

Optimal L_∞ Error Estimates for Galerkin Approximations to Solutions of Two-Point Boundary Value Problems*

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Abstract. A priori error estimates in the maximum norm are derived for Galerkin approximations to solutions of two-point boundary value problems. The class of Galerkin spaces considered includes almost all (quasiuniform) piecewise-polynomial spaces that are used in practice. The estimates are optimal in the sense that no better rate of approximation is possible in general in the spaces employed.

1. Introduction. Consider the two-point boundary value problem

$$-(a(x)y')' + b(x)y' + d(x)y = f(x), \quad x \in I = (0, 1), y(0) = y(1) = 0,$$

or, in weak form, the problem of finding $y \in \mathring{H}^1$ such that

$$(1.1) \quad (ay', v') + (by', v) + (dy, v) = (f, v), \quad v \in \mathring{H}^1.$$

To seek an approximate solution to the problem (1.1), consider a piecewise-polynomial spline space M_k^r , $-1 \leq k < r$, defined as

$$M_k^r = \{v \in C^k(I): v|_{I_i} \in \Pi_r(I_i), \quad i = 1, \dots, N\}.$$

Here, $I_i = (x_{i-1}, x_i)$, $0 = x_0 < x_1 < \dots < x_{N-1} < x_N = 1$, and $\Pi_r(I_i)$ denotes the set of polynomials on I_i of degree not greater than r . It is assumed that, as the meshes vary, they are quasiuniform; i.e., with $h_i = x_i - x_{i-1}$, there exists a constant c_0 such that

$$(1.2) \quad \max_{i,j} h_i h_j^{-1} \leq c_0.$$

Let $h = \max_i h_i$.

The approximate solution Y to (1.1) is sought in the space

$$M = \mathring{M}_k^r = M_k^r \cap \{v: v(0) = v(1) = 0\}$$

according to the rule

$$(1.3) \quad (aY', V') + (bY', V) + (dY, V) = (f, V), \quad V \in M.$$

Here, it is assumed that $0 \leq k < r$ so that $M \subset \mathring{H}^1$.

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Assume throughout the remainder of the paper that

- (i) $a(x) \geq c_1 > 0$, $x \in I$,
- (ii) $a, a', b, b', d \in L_\infty$,
- (iii) for all $f \in L_2$, there exists a unique $y \in \mathring{H}^1$ satisfying (1.1).

From these assumptions it follows that (1.3) has a unique solution $Y \in M$ for h sufficiently small (Schatz [5]; see also [3] for a proof). For h sufficiently small it is known (Nitsche [4]) that

$$(1.4) \quad \|y - Y\|_{L_2} + h \|y - Y\|_{H^1} \leq c_2 h^{r+1} \|y\|_{H^{r+1}},$$

where c_2 depends on the L_∞ -norms of the functions specified in assumption (ii). For simplicity, we shall also assume that the particular y of (1.1) that we shall approximate is an element of $W_\infty^{r+1} \cap \mathring{H}^1$.

Under the assumptions above, our result for the error in the maximum norm is:

THEOREM 1.1. *There exists a constant c*

$$c = c(c_0, c_1, c_2, \|a\|_{W_\infty^1}, \|b\|_{H^1}, \|d\|_{L_2})$$

such that

$$\|y - Y\|_{L_\infty} \leq c h^{r+1} \|y\|_{W_\infty^{r+1}}.$$

Theorem 1.1 was proved for $k = 0$, i.e., continuous piecewise-polynomial splines in [6] and, in that case, without the assumption of quasiuniformity.

Outline of the Paper. In Section 2, the notation used is defined, and a basic extension lemma is proved. In Section 3, the problem is reduced to the special case when $a(x) \equiv 1$, $b(x) \equiv d(x) \equiv 0$. In Section 4, it is first noted that, in this case, the derivative of the elliptic projection W of y into \mathring{M}_k^r is the L_2 -projection of y' into M_{k-1}^{r-1} . An estimate for the error in the L_2 -projection in the maximum norm is derived, giving an estimate for $y' - W'$. The proof of this estimate uses the extension lemma to prove that the L_2 -projection of a function of small support decreases rapidly outside that support. The estimate for $y' - W'$ then gives an estimate for $y - W$ via a duality argument.

Remark 1.1. The result (4.4) below (stability in the maximum norm of the L_2 -projection) also gives estimates for the error in the maximum norm for smooth spline interpolation, see [1, Lemma 2.1].

2. Notation and an Extension Lemma. For an open interval J , let $H^s(J)$ and $W_p^s(J)$ denote the closure of $C^\infty(\bar{J})$ in the norms

$$\|v\|_{H^s(J)} = \left(\sum_{i=0}^s \|v^{(i)}\|_{L_2(J)}^2 \right)^{1/2} \quad \text{and} \quad \|v\|_{W_p^s(J)} = \sum_{i=0}^s \|v^{(i)}\|_{L_p(J)},$$

respectively. When $J = I = (0, 1)$, we drop the dependence on the interval in the notation.

We note that

$$(2.1) \quad \|v\|_{L_\infty} \leq 2^{1/2} \|v\|_{H^1}.$$

Let $\mathring{H}^s(J)$ denote the closure of $C_0^\infty(J)$ in the norm $\|\cdot\|_{H^s(J)}$; then for $v \in \mathring{H}^1$,

$$(2.2) \quad \|v\|_{H^1} \leq 2^{1/2} \|v'\|_{L_2}.$$

Let (v, w) denote $\int_0^1 v(x)w(x)dx$, and for $i, l, m \in \mathbb{Z}$, let

$$I(i, l, m) = (I_{i-l} \cup I_{i-l+1} \cup \dots \cup I_{i+m}) \cap I,$$

where $\dots, I_{-2}, I_{-1}, I_0, I_{N+1}, \dots$ are arbitrarily defined.

The letters c and C will denote constants, not necessarily the same at each occurrence unless indexed.

The rest of this section is devoted to the proof of the following lemma, which allows us to construct appropriate piecewise-polynomial extensions.

LEMMA 2.1. *Given r and k , $-1 \leq k < r$, and M_k^r subject to (1.2), there exist constants $n = n(k, r) \in \mathbb{Z}$ and $c = c(c_0, k, r)$ such that, given $V \in \Pi_r(I_i)$, there exists a function $f_i \in M_k^r$ such that*

$$f_i = V \text{ on } I_i; \quad \text{supp } f_i \subset I(i, n, n); \quad \|f_i\|_{L_2} \leq c \|V\|_{L_2(I_i)}.$$

Proof. The case $k = -1$ is trivial, since we can set $f_i = 0$ outside I_i . Assume $0 \leq k < r$. We consider the problem of extending V to the right of I_i to fulfill the conditions of the lemma. Let $(k+1)/(r-k) = n - \sigma$, where n is an integer and $0 \leq \sigma < 1$. Put $s = \sigma(r-k)$. Assume for simplicity that $i+n < N$.

Define $f \in M_k^r(x_i, 1)$ (in obvious notation) by the requirements

$$(2.3) \quad f \equiv 0 \quad \text{outside } I_{i+n},$$

$$(2.4) \quad f^{(l)}(x_i) = V^{(l)}(x_i), \quad l = 0, \dots, k,$$

$$(2.5) \quad f^{(l)}(x_{i+n}) = 0, \quad l = 0, \dots, k+s.$$

We must show that these requirements determine f on $(x_i, 1)$. Let

$$f|_{I_m} = \sum_{j=0}^r f_{j,m} (x - x_{m-1})^j, \quad m = i+1, \dots, i+n.$$

We have $n(r+1)$ coefficients to determine and the requirements

$$(2.6) \quad l! f_{l,i+1} = V^{(l)}(x_i), \quad l = 0, \dots, k,$$

$$(2.7) \quad \sum_{j=0}^r j(j-1) \dots (j-l+1) f_{j,m} (x_m - x_{m-1})^{j-l} - l! f_{l,m+1} = 0, \\ l = 0, \dots, k; \quad m = i+1, \dots, i+n-1,$$

$$(2.8) \quad \sum_{j=0}^r j(j-1) \dots (j-l+1) f_{j,i+n} (x_{i+n} - x_{i+n-1})^{j-l} = 0, \\ l = 0, \dots, k+s,$$

to fulfill. These requirements total

$$\begin{aligned} k + 1 + (n - 1)(k + 1) + k + s + 1 &= n(k + 1) + (k + s + 1) \\ &= n(r + 1) + n(k - r) + k + s + 1 = n(r + 1), \end{aligned}$$

since $s = \sigma(r - k) = n(r - k) - (k + 1)$.

Hence, it suffices to show that if $V^{(l)}(x_i) = 0$, $l = 0, \dots, k$, then $f \equiv 0$. For this, consider the continuous function $f^{(k)}$: This function is a piecewise polynomial of degree not greater than $r - k$. On each of $\bar{I}_{i+2}, \dots, \bar{I}_{i+n-1}$ where $f^{(k)} \not\equiv 0$, it has at most $r - k$ roots. Similarly, if $f^{(k)} \not\equiv 0$ on the open interval I_{i+1} , it has at most $r - k - 1$ roots there, and on I_{i+n} , it has at most $r - k - s - 1$ roots. Altogether, on subintervals where $f^{(k)} \not\equiv 0$, it has at most

$$\begin{aligned} (n - 2)(r - k) + r - k - 1 + r - k - s - 1 &= n(r - k) - 2 - s \\ &= (n - \sigma)(r - k) - 2 = k - 1 \end{aligned}$$

roots not coinciding with x_i or x_{i+n} . Hence, we can find a polynomial $p(x)$ of degree $k - 1$ such that

$$f^{(k)}(x)p(x) \geq 0, \quad x_i \leq x \leq x_{i+n}, \quad \text{and} \quad f^{(k)}(x)p(x) > 0 \quad \text{if} \quad f^{(k)}(x) \neq 0.$$

However, by repeated partial integration, we find that, since $f^{(j)}(x_i) = f^{(j)}(x_{i+n}) = 0$, $j \leq k$,

$$\int_{x_i}^{x_{i+n}} f^{(k)} p \, dx = 0.$$

Thus, $f^{(k)} \equiv 0$, and $f \equiv 0$. Hence, (2.3)–(2.5) determine $f \in M_k^r(x_i, 1)$.

To establish the norm inequality of the lemma, multiply (2.6)–(2.8) by h_{m+1}^l ($h_{m+1} = x_{m+1} - x_m$). The corresponding linear system of equations for the quantities $g_{j,m} = f_{j,m} h_m^j$ is:

$$(2.6)' \quad l! g_{l,i+1} = V^{(l)}(x_i) h_{i+1}^l, \quad l = 0, \dots, k,$$

$$(2.7)' \quad \sum_{j=0}^r j(j-1) \cdots (j-l+1) \left(\frac{h_{m+1}}{h_m} \right)^l g_{j,m} - l! g_{l,m+1} = 0, \\ l = 0, \dots, k; \quad m = i+1, \dots, i+n-1,$$

$$(2.8)' \quad \sum_{j=0}^r j(j-1) \cdots (j-l+1) g_{j,i+n} = 0, \quad l = 0, \dots, k+s.$$

Since the determinant of this system is never zero, and since, by (1.2), h_{m+1}/h_m varies over a compact interval, it follows that there exists a constant $c = (c_0)$ such that

$$\max_{m,j} |f_{j,m} h_m^j| \leq c \max_l |V^{(l)}(x_i) h_{i+1}^l|.$$

Since

$$\|f\|_{L_2((x_i, x_{i+n}))} \leq c h^{1/2} \max_{m,j} |f_{j,m} h_m^j|$$

and

$$\max_l |V^{(l)}(x_i)h_i^l| \leq ch_i^{-1/2} \|V\|_{L_2(I_i)},$$

it follows that

$$(2.9) \quad \|f\|_{L_2(x_i, 1)} \leq c \|V\|_{L_2(I_i)}.$$

Apply the analogous construction leftwards; this concludes the proof.

3. Comparison of Different Elliptic Projections. We shall consider three different elliptic projections, Y , Z , and W , of the solution y of (1.1) into $M = \overset{\circ}{M}_k^r$. Here, Y is given by (1.3) or, equivalently, by

$$(3.1) \quad (a(y' - Y'), V') + (b(y' - Y'), V) + (d(y - Y), V) = 0, \quad V \in M,$$

and Z and W are given by

$$(3.2) \quad (a(y' - Z'), V') = 0, \quad V \in M,$$

$$(3.3) \quad (y' - W', V') = 0, \quad V \in M.$$

Since the bilinear forms corresponding to (3.2) and (3.3) are positive definite on M , Z and W are defined. We shall prove that the three elliptic projections defined above differ in H^1 by $O(h^{r+1})$.

LEMMA 3.1. *There exists a constant c ,*

$$c = c(c_1, c_2, \|b\|_{H^1}, \|d\|_{L_2}),$$

such that $\|Y - Z\|_{H^1} \leq c \|y - Y\|_{L_2}$.

Proof (cf. [6]). We have

$$\begin{aligned} 0 &= (a(y' - Y'), V') + (b(y' - Y'), V) + (d(y - Y), V) \\ &= (a(Z' - Y'), V') + ((y - Y), dV - (bV)'). \end{aligned}$$

Choosing $V = Z - Y$, we obtain

$$\begin{aligned} c_1 \|Z' - Y'\|_{L_2}^2 &\leq \|y - Y\|_{L_2} (\|d\|_{L_2} \|Z - Y\|_{L_\infty} + \|b\|_{L_\infty} \|Z' - Y'\|_{L_2} + \|b'\|_{L_2} \|Z - Y\|_{L_\infty}) \\ &\leq (\|d\|_{L_2} + \|b\|_{L_\infty} + \|b'\|_{L_2}) \|y - Y\|_{L_2} \|Z' - Y'\|_{L_2}. \end{aligned}$$

LEMMA 3.2. *There exists a constant c , $c = c(c_0, c_1, \|a\|_{W_\infty})$, such that $\|Z - W\|_{H^1} \leq ch \|y - W\|_{H^1}$.*

Proof. Let $\vartheta = Z - W$. From the definitions of Z and W , we see that for $\chi \in M$,

$$c_1 \|\vartheta'\|_{L_2}^2 \leq (a\vartheta', \vartheta') = (a(y - W)', \vartheta') = (y' - W', a\vartheta' - \chi').$$

Since $y' - W'$ has zero average value, we can use instead of χ' any $\nu \in M^* = M_{k-1}^{r-1}$.

Thus

$$(3.4) \quad \|\vartheta'\|_{L_2}^2 \leq c \|y - W\|_{H^1} \inf_{\nu \in M^*} \|a\vartheta' - \nu\|_{L_2}.$$

In order to prove the result, it suffices to show that for $V \in M^*$

$$(3.5) \quad \inf_{\nu \in M^*} \|aV - \nu\|_{L_2} \leq ch \|V\|_{L_2}.$$

In order to establish (3.5), we first remark that there is a constant c such that, if $W \in C^{k-1}(I)$ and $W|_{I_i} \in H^r(I_i)$, $i = 1, \dots, N$, then

$$(3.6) \quad \inf_{\nu \in M^*} \|W - \nu\|_{L_2} \leq ch^r \|W^{(r)}\|_{L_2} := \left(\sum_{i=1}^N \|W^{(r)}\|_{L_2(I_i)}^2 \right)^{1/2}$$

This is easily seen by adding a function $\nu_1 \in M^*$ to W so that $W + \nu_1 \in H^r$ and then noting that

$$\begin{aligned} \inf_{\nu \in M^*} \|W - \nu\|_{L_2} &= \inf_{\nu \in M^*} \|W + \nu_1 - \nu\|_{L_2} \\ &\leq ch^r \|(W + \nu_1)^{(r)}\|_{L_2} = ch^r \|W^{(r)}\|_{L_2}. \end{aligned}$$

Next, note that there exists a function $\psi \in M^*$ such that

$$(3.7) \quad \|a - \psi\|_{L_\infty} \leq c \|a'\|_{L_\infty} h$$

and

$$(3.8) \quad \|\psi^{(l)}\|_{L_\infty} \leq c \|a'\|_{L_\infty} h^{1-l}, \quad l = 1, 2, \dots, r-1;$$

this is easily seen by modifying a and applying an estimate like (4.1) of the next section.

Thus, from (3.6), (3.7) and (3.8), we see that for $V \in M^*$

$$\begin{aligned} \inf_{\chi \in M} \|aV - \chi\|_{L_2} &\leq \|(a - \psi)V\|_{L_2} + \inf_{\chi \in M^*} \|\psi V - \chi\|_{L_2} \\ &\leq ch \|V\|_{L_2} + ch^r \sum_{l=1}^{r-1} \|\psi^{(l)}\|_{L_\infty} \|V^{(r-l)}\|_{L_2} \\ &\leq ch \|V\|_{L_2}, \end{aligned}$$

where we used the quasiuniformity of the mesh to estimate the terms $\|V^{(r-l)}\|_{L_2}$.

4. Proof of Theorem 1.1. It is sufficient, as a consequence of the reduction of the last section, to prove Theorem 1.1 in the case $a \equiv 1$, $b \equiv d \equiv 0$. We begin by summarizing the approximation-theoretic properties of the space M_l^s , $-1 \leq l < s$, that we need.

LEMMA 4.1 (DE BOOR [2]). *There exists a constant c such that, if $u \in W_\infty^{s+1}$ and $v \in W_1^2 \cap \dot{H}^1$, there exists $\chi \in M_l^s$ and $\psi \in \dot{M}_l^s$ such that*

$$(4.1) \quad \|u - \chi\|_{L_\infty} \leq ch^{s+1} \|u\|_{W_\infty^{s+1}},$$

$$(4.2) \quad \|v - \psi\|_{W_1^1} \leq ch \|v\|_{W_1^2}.$$

Let Pu denote the L_2 -projection of a function $u \in L_2$ into M_l^s , $-1 \leq l < s$, defined by

$$(Pu - u, V) = 0, \quad V \in M_l^s.$$

LEMMA 4.2. *There exists a constant $c = c(c_0)$ such that, given $u \in W_\infty^{s+1}$,*

$$\|Pu - u\|_{L_\infty} \leq ch^{s+1} \|u\|_{W_\infty^{s+1}}.$$

The proof of this lemma is postponed until the end of this section.

Remark 4.1. It is easily seen by duality using Lemma 4.2 that P gives optimal approximation in the L_1 -norm. It then follows from interpolation that P gives optimal approximation in any L_p -norm, $1 \leq p \leq \infty$.

LEMMA 4.3. *There exists a constant $c = c(c_0)$ such that, given $y \in W_\infty^{r+1} \cap \mathring{H}^1$, and with W defined by (3.3),*

$$\|y - W\|_{L_\infty} \leq ch^{r+1} \|y\|_{W_\infty^{r+1}}.$$

Proof. Since $(y' - W', 1) = 0$, (3.3) implies that W' is the L_2 -projection of y' into M_{k-1}^{r-1} . By Lemma 4.2, it follows that

$$(4.3) \quad \|y' - W'\|_{L_\infty} \leq ch^r \|y'\|_{W_\infty^r}.$$

We now apply a duality argument [4]. Given $g \in L_1$, let G be such that $G'' = -g$, $G(0) = G(1) = 0$. Then

$$(y - W, g) = (y' - W', G') = (y' - W', G' - \chi'), \quad \chi \in M.$$

By Lemma 4.1, χ can be chosen so that

$$|(y - W, g)| \leq ch \|y' - W'\|_{L_\infty} \|g\|_{L_1},$$

and it follows from (4.3) that

$$\|y - W\|_{L_\infty} = \sup_{\|g\|_{L_1}=1} |(y - W, g)| \leq ch^{r+1} \|y\|_{W_\infty^{r+1}}.$$

Proof of Theorem 1.1. We have (cf. (3.2), (3.3))

$$\|y - Y\|_{L_\infty} \leq \|y - W\|_{L_\infty} + \|Y - Z\|_{L_\infty} + \|Z - W\|_{L_\infty}.$$

Using (2.1), Lemmas 3.1, 3.2, and 4.3, and (1.4) and its counterpart for $y - W$, the theorem follows.

It remains to prove Lemma 4.2.

Proof of Lemma 4.2. Let χ be as in (4.1). Since $P\chi = \chi$, we have

$$\|Pu - u\|_{L_\infty} \leq \|u - \chi\|_{L_\infty} + \|P(u - \chi)\|_{L_\infty},$$

and hence it suffices to show that there exists a constant c such that

$$(4.4) \quad \|Pu\|_{L_\infty} \leq c \|u\|_{L_\infty}, \quad u \in L_\infty.$$

Let $u = \sum_{i=1}^N u_i$, where

$$u_i(x) = \begin{cases} u(x), & x \in I_i, \\ 0, & x \notin I_i. \end{cases}$$

Consider Pu_i on I_m , $m < i$. By Lemma 2.1, there exists $f_{i,m} \in M_i^s$ which agrees with Pu_i on I_m , and satisfies

$$\|f_{i,m}\|_{L_2} \leq c \|Pu_i\|_{L_2(I_m)}.$$

Since the L_2 -projection Pu_i minimizes the L_2 -norm of the difference $V - u_i$ for V in M_i^s , and since $u_i = 0$ outside I_i , it follows that Pu_i minimizes the $L_2((0, x_{m-1}))$ -norm of elements of M_i^s agreeing with Pu_i on I_m . Thus,

$$\|Pu_i\|_{L_2((0, x_{m-1}))}^2 \leq \|f_{i,m}\|_{L_2}^2 \leq c_4 \|Pu_i\|_{L_2(I_m)}^2, \quad m \leq i.$$

Hence, with $p_{i,m} = \|Pu_i\|_{L_2(I_m)}^2$, we have

$$(4.5) \quad \sum_{\alpha < m} p_{i,\alpha} \leq c_4 p_{i,m}, \quad m \leq i.$$

From this it follows that

$$(4.6) \quad p_{i,m} \geq c_4^{-1} (1 + c_4^{-1})^{m-q-1} p_{i,q}, \quad 0 \leq q < m \leq i,$$

which we proceed to show by induction. Assume that (4.6) holds for q, m such that $0 \leq q < m \leq L$. (Note that for $L = 1$, i.e., $m = 1$ and $q = 0$, (4.6) is immediate from (4.5).) For any $q < L + 1$, we then obtain by (4.5) and the induction hypothesis

$$\begin{aligned} p_{i,L+1} &\geq c_4^{-1} \sum_{\alpha \leq L} p_{i,\alpha} \geq c_4^{-1} \left(\sum_{q < \alpha \leq L} p_{i,\alpha} + p_{i,q} \right) \\ &\geq c_4^{-1} p_{i,q} \left(\sum_{q < \alpha \leq L} c_4^{-1} (1 + c_4^{-1})^{\alpha-q-1} + 1 \right) \\ &= c_4^{-1} p_{i,q} (1 + c_4^{-1})^{(L+1)-q-1}. \end{aligned}$$

This establishes (4.6).

A similar result holds for intervals to the right of I_i , and, taking $m = i$ in (4.6), we find that there exist positive constants c and C , depending only on c_0 , such that

$$(4.7) \quad \|Pu_i\|_{L_2(I_q)} \leq C e^{-c|i-q|} \|Pu_i\|_{L_2(I_i)}.$$

Since Pu is a polynomial of fixed degree on I_q , we have

$$\|Pu\|_{L_\infty(I_q)} \leq c h_q^{-1/2} \|Pu\|_{L_2(I_q)}.$$

Using this, (1.2), (4.7), and the fact that $\|Pu\|_{L_2} \leq \|u\|_{L_2}$, we have, for any q ,

$$\begin{aligned} \|Pu\|_{L_\infty(I_q)} &\leq c h_q^{-1/2} \|Pu\|_{L_2(I_q)} \leq c h_q^{-1/2} \sum_i \|Pu_i\|_{L_2(I_q)} \\ &\leq C h_q^{-1/2} \sum_i e^{-c|i-q|} \|Pu_i\|_{L_2(I_i)} \leq C h_q^{-1/2} \sum_i e^{-|i-q|} \|u_i\|_{L_2} \\ &\leq C h_q^{-1/2} \sum_i e^{-c|i-q|} h_i^{1/2} \|u_i\|_{L_\infty} \leq c \|u\|_{L_\infty}. \end{aligned}$$

This proves (4.4) and establishes the lemma.

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