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Cubature Formulas of Degree Nine for Symmetric Planar Regions

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Abstract. A method of constructing 19-point cubature formulas with degree of exactness 9 is given for two-dimensional regions and weight functions which are symmetric in each variable. For some regions, e.g., the square and the circle, these formulas can be reduced to 18-point formulas.

1. Introduction. We consider cubature formulas,

$$\iint_R w(x, y) f(x, y)\, dx\, dy \cong \sum_{k=1}^{N} w_k f(x_k, y_k), \tag{1}$$

which are exact for all polynomials in x and y of degree $\leqslant d$ but not for all polynomials of degree $d + 1$. Such formulas are said to have degree d.

According to Rabinowitz and Richter [1], we say that formula (1) is a 'good' formula if it has all of its points (x_k, y_k) inside the region R and all of its coefficients w_k positive. We assume that R is a symmetric region (i.e., $(x, y) \in R$ implies $(\pm x, \pm y) \in R$) and that $w(x, y)$ is symmetric in x and y and nonnegative. In several recent publications [1]–[5], cubature formulas are computed which have the minimum number of points for their degree. Minimum-point formulas are closely connected with the theory of orthogonal polynomials [6]. This theory, however, is not yet sufficiently developed to give practical results in the case of high degree of exactness ($d > 7$). For $9 \leqslant d \leqslant 15$, Rabinowitz and Richter [1] have computed perfectly symmetric formulas, by solving a system of nonlinear equations. For $d = 9$, the number of points in their formulas is $N = 20$.

Until now, no formulas of degree 9 with less than 20 points were known for the square; for the circle, only one 19-point formula is computed by Albrecht [7]. In this note, we describe a numerical method for the construction of 18- and 19-point formulas, if they exist.

It is very likely, but not proved, that for the square and the circular domain with weight function $w(x) \equiv 1$ the 18-point formulas constructed in this way are minimum-point formulas.

2. Method of Construction. Since the region and the weight function are symmetric, it is reasonable to consider only symmetric formulas. Firstly, we construct 19-point formulas. As we shall show further, these formulas must have at least 12 points (x_k, y_k) with $x_k \neq 0$ and $y_k \neq 0$. We consider then the formula,

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$$\iint_R w(x, y) f(x, y)\, dx\, dy$$

$$\cong \sum_{k=1}^{3} w_k [f(x_k, y_k) + f(-x_k, y_k) + f(x_k, -y_k) + f(-x_k, -y_k)] \tag{2}$$

$$+ w_4 [f(x_4, 0) + f(-x_4, 0)] + \sum_{k=5}^{6} w_k [f(0, y_k) + f(0, -y_k)] + w_7 f(0, 0),$$

where $x_k > 0$ and $y_k > 0$.

For the computation of the coefficients w_k and the points (x_k, y_k) of (2), we have the following set of equations

$$4 \sum_{k=1}^{3} w_k x_k^{2\alpha} y_k^{2\beta} = m_{2\alpha,2\beta}, \qquad 0 < \alpha + \beta \leqslant 4, \tag{3.a}$$

$$4 \sum_{k=1}^{3} w_k x_k^{2\alpha} + 2 w_4 x_4^{2\alpha} = m_{2\alpha,0}, \qquad 0 < \alpha \leqslant 4, \tag{3.b}$$

$$4 \sum_{k=1}^{3} w_k y_k^{2\beta} + 2 \sum_{k=5}^{6} w_k y_k^{2\beta} = m_{0,2\beta}, \qquad 0 < \beta \leqslant 4, \tag{3.c}$$

$$4 \sum_{k=1}^{3} w_k + 2 \sum_{k=4}^{6} w_k + w_7 = m_{0,0}. \tag{3.d}$$

Here α and β are natural numbers and

$$m_{\alpha,\beta} = \iint_R w(x, y) x^\alpha y^\beta \, dx\, dy.$$

Firstly, we choose x_2, x_3 and y_3 arbitrarily. The system of six equations (3.a) can then be solved explicitly

$$\begin{aligned}
w_3 = {} & \frac{(m_{2,4}^2 m_{6,2} + m_{4,4}^2 m_{2,2} - 2m_{4,4}m_{2,4}m_{4,2} - m_{6,2}m_{2,2}m_{2,6} + m_{4,2}^2 m_{2,6})}{\{[(m_{2,4}^2 - m_{2,2}m_{2,6})x_3^4 + (m_{4,2}^2 - m_{6,2}m_{2,2})y_3^4} \\
& \quad + 2(m_{4,2}m_{2,6} - m_{4,4}m_{2,4})x_3^2 + 2(m_{2,4}m_{6,2} - m_{4,4}m_{4,2})y_3^2 \\
& \qquad + 2(m_{4,4}m_{2,2} - m_{2,4}m_{4,2})x_3^2 y_3^2 + m_{4,4}^2 - m_{6,2}m_{2,6}]\, 4x_3^2 y_3^2\}, \\
& x_1^2 = (m_{4,2}^* x_2^2 - m_{6,2}^*)/(m_{2,2}^* x_2^2 - m_{4,2}^*), \\
& y_1^2 = (m_{2,4}^* x_2^2 - m_{4,4}^*)/(m_{2,2}^* x_2^2 - m_{4,2}^*), \\
y_2^2 = {} & [(m_{4,4}^* m_{2,2}^* - m_{2,4}^* m_{4,2}^*)x_2^2 + m_{2,4}^* m_{6,2}^* - m_{4,4}^* m_{4,2}^*]/(m_{6,2}^* m_{2,2}^* - m_{4,2}^{*\,2}), \\
w_1 = {} & (m_{2,2}^* x_2^2 - m_{4,2}^*)^2 / [4x_1^2 y_1^2 (m_{2,2}^* x_2^4 - 2m_{4,2}^* x_2 + m_{6,2}^*)], \\
w_2 = {} & (m_{6,2}^* m_{2,2}^* - m_{4,2}^{*\,2}) / [4x_1^2 y_1^2 (m_{2,2}^* x_2^4 - 2m_{4,2}^* x_2 + m_{6,2}^*)],
\end{aligned} \tag{4}$$

where $m^*_{\alpha,\beta} = m_{\alpha,\beta} - 4w_3^2 x_3^\alpha y_3^\alpha$. Formula (4) shows that the coefficient w_3 corresponding to the arbitrarily chosen point (x_3, y_3) is independent of the position of the other points and also different from zero (except perhaps for exceptional regions or weight functions). This means that a symmetric formula of degree 9 requires at least three points in each quadrant of the region.

We consider now the set of four equations (3.b) into which we substitute the free parameters x_2, x_3 and y_3 and the computed values w_1, x_1, y_1, w_2 and y_3. The first two of these equations can be used for the computation of w_4 and x_4. The remaining two equations of this set are then considered as a system of two simultaneous equations in the unknowns x_2, x_3 and y_3. This will generally leave one free parameter, say y_3. The system of nonlinear equations in the unknowns x_2 and x_3 must be solved numerically.

The parameters y_5, y_6, w_5 and w_6 are then computed by solving the system (3.c). The numbers y_5^2 and y_6^2 are the roots of the quadratic equation,

$$({m^*_{0,4}}^2 - m^*_{0,6}m^*_{0,2})z^2 + (m^*_{0,2}m^*_{0,8} - m^*_{0,4}m^*_{0,6})z + ({m^*_{0,6}}^2 - m^*_{0,4}m^*_{0,8}) = 0,$$

while

$$w_5 = (m^*_{0,2}y_6^2 - m^*_{0,4})/[2y_5^2(y_6^2 - y_5^2)]$$

and

$$w_6 = (m^*_{0,4} - m^*_{0,2}y_5^2)/[2y_6^2(y_6^2 - y_5^2)]$$

where

$$m^*_{0,\beta} = m_{0,\beta} - 4\sum_{k=1}^{3} w_k y_k^\beta.$$

Finally, w_7 is computed from (3.d).

From this method of solution, we conclude that there are generally more solutions, since y_3 is still a free parameter. However, it is not impossible that several or even all solutions are complex-valued. In this last case, a 19-point formula of the form (2) does not exist. However, there may still exist a formula of the form,

$$\iint_R w(x, y) f(x, y)\, dx\, dy$$

$$\cong \sum_{k=1}^{4} w_k [f(x_k, y_k) + f(-x_k, y_k) + f(x_k - y_k) + f(-x_k, -y_k)] \tag{5}$$

$$+ w_5 [f(x_5, 0) + f(-x_5, 0)] + w_6 f(0, 0).$$

In order to obtain an 18-point formula of degree 9, we consider w_7 as a function of y_3 and we solve the equation $w_7(y_3) = 0$ (if there exists a real-valued solution). We conjecture that the 18-point formulas computed in this way are minimum-point formulas.

For several regions and weight functions, we have carried out numerical experiments. We summarize the most important results in the following section.

3. Some Results. (i) For the square $R = C_2 = \{(x, y): -1 \leqslant x, y \leqslant 1\}$ and $w(x, y) \equiv 1$ infinitely many 'good' 19-point formulas of degree 9 exist. There are also at least two 'good' 18-point formulas, the parameters of which we give in Tables 1 and 2 to 20 significant digits. The points of both formulas are common zeros of three orthogonal polynomials of degree 5. These orthogonal polynomials are $P_0 + \lambda_1 P_2$, $P_0 + \lambda_2 P_4$ and $P_1 + \mu_1 P_3 + \mu_2 P_5$ where, for the first formula,

$$\lambda_1 = 0.24819696, \qquad \lambda_2 = 0.25574007,$$
$$\mu_1 = 0.00469095, \qquad \mu_2 = -0.96079906,$$

and for the second formula,

$$\lambda_1 = -0.22003380, \qquad \lambda_2 = -0.30398642,$$
$$\mu_1 = -0.33809909, \qquad \mu_2 = -1.33579356,$$

and where

$$P_0(x, y) = x^5 - 10x^3/9 + 5x/21,$$
$$P_1(x, y) = x^4 y - 6x^2 y/7 + 3y/35,$$
$$P_2(x, y) = x^3 y^2 - x^3/3 - 3xy^2/5 + x/5,$$
$$P_3(x, y) = P_2(y, x),$$
$$P_4(x, y) = P_1(y, x),$$
$$P_5(x, y) = P_0(y, x),$$

are the basic orthogonal polynomials for C_2.

(ii) For the circle $R = S_2 = \{(x, y): x^2 + y^2 \leqslant 1\}$ with $w(x, y) \equiv 1$, infinitely many 'good' 19-point formulas of degree 9 exist. We have also computed one 18-point formula which has, however, four of its points outside S_2 (see Table 3). The points of this formula are the common zeros of the orthogonal polynomials $P_0 + \lambda_1 P_2$, $P_0 + \lambda_2 P_4$ and $P_1 + \mu_1 P_3 + \mu_2 P_5$ with

$$\lambda_1 = 0.56685388, \qquad \lambda_2 = 0.16924976,$$
$$\mu_1 = 0.56433371, \qquad \mu_2 = -0.65980030,$$

and where

$$P_0(x, y) = x^5 - x^3 + 3x/16,$$
$$P_1(x, y) = x^4 y - 3x^2 y/5 + 3y/80,$$
$$P_2(x, y) = x^3 y^2 - x^3/10 - 3xy^2/10 + 3x/80,$$
$$P_3(x, y) = P_2(y, x),$$
$$P_4(x, y) = P_1(y, x),$$
$$P_5(x, y) = P_0(y, x).$$

(iii) For the entire plane $R = E_2^{r^2} = \{(x, y): -\infty \leqslant x, y \leqslant \infty\}$ with weight function $w(x, y) = \exp(-x^2 - y^2)$, infinitely many 'good' 19-point formulas exist. However, we have not found any 18-point formula.

(iv) For the entire plane $R = E_2^r = \{(x, y): -\infty \leqslant x, y \leqslant \infty\}$ with weight function $w(x, y) = \exp(-(x^2 + y^2)^{1/2})$, we have not found any real solution of the system of equations (3.a, b, c, d).

A number of the 19-point formulas for C_2, S_2 and $E_2^{r^2}$ are tabulated in [8].

TABLE 1. First 18-point formula for the square

k	x_k	y_k	w_k
1	0.87980721399752853896	0.92797961509268528861	0.68416522462309305679 (-1)
2	0.50445910315479838456	0.75347199103161505380	0.27903384209687301395
3	0.91531235408227324183	0.42299357094876513066	0.16806533822999587126
4	0.57882826011929170546	0	0.40927359555433144329
5	0	0.97700090158004246059	0.10648011781560231854
6	0	0.39364057271848893512	0.45321488105170985638

TABLE 2. Second 18-point formula for the square

k	x_k	y_k	w_k
1	0.93742666622066710914	0.94145119299928430974	0.42853317248897088536 (-1)
2	0.57077001686857404415	0.79214654516847247531	0.25788406360659644304
3	0.89774224179848572970	0.40001733897633692860	0.19397744037003970872
4	0.49471787965159623409	0	0.45212398131214854997
5	0	0.98085697194664054422	0.10243215270991495821
6	0	0.48311469619727965642	0.45601422352687001122

TABLE 3. 18-point formula for the circle

k	x_k	y_k	w_k
1	0.86686876801492291622	0.28376671812094800827	0.12937261598422958670
2	0.63925306939199114680	0.95409639862933054563	0.77785540900483355115 (-2)
3	0.48645191470776426796	0.63982457013387676359	0.22713305094453060651
4	0.51286789206607718656	0	0.32090673961381781518
5	0	0.88859953503035797854	0.16042870730308439624
6	0	0.35335517369353007690	0.36089243784037735036

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