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A Lower Bound for the Set of Odd Perfect Numbers

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Abstract. It is proved here that if n is odd and perfect, then $n > 10^{50}$.

Whether or not the set of odd perfect numbers is empty is still an open, and apparently very difficult, question. However, many properties of the elements of this set *have* been determined. For example, it is well known that if n is both odd and perfect, then

(1) $$n = p_0^{\alpha_0} p_1^{\alpha_1} \cdots p_t^{\alpha_t}$$

where the p_i are distinct primes, $p_0 \equiv \alpha_0 \equiv 1 \pmod 4$ and $2 \mid \alpha_i$ if $i > 0$. It has recently been shown [8], [9] that

(2) $$t \geqq 6,$$

while the author and McDaniel [2] have established that

(3) $$p_i > 10^4 \quad \text{for } some\ i \geqq 0,$$

and Tuckerman [10], [11] has proved that

(4) $$p_i^{\alpha_i} > 10^{18} \quad \text{for } some\ i > 0\ if\ (15, n) \neq 1.$$

In 1957, Kanold [4] proved that $n > 10^{20}$, while in 1967 Tuckerman [10], [11] proved that $n > 10^{36}$. The purpose of the present paper is to establish a still better lower bound for the set of odd perfect numbers. To be precise, we shall prove the following result.

THEOREM. *If n is odd and perfect, then $n > 10^{50}$.*

Our proof rests on a case study which was carried out with the aid of the CDC 6400 at the Temple University Computing Center. There are five "basic" cases which are characterized by the following mutually exclusive and exhaustive divisibility restrictions on n, where n is an element of the (possibly empty) set of odd perfect numbers: (I) $3 \nmid n$, $5 \nmid n$; (II) $3 \nmid n$, $5 \mid n$, $7 \nmid n$; (III) $3 \nmid n$, $5 \mid n$, $7 \mid n$; (IV) $3^2 \parallel n$; (V) $3^\beta \mid n$ where $2 \mid \beta$ and $\beta > 2$. Except for the first two, these basic cases "branch" into numerous subcases in which additional restrictions are imposed on n. In all, a total of approximately 175 individual cases are considered, each of which leads to an inequality of the form $n > 10^m$ with $m \geqq 50$. Since it is clearly not possible to discuss all of these cases here, we shall confine ourselves to a presentation of a few rather typical cases. The complete case study [1] has been deposited in the UMT file.

Referring to (1) we note first that since $\sigma(n) = 2n$ and since the σ-function is multiplicative it follows that every odd prime which divides $\sigma(p_i^{\alpha_i})$ also divides n. Q will denote a prime divisor of n which exceeds 10^4. The existence of such a prime is insured by (3).

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Case 0. $3 \nmid n$, $5 \nmid n$. Since the smallest prime which divides n is at least 7, we see from the table to be found in [7] that $t \geqq 14$. According to a theorem due to Muskat [5], at least one of the prime powers appearing in (1) exceeds 10^{12}. It follows easily that $n > 61 \cdot 7^4 R^2 10^{12} > 10^{51}$, where R is the product of the 12 primes which lie between 11 and 53, inclusive.

Case 1. $3 \nmid n$, $5 \mid n$, $7 \nmid n$. Since $\sigma(p^\alpha)/p^\alpha < p/(p-1)$ and $x/(x-1)$ is a monotonic decreasing function of x we see that if $t < 11$ then

$$\begin{aligned}&\sigma(n)/n\\ &\quad < (5 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 Q)/(4 \cdot 10 \cdot 12 \cdot 16 \cdot 18 \cdot 22 \cdot 28 \cdot 30 \cdot 36 \cdot 40(Q-1))\\ &\quad < 2.\end{aligned}$$

This contradiction shows that $t \geqq 11$. Since $(p_0 + 1) \mid \sigma(p_0^{\alpha_0})$ and since $3 \mid (p_0 + 1)$ if $p_0 \equiv -1 \pmod 3$ it follows that $p_0 \equiv 1 \pmod{12}$. We also note that $3 \mid \sigma(p^2)$ if $p \equiv 1 \pmod 3$; $7 \mid \sigma(p^2)$ if $p \equiv 2, 4 \pmod 7$. Recalling (4), we see that

$$n > 13(5 \cdot 17 \cdot 29 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \cdot 83 \cdot 89 \cdot 101)^2 \cdot 10^{18} > 10^{51}.$$

Case 7. $3 \nmid n$, $5 \mid n$, $7^{18} \| n$. Then $\sigma(7^{18}) = 419P \mid n$ where $P = 4534166740403$. Since $3 \nmid n$, we have $t \geqq 8$ by a theorem due to Kanold [3]. As in Case 1, $p_0 \equiv 1 \pmod{12}$ and $p^2 \nparallel n$ if $p \equiv 1 \pmod 3$. It follows that

$$n \geqq 13(5 \cdot 11 \cdot 17 \cdot 23 \cdot 29 \cdot 419P)^2 7^{18} > 10^{58}.$$

Case 100. $3^2 \| n$, $p_0 = 13$, $11 \mid n$, $7^4 \| n$. It is proved in [1] that, if $3 \cdot 7 \cdot 11 \cdot 13 \mid n$ and P is another prime divisor of n, then $P > 523$. Since $\sigma(7^4) = 2801$ and $37 \mid \sigma(2801^2)$, $5 \mid \sigma(2801^4)$, $71 \mid \sigma(2801^6)$, $37 \mid \sigma(2801^8)$, $23 \mid \sigma(2801^{10})$ we see that $2801^{12} \mid n$. Therefore, $n \geqq 2801^{12} \cdot \sigma(2801^{12}) > 2801^{24} > 10^{82}$.

Case 104, F. $3^2 \| n$, $p_0 = 13$, $11 \nmid n$, $7^\alpha \| n$ where $\alpha = 2, 8, 14, 20, 26$ (note that $3 \cdot 19 \mid \sigma(7^\alpha)$), $19^{16} \| n$. If $M = \sigma(19^{16})$ then $M \mid n$. It was found that every prime which divides M exceeds 10^5. Since $M < 10^{21}$ and M is not a square ($M \equiv -1 \pmod 3$), it follows that $M = P_1$ or P_1P_2 or $P_1P_2P_3$ or P_1^3 or $P_1P_2^2$ or $P_1P_2P_3P_4$ or $P_1P_2P_3^2$ or $P_1P_2^3$ where each P_i is a prime greater than 10^5. Recalling (2), it is not difficult to see that the fifth form yields the "minimal" value for n and that

$$n \geqq 13(3 \cdot 7 \cdot 17)^2 \cdot 19^6 \cdot P_1 \cdot P_1P_2^2 > 13(3 \cdot 7 \cdot 17)^2 \cdot 19^6 \cdot 10^5 M > 10^{52}.$$

Case 205. $3^4 \mid n$, $11^{18} \| n$. Then $\sigma(11^{18}) = M \mid n$ and it was determined that every prime factor of M exceeds 10^7. Since $M < 10^{19}$ and M is not a square ($M \equiv 5 \pmod 8$), it follows that either $M = P$ or $M = P_1P_2$. We consider these possibilities separately in the following two cases.

Case 205, A. $M = P$. If $P = p_0$ then, since $(p_0 + 1) \mid \sigma(p_0^{\alpha_0})$ and $5 \mid (P + 1)$, we see that $5^2 \mid n$. But, according to a theorem of Kanold [3], $3 \cdot 5^2 \cdot 11 \nmid n$. Therefore, $P \neq p_0$ and it follows that $n > P^2 \cdot 3^4 \cdot 11^{18} > 10^{58}$.

Case 205, B. $M = P_1P_2$. Without loss of generality we can assume that $p_0 \neq P_2$ and, since n has at least seven prime factors,

$$n \geqq P_1(7 \cdot 13 \cdot 17P_2)^2 \cdot 3^4 \cdot 11^{18} > M(7 \cdot 13 \cdot 17)^2 \cdot 10^7 \cdot 3^4 \cdot 11^{18} > 10^{52}.$$

Case 650. $3^{16} \mid n$, $3^3 \nmid (p_0 + 1)$. For convenience, we omit the subscript 0 on p

and α here. We note first that

$$\sigma(p^\alpha) = (p + 1)(1 + p^2 + p^4 + \cdots + p^{\alpha-1}) = (p + 1)\cdot f(p, \alpha);$$

it is easy to show that $3 \mid f(p, \alpha)$ if and only if $\alpha \equiv 5 \pmod{12}$. Moreover, it can be verified that $3 \| f(p, \alpha)$ for $\alpha = 5, 29, 41$; $3^2 \| f(p, 17)$; and $3^3 \| f(p, 53)$. Now, $2n = \sigma(n) = (p + 1)\cdot f(p, \alpha)\cdot\sigma(n/p^\alpha)$ and it follows from the stated restrictions that (i) $3^3 \mid f(p, \alpha)$ or (ii) $3^{12} \mid \sigma(n/p^\alpha)$. If (i), then, from the remarks just made, $n > 3^{16}\cdot 5^{53}\cdot Q^2 > 10^{52}$. Now assume (ii). According to the theorems found in Chapter V of [6], if β is even and q is an odd prime, then $3 \mid \sigma(q^\beta)$ if and only if $q \equiv 1 \pmod 3$ *and* $\beta + 1 = 3^k v$ where $(3, v) = 1$ and $k > 0$. Moreover, $3^k \| \sigma(q^\beta)$. Therefore, $n > QS^2\cdot 3^{16} > 10^{50}$ where S is the product of the 12 primes between 7 and 103 inclusive which are congruent to 1 modulo 3.

Case 1400. $3^{40} \| n$, $3^3 \mid (p_0 + 1)$. Then $\sigma(3^{40}) = 83M \mid n$ where every prime divisor of M exceeds 10^5. Since $M < 10^{18}$ and M is not a square ($M \equiv 3 \pmod 4$), we see that $M = P_1$ or P_1P_2 or $P_1P_2^2$ or $P_1P_2P_3$ or P_1^3. From (2) it follows that

$$n \geqq 53(5\cdot 13\cdot 83)^2\cdot 3^{40}\cdot P_1\cdot P_1P_2^2 > 53(5\cdot 13\cdot 83)^2\cdot 3^{40}\cdot 10^5 M > 10^{50}.$$

Case 1700. $3^{54} \mid n$. Then $n \geqq 3^{54}\cdot\sigma(3^{54}) > 3^{108} > 10^{51}$.

Remarks. In an earlier version of this study in which (2) and (4) were *not* used, a lower bound of 10^{45} was obtained for n. This is reflected in the complete case study [1]. A limited number of copies of the complete study are available from the author upon request.

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