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# A Class of Quadrature Formulas*

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**Abstract.** It is proved that there exists a set of polynomials orthogonal on $[-1, 1]$ with respect to the weight function

$$w(t)/(t - x) \tag{1}$$

corresponding to the polynomials orthogonal on $[-1, 1]$ with respect to the weight function $w$. Simplified forms of such polynomials are obtained for the special cases

$$\begin{aligned} w(t) &= (1 - t^2)^{-1/2}, \\ &= (1 - t^2)^{1/2}, \\ &= ((1 - t)/(1 + t))^{1/2}, \end{aligned} \tag{2}$$

and the generating functions and the recurrence relation are also given. Subsequently, a set of quadrature formulas given by

$$\int_{-1}^{1} (1 + t)^{p-1/2}(1 - t)^{q-1/2}(1 + a^2 + 2at)^{-1}f(t)\,dt = \sum_{k=1}^{n} H_k f(t_k) + E_n(f) \tag{3}$$

for $(p, q) = (0, 0)$, $(0, 1)$ and $(1, 1)$ is established; these formulas are valid for analytic functions. Convergence of the quadrature rules is discussed, using a technique based on the generating functions. This method appears to be simpler than the one suggested by Davis [2, pp. 311-312] and used by Chawla and Jain [3]. Finally, bounds on the error are obtained.

**1. Introduction.** Szegö [1] has pointed out the possible existence of orthonormal polynomials in $[-1, 1]$ corresponding to weight functions of the kind

$$w/\rho \tag{4}$$

where $w$ is given by (2) and $\rho$ is a polynomial satisfying certain conditions in $[-1, 1]$. A suitable choice for $\rho$ is found to be

$$\rho(t) = 1 + a^2 + 2at \tag{4'}$$

which further suggests the existence of polynomials orthogonal on $[-1, 1]$ with respect to the weight function (1).

In this paper, a theorem is established which shows that the polynomials orthogonal on $[-1, 1]$ with regard to (1) are linear combinations of the polynomials which are orthogonal on $[-1, 1]$ with regard to $w$. Particular cases of $w$ given in (2) are of special interest and they are dealt with in detail in the following sections.

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Finally, the corresponding quadrature formulas are developed and their convergence is discussed by a different method. This method, depending on the use of generating functions, is a simplification of the one used in [3]. Certain lemmas are proved which are subsequently used to find bounds on the error in formulas (3).

**2. Derivation of Formulas.** Let $w$ be a fixed positive, integrable function defined on $[-1, 1]$ and let $\{\psi_n\}$ be the polynomials that are orthogonal on $[-1, 1]$ with regard to the weight function $w$. Then

$$\int_{-1}^{1} w(t)\psi_n(t)t^r\,dt = 0, \qquad r = 0, 1, \ldots, n-1. \tag{5}$$

We propose to find the polynomial $\phi_n$ of degree $n$ in $t$ such that

$$\int_{-1}^{1} \frac{w(t)}{t-x}\phi_n(t)t^r\,dt = 0, \qquad r = 0, 1, \ldots, n-1, \tag{6}$$

where $x$ is a constant such that $|x| > 1$.

From (6) we have

$$\begin{aligned} &\int_{-1}^{1} w(t)\phi_n(t)\frac{t^{r+1}-xt^r}{t-x}\,dt = 0, && r = 0, 1, \ldots, n-2, \\ \Rightarrow &\int_{-1}^{1} w(t)\phi_n(t)t^r\,dt = 0, && r = 0, 1, \ldots, n-2, \\ \Rightarrow &\int_{-1}^{1} w(t)\phi_n(t)\psi_r(t)\,dt = 0, && r = 0, 1, \ldots, n-2. \end{aligned} \tag{7}$$

By expressing $\phi_n$ in the form $\sum_{s=0}^{n} a_s\psi_s$ and substituting in (7), we see that, since $a_n \neq 0$, we may write

$$\phi_n = \psi_n - \alpha_n\psi_{n-1} \tag{8}$$

where $\alpha_n$ is some constant depending on $n$.

Introduction of (8) in (6) with $r = 0$ and a little manipulation gives

$$\alpha_n = I_n/I_{n-1}, \tag{9}$$

$$I_n = \int_{-1}^{1} \frac{w(t)}{t-x}\psi_n(t)\,dt. \tag{10}$$

We have thus established the following result

THEOREM 1. *Given a set of polynomials $\{\psi_n\}$ such that*

$$\int_{-1}^{1} w(t)\psi_m(t)\psi_n(t)\,dt = 0, \qquad m \neq n,$$

*there is defined a set of polynomials $\{\phi_n\}$ given by*

$$\phi_n = \psi_n - \alpha_n\psi_{n-1}$$

*such that*

$$\int_{-1}^{1} \frac{w(t)}{t-x}\phi_m(t)\phi_n(t)\,dt = 0, \qquad m \neq n,$$

*where*

$$\alpha_n = \frac{I_n}{I_{n-1}}, \qquad I_n = \int_{-1}^{1} \frac{w(t)}{t-x}\psi_n(t)\,dt, \qquad |x| > 1.$$

The following particular cases follow from above.

**3. Case I.** Let $w(t) = (1-t^2)^{-1/2}$ so that $\psi_n = T_n$ is the Chebyshev polynomial of degree $n$ of the first kind. Let

$$x = -\tfrac{1}{2}(a + 1/a), \quad a \text{ being real,}$$

so that $|x| > 1$, whatever $a$. With this, (10) gives

$$I_n = 2a \int_{-1}^{1} \frac{(1-t^2)^{-1/2}}{1+a^2+2at} T_n(t)\,dt. \tag{11}$$

The generating function for Chebyshev polynomials can be written as

$$\frac{1}{2}\frac{1-w^2}{1-2tw+w^2} = \frac{1}{2} + \sum_{n=1}^{\infty} T_n(t)w^n, \qquad |w| < 1.$$

With $w = -1/a$, this becomes

$$\frac{1}{1+a^2+2at} = \frac{2}{a^2-1}\left[\frac{1}{2} + \sum_{r=1}^{\infty} (-1)^r a^{-r} T_r(t)\right], \qquad |a| > 1.$$

Inserting this in (11), using the orthogonality property of the Chebyshev polynomials and the result

$$\int_{-1}^{1} (1-t^2)^{-1/2} T_n^2(t)\,dt = \frac{\pi}{2}, \qquad n \geq 1,$$

we get

$$I_n = (-1)^n a^{-n+1} \cdot \frac{2\pi}{a^2-1} \quad \text{and} \quad \alpha_n = \frac{I_n}{I_{n-1}} = -\frac{1}{a}.$$

Thus, from (8), we get

$$p_n = a \cdot \phi_n = aT_n + T_{n-1}, \qquad n \geq 1, |a| > 1.$$

It is easy to prove that the corresponding orthonormal polynomials are

$$p_0^* = \left(\frac{a^2-1}{\pi}\right)^{1/2}, \qquad p_n^* = \left(\frac{2}{\pi}\right)^{1/2} [aT_n + T_{n-1}], \quad n \geq 1, \tag{12}$$

which satisfy the orthonormality condition

$$\int_{-1}^{1} (1-t^2)^{-1/2}(1+a^2+2at)^{-1} p_m^*(t) p_n^*(t)\,dt = \delta_{mn}, \qquad a \neq 1, m, n \geq 0, \tag{13}$$

and the recurrence relation

$$p_{n+1}^*(t) = 2tp_n^*(t) - p_{n-1}^*(t), \qquad n \geq 2. \tag{14}$$

The generating function for the Chebyshev polynomials can be written as

$$\frac{1}{2}\frac{1-w^2}{1-2tw+w^2} = \sum_{n=0}^{\infty}{}' w^n T_n(t) = \frac{1}{2} + w\sum_{n=0}^{\infty} w^n T_{n+1}(t) = -\frac{1}{2} + \sum_{n=0}^{\infty} w^n T_n(t).$$

This gives

$$\begin{aligned}(15)\quad \frac{1}{2}\frac{1-w^2}{1-2tw+w^2}(a+w) &= a\left[\frac{1}{2}+w\sum_{n=0}^{\infty} w^n T_{n+1}(t)\right]+w\left[-\frac{1}{2}+\sum_{n=0}^{\infty} w^n T_n(t)\right]\\ &= \frac{a-w}{2}+\sum_{n=0}^{\infty} w^{n+1}[aT_{n+1}(t)+T_n(t)].\end{aligned}$$

Insertion of (12) in (15) and a little manipulation leads to the generating function (16) for the polynomials (12)

$$(16)\qquad \frac{1}{2}\frac{(1-w^2)(a+w)}{1-2tw+w^2}=\frac{a-w}{2}+\sqrt{\frac{\pi}{2}}\sum_{n=0}^{\infty} w^{n+1}p_{n+1}^*(t).$$

Polynomials (12) give rise to the quadrature formulas

$$(17)\qquad \int_{-1}^{1}(1-t^2)^{-1/2}(1+a^2+2at)^{-1}f(t)\,dt=\sum_{k=1}^{n} H_k f(t_k)+E_n(f)$$

which are exact for all polynomials of degree $\leq 2n-1$. The weight coefficients and the error term in (17) are calculated through standard methods to be given by

$$(18)\qquad H_k=-2/[p_{n+1}^*(t_k)p_n^{*\prime}(t_k)],$$

and

$$(19)\qquad E_n(f)=\frac{\pi}{(2n)!\,2^{2n-1}a^2}f^{(2n)}(\xi),\qquad -1<\xi<1,$$

where the prime denotes the derivative and $\{t_k\}$ are the zeros of the $n$th degree polynomial $p_n^*$.

**4. Case II.** Let $w(t)=(1-t^2)^{1/2}$ so that $\psi_n=U_n$ is the Chebyshev polynomial of degree $n$ of the second kind. Following the procedure of Section 3, relations (12) to (14) become

$$(20)\qquad q_0^*=(2/\pi)^{1/2},\qquad q_n^*=(2/\pi)^{1/2}[aU_n+U_{n-1}],\quad n\geq 1,$$

$$(21)\qquad \int_{-1}^{1}(1-t^2)^{1/2}(1+a^2+2at)^{-1}q_m^*(t)q_n^*(t)\,dt=\delta_{mn},\quad a\neq 1, m,n\geq 0,$$

$$(22)\qquad q_{n+1}^*(t)=2tq_n^*(t)-q_{n-1}^*(t),\qquad n\geq 2.$$

The generating function for $q_n^*$ can similarly be written as

$$(23)\qquad \frac{a+w}{1-2tw+w^2}=a+\sqrt{\frac{\pi}{2}}\sum_{n=0}^{\infty} w^{n+1}q_{n+1}^*(t).$$

The corresponding quadrature formula is given by

$$(24)\qquad \int_{-1}^{1}(1-t^2)^{1/2}(1+a^2+2at)^{-1}f(t)\,dt=\sum_{k=1}^{n} H_k f(t_k)+E_n(f),$$

where

$$(25)\qquad H_k=-2/[q_{n+1}^*(t_k)q_n^{*\prime}(t_k)],$$

$$E_n(f) = \frac{\pi}{(2n)!\,2^{2n+1}a^2} f^{(2n)}(\xi), \qquad -1 < \xi < 1, \tag{26}$$

and $\{t_k\}$ are the zeros of $q_n^*$.

**5. Case III.** With $w(t) = ((1-t)/(1+t))^{1/2}$ and orthonormal polynomials $r_n^*$, the corresponding results are as follows:

$$\begin{aligned} r_0^* &= \frac{1}{\sqrt{\pi}}(a-1), \quad r_1^*(t) = \frac{1}{\sqrt{\pi}}(2at + a + 1), \\ r_n^* &= \frac{1}{\sqrt{\pi}}[aU_n + (1+a)U_{n-1} + U_{n-2}], \qquad n \geq 2, \end{aligned} \tag{27}$$

$$\int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{1/2} (1 + a^2 + 2at)^{-1} r_m^*(t) r_n^*(t)\, dt = \delta_{mn}, \qquad a \neq 1,\ m, n \geq 0, \tag{28}$$

$$r_{n+1}^*(t) = 2t r_n^*(t) - r_{n-1}^*(t), \qquad n \geq 1. \tag{29}$$

$$\frac{a + (1+a)w + w^2}{1 - 2tw + w^2} = a + 2atw + (1+a)w + (\pi)^{1/2} \sum_{n=0}^{\infty} r_{n+2}^*(t) w^{n+2}. \tag{30}$$

The relations corresponding to (17), (18) and (19) are

$$\int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{1/2} (1 + a^2 + 2at)^{-1} f(t)\, dt = \sum_{k=1}^{n} H_k f(t_k) + E_n(f), \tag{31}$$

$$H_k = -2/[r_{n+1}^*(t_k) r_n^{*\prime}(t_k)], \tag{32}$$

$$E_n(f) = \frac{\pi}{(2n)!\,2^{2n}a^2} f^{(2n)}(\xi), \qquad -1 < \xi < 1, \tag{33}$$

where $\{t_k\}$ are the zeros of $r_n^*$.

We now discuss the convergence of the quadrature rules.

**6. Case I.** Let $L$ be a closed contour enclosing the interval $[-1, 1]$ in the $z$-plane and let the zeros of the polynomials $p_n^*$ be denoted by $\{t_i\}_1^n$. Application of the residue theorem to the contour integral

$$\frac{1}{2\pi i} \int_L \frac{f(z)\, dz}{(z-t) p_n^*(z)}$$

gives

$$f(t) = \sum_{i=1}^{n} \frac{p_n^*(t)}{(t - t_i) p_n^{*\prime}(t_i)} f(t_i) + \frac{1}{2\pi i} \int_L \frac{f(z) p_n^*(t)\, dz}{(z-t) p_n^*(z)}, \tag{34}$$

assuming that $f(z)$ is regular within $L$.

Multiplying both sides of (34) with $(1-t^2)^{-1/2}(1 + a^2 + 2at)^{-1}$, integrating with regard to $t$ on $[-1, 1]$ and interchanging the order of integration on the right-hand side, we get

$$\int_{-1}^{1} \frac{f(t)\, dt}{(1-t^2)^{1/2}(1 + a^2 + 2at)} = \sum_{i=1}^{n} \mu_i f(t_i) + E_n(f) \tag{35}$$

where

$$\mu_i = \frac{1}{p_n^{*\prime}(t_i)} \int_{-1}^{1} \frac{p_n^*(t)\,dt}{(t - t_i)(1 - t^2)^{1/2}(1 + a^2 + 2at)} \tag{36}$$

and

$$E_n(f) = \frac{1}{2\pi i} \int_L \frac{f(z)}{p_n^*(z)} \int_{-1}^{1} \frac{p_n^*(t)\,dt}{(z - t)(1 - t^2)^{1/2}(1 + a^2 + 2at)} dz.$$

This is the quadrature formula (3) with $(p, q) = (0, 0)$ for analytic functions with abscissas $t_i$ and weights $\mu_i$.

The error of the quadrature formula can be written as

$$E_n(f) = \frac{1}{\pi i} \int_L \frac{f(z) Q_n^*(z)}{p_n^*(z)} dz \tag{37}$$

where

$$Q_n^*(z) = \frac{1}{2} \int_{-1}^{1} \frac{p_n^*(t)\,dt}{(1 - t^2)^{1/2}(z - t)(1 + a^2 + 2at)} \tag{38}$$

is a single-valued function for all $z$ in the plane with the interval $[-1, 1]$ deleted.

The mapping $z = \frac{1}{2}(\xi + \xi^{-1})$, $\xi = \rho e^{i\theta}$ $(0 \le \theta \le 2\pi)$ is now introduced which maps the exterior of the unit circle $|\xi| = 1$ conformally onto the $z$-plane with the interval $[-1, 1]$ deleted. The circle $|\xi| = \rho$ $(\rho > 1)$ is mapped onto an ellipse $\varepsilon_\rho$ with foci at $z = \pm 1$ and semi-axes $\frac{1}{2}(\rho + \rho^{-1})$ and $\frac{1}{2}(\rho - \rho^{-1})$.

**7. A Lemma for $Q_n^*(z)$.** Relation (38) with $\eta = \xi^{-1}$ now becomes

$$Q_n^*(z) = \eta \int_{-1}^{1} \frac{p_n^*(t)\,dt}{(1 - t^2)^{1/2}(1 + a^2 + 2at)(1 - 2\eta t + \eta^2)}. \tag{39}$$

Relation (16) with $\eta$ for $w$ gives

$$\frac{1}{1 - 2\eta t + \eta^2} = \frac{2}{(a + \eta)(1 - \eta^2)} \left\{ \frac{a - \eta}{2} + \sqrt{\frac{\pi}{2}} \sum_0^\infty \eta^{r+1} p_{r+1}^*(t) \right\}.$$

Inserting this in (39) and using the orthonormality property of the polynomials $p_n^*$, we get

$$Q_n^*(z) = \frac{\sqrt{2\pi}}{(1 - \eta^2)(a + \eta)} \eta^{n+1} = \frac{\sqrt{2\pi}}{(1 - 1/\xi^2)(a + 1/\xi)} \xi^{-n-1}.$$

Hence, for $z$ on $\epsilon_\rho$, we have

$$|Q_n^*(z)| \le \frac{\sqrt{2\pi}}{(1 - 1/\rho^2)(a - 1/\rho)} \rho^{-n-1} = \frac{\sqrt{2\pi}}{(\rho^2 - 1)(a\rho - 1)} \rho^{2-n}.$$

We have thus proved the following lemma.

LEMMA. *For $z$ on $\varepsilon_\rho$,*

$$|Q_n^*(z)| \le \frac{\sqrt{2\pi}}{(\rho^2 - 1)(a\rho - 1)} \rho^{2-n}. \tag{40}$$

**8. Convergence of the Quadrature Formula.** Since, for $z$ on $\varepsilon_\rho$, $T_n(z) = \frac{1}{2}(\xi^n + \xi^{-n})$, we have

$$|T_n(z)| \geq \tfrac{1}{2}(\rho^n - \rho^{-n}) \quad \text{and} \quad |T_{n-1}(z)| \leq \tfrac{1}{2}(\rho^{n-1} + \rho^{1-n}).$$

Also

$$p_n^*(z) = (2/\pi)^{1/2}[aT_n(z) + T_{n-1}(z)].$$

Therefore

$$|p_n^*(z)| \geq (2/\pi)^{1/2} \cdot \tfrac{1}{2} \cdot [a(\rho^n - \rho^{-n}) - (\rho^{n-1} + \rho^{1-n})]. \tag{41}$$

From (37), by selecting the contour as an ellipse $\varepsilon_\rho$ ($\rho > 1$), it follows that

$$|E_n(f)| \leq \frac{1}{\pi}\int_{\varepsilon_\rho} \frac{|f(z)| \cdot |Q_n^*(z)|}{|p_n^*(z)|} ds \qquad (|dz| = ds). \tag{42}$$

Let

$$M(\rho) = \max_{z \in \epsilon_\rho} |f(z)| \quad \text{and} \quad l(\varepsilon_\rho) = \text{length of } \varepsilon_\rho. \tag{43}$$

Inserting (40), (41) and (43) in (42), we get

$$|E_n(f)| \leq \frac{2lM}{(\rho^2 - 1)(a\rho - 1)} \frac{\rho^{2-n}}{a(\rho^n - \rho^{-n}) + (\rho^{n-1} - \rho^{1-n})}.$$

Thus, the following result has been established.

THEOREM 2. *Let* $f \in A(\epsilon_\rho)$ *and let* $\rho > 1$. *Then*

$$|E_n(f)| \leq \frac{2lM}{(\rho^2 - 1)(a\rho - 1)} \frac{\rho^{2-n}}{a(\rho^n - \rho^{-n}) + (\rho^{n-1} - \rho^{1-n})}. \tag{44}$$

**9. Case II.** Corresponding to $(p, q) = (1, 1)$ in formula (3), relations (35) to (39) are revised as follows:

$$\int_{-1}^{1} \frac{(1 - t^2)^{1/2}}{1 + a^2 + 2at} f(t)\,dt = \sum_{i=1}^{n} \mu_i f(t_i) + E_n(f), \tag{45}$$

$$\mu_i = \frac{1}{q_n^{*\prime}(t_i)} \int_{-1}^{1} \frac{(1 - t^2)^{1/2} q_n^*(t)}{(t - t_i)(1 + a^2 + 2at)} dt, \tag{46}$$

$$E_n(f) = \frac{1}{\pi i}\int_L \frac{f(z)Q_n^*(z)}{q_n^*(z)} dz, \tag{47}$$

$$Q_n^*(z) = \frac{1}{2}\int_{-1}^{1} \frac{(1 - t^2)^{1/2} q_n^*(t)\,dt}{(z - t)(1 + a^2 + 2at)}, \tag{48}$$

$$Q_n^*(z) = \eta \int_{-1}^{1} \frac{(1 - t^2)^{1/2}}{1 + a^2 + 2at} \frac{q_n^*(t)\,dt}{1 - 2\eta t + \eta^2}, \tag{49}$$

where $t_i$ are the zeros of $q_n^*$.

Inserting (23) with $\eta$ for $w$ in (49) and using the orthonormality property of the polynomials $q_n^*$, we get

$$Q_n^*(z) = \sqrt{\frac{\pi}{2}}\,\frac{\eta^{n+1}}{a+\eta} = \sqrt{\frac{\pi}{2}}\,\frac{\xi^{-n-1}}{a+1/\xi}$$

which proves the following lemma.

LEMMA. *For $z$ on $\varepsilon_\rho$,*

$$|Q_n^*(z)| \le \sqrt{\frac{\pi}{2}}\,\frac{\rho^{-n}}{a\rho - 1}. \tag{50}$$

**10. Bounds on Error.** Since

$$|z_1 - z_2| \ge \big||z_1|-|z_2|\big| \quad \text{and} \quad q_n^*(z) = (2/\pi)^{1/2}[aU_n(z) + U_{n-1}(z)]$$

we have

$$|q_n^*(z)| \ge (2/\pi)^{1/2}[a|U_n(z)| - |U_{n-1}(z)|].$$

Now, for $z$ on $\varepsilon_\rho$,

$$U_n(z) = (\xi^{n+1} - \xi^{-n-1})/(\xi - \xi^{-1}).$$

Therefore

$$|U_n(z)| \ge \frac{\rho^{n+1} - \rho^{-n-1}}{\rho + \rho^{-1}} \quad \text{and} \quad |U_{n-1}(z)| \le \frac{\rho^n + \rho^{-n}}{\rho - \rho^{-1}}.$$

Hence

$$|q_n^*(z)| \ge \left(\frac{2}{\pi}\right)^{1/2}\left[a\frac{\rho^{n+1} - \rho^{-n-1}}{\rho + \rho^{-1}} - \frac{\rho^n + \rho^{-n}}{\rho - \rho^{-1}}\right]. \tag{51}$$

From (47), we have

$$|E_n(f)| \le \frac{1}{\pi}\int_{\varepsilon_\rho} \frac{|f(z)|\,|Q_n^*(z)|}{|q_n^*(z)|}\,ds \qquad (|dz| = ds). \tag{52}$$

Inserting (50), (51) and (43) in (52), we get, on simplification, the following result:

THEOREM 3. *Let $f \in A(\epsilon_\rho)$ and let $\rho > 1$. Then*

$$|E_n(f)| \le \frac{M(\rho)l(\varepsilon_\rho)}{2}\,\frac{\rho^{-n}}{a\rho^{-1}}\cdot\left(a\left(\frac{\rho^{n+1} - \rho^{-n-1}}{\rho + \rho^{-1}}\right) - \left(\frac{\rho^n + \rho^{-n}}{\rho - \rho^{-1}}\right)\right)^{-1} \tag{53}$$

*where $M(\rho)$ and $l(\varepsilon_\rho)$ are given by* (43).

**11. Case III.** Corresponding to $(p, q) = (0, 1)$ in formula (3), relations (35) to (39) are revised as follows:

$$\int_{-1}^{1}\left(\frac{1-t}{1+t}\right)^{1/2}(1 + a^2 + 2at)^{-1}f(t)\,dt = \sum_{i=1}^{n}\mu_i f(t_i) + E_n(f), \tag{54}$$

$$\mu_i = \frac{1}{r_n^{*\prime}(t_i)}\int_{-1}^{1}\left(\frac{1-t}{1+t}\right)^{1/2}\frac{r_n^*(t)\,dt}{(t - t_i)(1 + a^2 + 2at)}, \tag{55}$$

(56) $$E_n(f) = \frac{1}{\pi i}\int_L \frac{f(z)Q_n^*(z)}{r_n^*(z)}dz,$$

(57) $$Q_n^*(z) = \frac{1}{2}\int_{-1}^{1}\left(\frac{1-t}{1+t}\right)^{1/2}\frac{r_n^*(t)}{(z-t)(1+a^2+2at)}dt,$$

where $t_i$ are the zeros of $r_n^*(t)$,

(58) $$Q_n^*(z) = \eta\int_{-1}^{1}\left(\frac{1-t}{1+t}\right)^{1/2}\frac{r_n^*(t)}{1+a^2+2at}\,\frac{dt}{1-2\eta t+\eta^2}.$$

Introduction of (30) in (58), with $\eta$ for $w$, and the use of orthonormality property of the polynomials $r_n^*$, we get

$$Q_n^*(z) = (\pi)^{1/2}\frac{\eta^{n+1}}{a+(1+a)\eta+\eta^2} = (\pi)^{1/2}\frac{\xi^{-n+1}}{a\xi^2+(1+a)\xi+1},$$

which proves the following lemma:

LEMMA. *For $z$ on $\varepsilon_\rho$,*

(59) $$|Q_n^*(z)| \le (\pi)^{1/2}\frac{\rho^{-n+1}}{a\rho^2-(1+a)\rho+1} \qquad (n>1).$$

**12. Bounds on Error.** Since

$$|z_1+z_2| \ge ||z_1|-|z_2|| \quad\text{and}\quad r_n^*(z) = (\pi)^{-1/2}[aU_n(z) + \{(1+a)U_{n-1}(z)+U_{n-2}(z)\}]$$

we have

$$|r_n^*(z)| \ge (\pi)^{-1/2}[a|U_n(z)| - \{(1+a)|U_{n-1}(z)| + |U_{n-2}(z)|\}].$$

Now, for $z$ on $\varepsilon_\rho$,

$$U_n(z) = (\xi^{n+1}-\xi^{-n-1})/(\xi-\xi^{-1}).$$

Therefore

$$|U_n(z)| \ge \frac{\rho^{n+1}-\rho^{-n-1}}{\rho+\rho^{-1}},$$

$$|U_{n-1}(z)| \le \frac{\rho^n+\rho^{-n}}{\rho-\rho^{-1}} \quad\text{and}\quad |U_{n-2}(z)| \le \frac{\rho^{n-1}+\rho^{-n+1}}{\rho-\rho^{-1}}.$$

Hence

(60) $$|r_n^*(z)| \ge (\pi)^{-1/2}\left[a\frac{\rho^{n+1}-\rho^{-n-1}}{\rho+\rho^{-1}} - (1+a)\frac{\rho^n+\rho^{-n}}{\rho-\rho^{-1}} - \frac{\rho^{n-1}+\rho^{-n+1}}{\rho-\rho^{-1}}\right].$$

From (56) we have

(61) $$|E_n(f)| \le \frac{1}{\pi}\int_{\varepsilon_\rho}\frac{|f(z)|\cdot|Q_n^*(z)|}{|r_n^*(z)|}ds \qquad (|dz| = ds).$$

Inserting (59), (60) and (43) in (61), we get, on simplification, the following result:

THEOREM 4. *Let* $f \in A(\epsilon_\rho)$ *and let* $\rho > 1$. *Then*

$$|E_n(f)| \leq \frac{M(\rho)l(\varepsilon_\rho)\rho^{-n+1}}{a\rho^2 - (1 + a)\rho + 1} \tag{62}$$
$$\cdot \left(a\left(\frac{\rho^{n+1} - \rho^{-n-1}}{\rho + \rho^{-1}}\right) - (1 + a)\left(\frac{\rho^n + \rho^{-n}}{\rho - \rho^{-1}}\right) - \left(\frac{\rho^{n-1} + \rho^{-n+1}}{\rho - \rho^{-1}}\right)\right)^{-1}.$$

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