MATHEMATICS OF COMPUTATION, VOLUME 28, NUMBER 128
OCTOBER 1974, PAGES 1053–1055

A Nonexistence Theorem for Explicit A-Stable Methods

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Abstract. It is proved that there are no A-stable explicit methods in a general class of "linear" methods. The class contains, for example, Runge-Kutta methods, linear multistep methods, predictor-corrector formulas, cyclic multistep methods and linear multistep methods with higher derivatives.

It is obvious that explicit methods for stiff problems of ordinary differential equations have to be nonlinear in some sense. In this paper, we prove that there are no A-stable explicit methods in a general class of "linear" methods. Consider the following methods

$$\sum_{r=0}^{s} \sum_{i=0}^{k} h^r A_i^r \mathrm{y}_{n-k+i}^{(r)} = 0 \tag{1}$$

for the solution of $x' = f(t, x)$, $x(0) = x_0$, $t \in [0, 1]$. In Eq. (1), $\mathrm{y}_n^{(0)}$ is an approximation to

$$\mathrm{x}(t_n) = \begin{pmatrix} x(t_n^1) \\ \vdots \\ x(t_n^m) \end{pmatrix}$$

and $\mathrm{y}_n^{(r)}$, $r = 1(1)s$, a corresponding approximation to $\mathrm{x}^{(r)}(t_n)$, the $t_n^j = (n - 1 + \tau_j)h$, τ_j fixed constants. The $A_i^r = (a_{\mu\lambda}^{ir})$ are fixed matrixes with

$$a_{\mu\lambda}^{k0} = 0 \quad \text{for } \mu < \lambda, \qquad a_{\mu\mu}^{k0} \neq 0, \qquad \mu = 1(1)m,$$

$$\left(\sum_{i=0}^{k} A_i^0\right) e = 0, \quad \text{where } e = (1, 1, \cdots, 1)^T.$$

The method is explicit if

$$a_{\mu\lambda}^{kr} = 0 \quad \text{for } \mu \leqslant \lambda, \qquad r = 1(1)s.$$

If $s = 1$, method (1) reduces to Stetter's simple m-stage k-step method, cf. [2, p. 275]. It thus contains Runge-Kutta methods, linear multistep methods, predictor-corrector

Received December 27, 1973.

AMS (MOS) subject classifications (1970). Primary 65L05.

formulas and cyclic multistep methods. In addition, method (1) contains, e.g., multistep methods with higher derivatives. We shall prove

THEOREM. *There are no explicit A-stable methods* (1).

Remark 1. For linear multistep methods, the result has been proved by Dahlquist [1].

Remark 2. The predictor-corrector methods, when the corrector is iterated m-times, are explicit in formulation (1). Thus, they are not A-stable, although the corrector formula itself might be A-stable.

Remark 3. In fact, we shall prove that there are no explicit $A(0)$-stable [3] methods (1).

Proof. Let us consider the scalar equation $x' = \lambda x$, $\operatorname{Re} \lambda < 0$. Applying method (1) to this equation we get the linear difference equation of order k

$$\sum_{r=0}^{s} \sum_{i=0}^{k} (\lambda h)^r A_i^r y_{r-k+i} = 0. \tag{2}$$

We can assume that $\Sigma_{r=0}^{s} (\lambda h)^r A_k^r$ is nonsingular. Thus, Eq. (2) can be expressed in the form

$$\begin{pmatrix} y_{n-k+1} \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{pmatrix} = \begin{pmatrix} 0 & I & & & 0 \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \\ 0 & & & 0 & I \\ -B_k^{-1} B_0 & & \cdots & & -B_k^{-1} B_{k-1} \end{pmatrix} \begin{pmatrix} y_{n-k} \\ \cdot \\ \cdot \\ \cdot \\ y_{n-1} \end{pmatrix}$$

$$= C(\lambda h) \begin{pmatrix} y_{n-k} \\ \cdot \\ \cdot \\ \cdot \\ y_{n-1} \end{pmatrix},$$

where $B_i(\lambda h) = \Sigma_{r=0}^{s} (\lambda h)^r A_i^r$, $i = 0(1)k$. A-stability requires $\|C(\lambda h)\| < 1$, which demands that all eigenvalues of $C(\lambda h)$ have modulus smaller than 1. Hence, we consider these eigenvalues, which are the roots of

$$\det \left[\sum_{r=0}^{s} \sum_{i=0}^{k} (\lambda h)^r A_i^r z^i \right] = 0, \tag{3}$$

since

$$\det [zI - C(\lambda h)] = \det \left(\sum_{i=0}^{k} B_k^{-1} B_i z^i \right) = \det (B_k^{-1}) \cdot \det \left(\sum_{i=0}^{k} B_i z^i \right).$$

Next, we show that for explicit methods (1) Eq. (3) is of the form

$$z^{km} + \sum_{\nu=0}^{km-1} p_\nu(\lambda h) z^\nu = 0, \tag{4}$$

where the p_ν's are polynomials. Remember that for an explicit method $a_{\mu\lambda}^{kr} = 0$ for $\mu \leqslant \lambda$, $r = 1(1)s$ and $a_{\mu\lambda}^{k0} = 0$ for $\mu < \lambda$. Let us write Eq. (3) as

$$\det \begin{pmatrix} a_{11}^{k0} z^k + q_{11}^{k-1} & q_{12}^{k-1} & \cdots & q_{1m}^{k-1} \\ q_{21}^{k} & a_{22}^{k0} z^k + q_{22}^{k-1} & \cdots & q_{2m}^{k-1} \\ \vdots & \vdots & & \vdots \\ q_{m1}^{k} & q_{m2}^{k} & \cdots & a_{mm}^{k0} z^k + q_{mm}^{k-1} \end{pmatrix} = 0,$$

where $q_{\mu\omega}^j = \Sigma_{r=0}^s \Sigma_{i=0}^j (\lambda h)^r a_{\mu\omega}^{ir} z^i$. Expanding according to the elements of the first column, we observe that the only cofactor containing z^k in the elements of the first row is that corresponding to the first element. Denote this cofactor by C_{11}. Thus, the only term in the expansion that can contain z^{km} is $a_{11} z^k C_{11}$. The same reasoning applies directly to C_{11} and so on, giving the coefficient of z^{km} as $\Pi_{\mu=1}^m a_{\mu\mu}^{k0}$. Dividing the expansion by $\Pi_{\mu=1}^m a_{\mu\mu}^{k0}$, which is different from zero by assumption, gives Eq. (3) in form (4).

Suppose not all p_ν's are constant in representation (4) of Eq. (3). Then at least one of the roots of Eq. (3) grows without bound in modulus, for example when $\lambda h \longrightarrow -\infty$ in the real axis. On the other hand, if none of the p_ν's depend on λh, then Eq. (3) does not depend on λh and is thus equivalent to

$$\det \left[\sum_{i=0}^{k} A_i^0 z^i \right] = 0, \tag{5}$$

which is obtained from (3) by substitution of $\lambda h = 0$. Equation (5) is clearly satisfied by $z = 1$ since, by assumption, $(\Sigma_{i=0}^k A_i^0)e = 0$, where $e = (1, 1, \cdots, 1)^T$. This is again in contradiction with the requirement of A-stability.

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