbers, then* $S = |a_1|$.

In the following section, we will construct the function f of Theorem 1 as a uniformly convergent sum of linear spline functions (broken linear functions) on the interval [0, 1]. This constructive proof of Theorem 1 leads to an elementary proof of Corollary 2. In the final section, we consider a particular sequence of quadrature formulas and show the easier calculation involved in this case.

Some historical remarks are in order. Let P_n denote the set of polynomials of degree $n - 1$ in x, $n = 1, 2, 3, \cdots$. In 1938, Bernstein [1] proved that, given any sequence of positive numbers $a_1 \geqq a_2 \geqq a_3 \geqq \cdots \geqq a_n \geqq \cdots \geqq 0$, where $a_n \to 0$ as $n \to \infty$, there exists a function $f \in C[0, 1]$ such that

$$\inf_{p \in P_n} \left\{ \sup_{x \in (0,1)} |f(x) - p(x)| \right\} = a_n, \qquad n = 1, 2, 3, \cdots .$$

This result has since been cast in the terminology of best approximation in normed linear spaces (Timan [2, p. 40]). From the point of view of practical applications, Bernstein's theorem tells us that there are continuous functions defined on the interval [0, 1] which cannot be approximated to a desired accuracy by polynomials.

Our paper extends a recent result of Chui [5] who proved that given $\{a_n\}_{n=1}^{\infty}$, a sequence of positive numbers which converges monotonically to zero, there exists a Riemann integrable function f such that

$$\left| \int_0^1 f(x)\,dx - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \right| \geqq a_n \quad \text{for } n = 1, 2, \cdots .$$

In 1933 Pólya [3] constructed an analytic function for which the Newton-Cotes quadrature scheme diverges. Pólya's result was preceded by an interesting asymptotic estimate of the remainder of Newton-Cotes quadrature due to Ouspensky [6]; in [6], Ouspensky concluded that the Newton-Cotes formulas were devoid of any practical value. Thus, while proofs of convergence of quadrature schemes applied to continuous functions are very interesting (see Pólya [3] and Espinoza-Maldonado and Byrne [4]), the point of our paper in the spirit of Ouspensky is that when quadrature formulas are applied to continuous functions, the results may be devoid of any practical value.

2. Constructive Proofs. Let Π_i denote the set of evaluation points of Q_{n_i}. The function f will be of the form

$$f(x) = \sum_{i=1}^{\infty} \alpha_i s_i(x) \tag{2.1}$$

where the s_i are to be constructed and the α_i are to be determined.

Choice of n_i and Construction of $s_i(x)$. We fix ϵ such that $0 < \epsilon < \frac{1}{4}$. Then, by (1.2), there is an integer n_1 such that

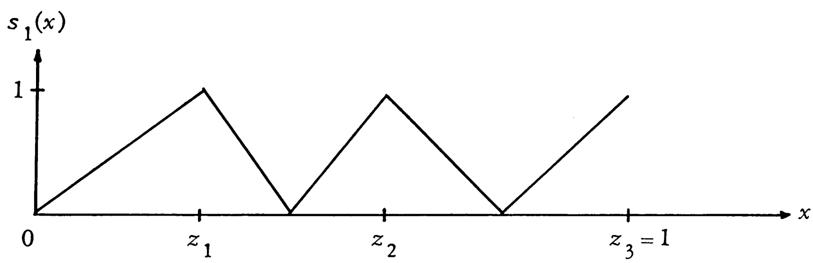
$$|Q_{n_1} 1 - I(1)| < \epsilon/2. \tag{2.2}$$

We now define s_1 to be the linear spline function whose graph has the vertices

(i) $s_1(x) = 1$ for all $x \in \Pi_1$, and

(ii) $s_1(x) = 0$ for all x midway between consecutive points of Π_1, and also for $x \in \{0, 1\} - \Pi_1$.

Thus, if $\Pi_1 = \{z_1, z_2, z_3\}$, we would have the following graph:



Since $s_1(x) = 1$ on Π_1, we have, by (2.2),

$$Q_{n_1} s_1 = 1 + \eta_{11} \quad \text{where } |\eta_{11}| < \epsilon/2. \tag{2.3}$$

Also, clearly,

$$\int_0^1 s_1(x)\,dx = \tfrac{1}{2}. \tag{2.4}$$

We now proceed to the pth stage and suppose we have picked $n_1, n_2, \cdots, n_{p-1}$ so that

$$\int_0^1 s_i(x)\,dx = \tfrac{1}{2} \quad \text{for } i = 1, 2, \cdots, p-1. \tag{2.5}$$

Let $\delta_p = \epsilon/p2^p$. We cover the union of the Π_i $(i = 1, 2, \cdots, p-1)$ with a finite union of open (and, possibly, half-open) intervals of total length $\delta_p/3$. We call this union C_p and cover its closure with another union of open (and half-open) intervals, B_p, this time of total length $2\delta_p/3$.

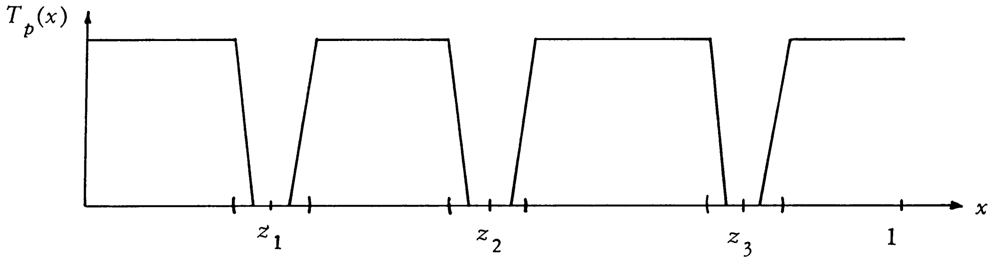
We now define a preliminary function T_p to be continuous on the interval [0, 1] and such that

(i) $T_p(x) = 0$ for x in C_p,

(ii) $T_p(x) = 1$ for x in $[0, 1] - B_p$, and

(iii) $T_p(x)$ is linear elsewhere.

Thus, we might have the graph



where the union of the Π_i is $\{z_1, z_2, z_3\}$. Then, from the definition of T_p, we clearly have

$$\int_0^1 (1 - T_p(x))\,dx \leqq 2\delta_p/3. \tag{2.6}$$

By (1.2), (2.5) and (2.6) there is an integer n_p such that

$$\begin{aligned} &Q_{n_p}s_j = \tfrac{1}{2} + \eta_{pj} \quad \text{for } j = 1, 2, \cdots, p-1 \quad \text{where } |\eta_{pj}| < \delta_p, \quad \text{and} \\ &Q_{n_p}T_p = 1 + \eta_{pp} \quad \text{where } |\eta_{pp}| < \delta_p. \end{aligned} \tag{2.7}$$

Finally, we define $s_p(x)$ to be the linear spline function which is identical to $T_p(x)$ on the closure of the set $B_p - C_p$ and whose graph has the following additional vertices:

(i) $s_p(x) = 1$ for $x \in \Pi_p \cap ([0, 1] - B_p)$,

(ii) $s_p(x) = 0$ for $x \in \bigcup_{i=1}^{p-1} \Pi_i$, $x \in \{0, 1\} - \Pi_p$ and $x \in \Pi_p \cap C_p$,

(iii) $s_p(x) = 0$ for all x halfway between two consecutive "one" vertices so far defined, and,

(iv) $s_p(x) = 1$ for all x halfway between two consecutive "zero" vertices defined in (ii).

Then, $s_p(x)$ has the following properties:

$$Q_{n_p}s_p = Q_{n_p}T_p \quad \text{since } s_p = T_p \text{ on } \Pi_p. \tag{2.8}$$

$$\int_0^1 s_p(x)\,dx = \tfrac{1}{2}. \tag{2.9}$$

$$Q_{n_i}s_p = 0 \quad \text{for } i = 1, 2, \cdots, p-1 \quad \text{by (ii)}. \tag{2.10}$$

In this manner, we pick the subsequence $\{n_i\}_{i=1}^{\infty}$ and the functions $\{s_i\}_{i=1}^{\infty}$.

Determination of α_i. We can now apply the linear functional $I - Q_{n_p}$ to the function f of (2.1), and, at the same time, impose the condition (1.5). We then obtain, using (2.6)–(2.10), the following infinite system of linear equations:

$$\frac{1}{2}\sum_{i=1}^{\infty} \alpha_i - \left(\sum_{j=1}^{p-1} \alpha_j(\tfrac{1}{2} + \eta_{pj}) + \alpha_p(1 + \eta_{pp})\right) = a_p \quad \text{for } p = 1, 2, \cdots. \tag{2.11}$$

If we subtract two such consecutive equations, we obtain the recurrence relation

$$\begin{aligned} \alpha_{p+1}(1 + \eta_{p+1,p+1}) = a_p - a_{p+1} &+ \alpha_p(\tfrac{1}{2} + \eta_{pp} - \eta_{p+1,p}) \\ &+ \sum_{j=1}^{p-1} \alpha_j(\eta_{pj} - \eta_{p+1,j}). \end{aligned} \tag{2.12}$$

Setting $\alpha_1 = 0$, we can use (2.12) to solve for all the α_j.

It should be noted here that the passage from (2.11) to (2.12) is so far only a formal one, since the infinite series on the left-hand side of (2.11) has not yet been shown to converge. This will be shown in the next part of the proof.

f is Continuous. In order to complete the proof of Theorem 1 we need only show that the α_i, as determined above, make the function f of (2.1) continuous on the interval [0, 1]. Thus, we must show that the infinite series (2.1) converges uniformly on the interval [0, 1]. Since $|s_i(x)| \leqq 1$ for all x and for every integer i, we need only show that

$$\sum_{p=1}^{\infty} |\alpha_p| < \infty. \tag{2.13}$$

Indeed, from (2.12) and the bounds on η_{pj} given in (2.7),

$$|\alpha_{p+1}| \leqq |a_p - a_{p+1}| + \frac{1}{2}|\alpha_p| + \frac{\epsilon}{2^{p-1}} \sum_{j=1}^{p+1} |\alpha_j|. \tag{2.14}$$

We now sum each side of (2.14) from 1 to N, where N is an arbitrary but fixed positive integer. Replacing the sum $\sum_{j=1}^{p+1} |\alpha_j|$ on the right-hand side by $\sum_{j=1}^{N+1} |\alpha_j|$ and adding $\frac{1}{2}|\alpha_{N+1}|$ to the right-hand side, we obtain

$$\sum_{p=1}^{N} |\alpha_{p+1}| \leqq \sum_{p=1}^{N} |a_p - a_{p+1}| + \frac{1}{2}\sum_{p=1}^{N+1} |\alpha_p| + \epsilon \sum_{p=1}^{N} \frac{1}{2^{p-1}} \sum_{j=1}^{N+1} |\alpha_j|. \tag{2.15}$$

Transposing and simplifying gives

$$\sum_{p=1}^{N} |\alpha_{p+1}| \left\{1 - \tfrac{1}{2} - 2\epsilon\left(1 - \frac{1}{2^N}\right)\right\} \leqq \sum_{p=1}^{\infty} |a_p - a_{p+1}| < \infty \quad \text{by (1.4)}. \tag{2.16}$$

Now letting $N \to \infty$ on the left, we obtain

$$\sum_{p=1}^{\infty} |\alpha_{p+1}| \leqq \frac{2}{1 - 4\epsilon} \sum_{p=1}^{\infty} |a_{p+1} - a_p|. \tag{2.17}$$

This completes the proof of Theorem 1.

Proof of Corollary 2. The proof of Corollary 2 is similar to that of Theorem 1, with one exception. Here we also need to use Weierstrass' approximation theorem, which enables us to approximate each continuous function s_p that we constructed in the proof of Theorem 1 by a polynomial q_p, such that $\max_{x \in [0,1]} |s_p(x) - q_p(x)| < \delta_p/2$ for $p = 1, 2, \cdots, N$. We omit the details of this proof.

3. A Special Case. We consider the following sequence of midpoint quadrature formulas which certainly satisfy (1.2):

$$Q_n f = \frac{1}{2^{n-1}} \left[f\left(\frac{1}{2^n}\right) + f\left(\frac{3}{2^n}\right) + f\left(\frac{5}{2^n}\right) + \cdots + f\left(\frac{2^n - 1}{2^n}\right)\right]. \tag{3.1}$$

In this case there are no repeating evaluation points, and we can therefore choose $n_i = i$, and define s_i to be the linear spline whose graph has the vertices

(i) $s_i(x) = 1$ for $x \in \Pi_i$, and

(ii) $s_i(x) = 0$ for $x \in \bigcup_{j=1}^{i-1} \Pi_j$ and also for $x = 0, 1$.

Then, as in (2.9),

$$\int_0^1 s_i(x)\,dx = \tfrac{1}{2} \quad \text{for all } i. \tag{3.2}$$

Instead of (2.6) and (2.7), we have the exact relations

$$\begin{aligned} Q_i s_j &= \tfrac{1}{2} \quad \text{if } j < i, \\ &= 1 \quad \text{if } j = i, \\ &= 0 \quad \text{if } j > i. \end{aligned} \tag{3.3}$$

Instead of (2.11), we have

$$\frac{1}{2}\sum_{i=1}^{\infty} \alpha_i - \frac{1}{2}\sum_{i=1}^{p-1} \alpha_i - \alpha_p = a_p \quad \text{for } p = 1, 2, \cdots. \tag{3.4}$$

Subtracting two such consecutive equations gives the recurrence relation

$$\alpha_{p+1} = a_p - a_{p+1} + \tfrac{1}{2}\alpha_p, \tag{3.5}$$

which can be solved explicitly in terms of α_1 to give

$$\begin{aligned} \alpha_{p+1} = a_p - a_{p+1} &+ \frac{1}{2}(a_{p-1} - a_p) \\ &+ \frac{1}{2^2}(a_{p-2} - a_{p-1}) + \cdots + \frac{1}{2^{p-1}}(a_1 - a_2) + \frac{1}{2^p}\alpha_1. \end{aligned} \tag{3.6}$$

The function f of the form (2.1) is easily seen to be continuous this time, since

$$\sum_{p=1}^{\infty} |\alpha_p| \leqq 2\sum_{p=1}^{\infty} |a_p - a_{p+1}| + 2\,|\alpha_1|. \tag{3.7}$$

If we take $\alpha_1 = 0$, we note that

$$|f(x)| \leqq 2\sum_{p=1}^{\infty} |a_{p+1} - a_p|. \tag{3.8}$$

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