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Models of Difference Schemes for $u_t + u_x = 0$ by Partial Differential Equations*

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Abstract. It is well known that difference schemes for hyperbolic equations display dispersion of waves. For a general dissipative difference scheme, we present a dispersive wave equation and show that the dispersions are essentially the same when the initial data is a step function.

I. Introduction. For the equation $u_t + u_x = 0$, consider the difference scheme,

$$v(x, t + \Delta t) = \sum c_j v(x + jh, t). \tag{1.1}$$

We assume, of course, that the difference scheme is consistent. The scheme (1.1) is explicit if the sum is over only a finite number of terms and implicit otherwise. The symbol or amplification function for (1.1) is

$$G(\xi) = \sum c_j e^{ij\xi}. \tag{1.2}$$

If $\hat{v}(\xi, t)$ denotes the Fourier transform of $v(x, t)$, then it is known ([1, p. 67] or [2]) that

$$\hat{v}(\xi, n\Delta t) = G^n(h\xi)\hat{v}(\xi, 0). \tag{1.3}$$

If the difference scheme (1.1) is dissipative, that is, if

$$|G(\xi)| \leqslant \exp\{-\gamma_0 \xi^s\} \qquad (|\xi| \leqslant \pi)$$

for some positive γ_0 and some even integer s, then either

$$G(\xi) = \exp\{-i\rho\xi - \gamma\xi^s + O(|\xi|^{s+1})\} \qquad (|\xi| \longrightarrow 0) \tag{1.4}$$

or

$$G(\xi) = \exp\left\{-i\rho\xi + i\sum_r^{s-1} \beta_j \xi^j - \gamma\xi^s + O(|\xi|^{s+1})\right\} \qquad (|\xi| \longrightarrow 0), \tag{1.5}$$

where $\rho = \Delta t/h$, $\operatorname{Re} \gamma > 0$, and the β_j are real.

If we combine (1.5) and (1.3) and set $t = n\Delta t$, we find that

$$\hat{v}(\xi, t) = \exp\left\{t\left(-i\xi + (i/\rho)\sum_r^{s-1} \beta_j h^{j-1}\xi^j - (\gamma/\rho)h^{s-1}\xi^s + O(h^s|\xi|^{s+1})\right)\right\} \hat{v}(\xi, 0). \tag{1.6}$$

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It is easy to write down a partial differential equation for which the Fourier transform of the solution has a representation like (1.6), namely,

$$u_t + u_x = \sum_{r}^{s-1} (\beta_j/\rho)(h/i)^{j-1} \partial^j u/\partial x^j - (\gamma/\rho)(-1)^{s/2} h^{s-1} \partial^s u/\partial x^s, \tag{1.7}$$

where $i^2 = -1$. In the case when (1.4) holds, we use

$$u_t + u_x = -(\gamma/\rho)(-1)^{s/2} h^{s-1} \partial^s u/\partial x^s. \tag{1.8}$$

For the Lax-Wendroff difference scheme [1, p. 302],

$$v(x, t + \Delta t) = \rho(\rho - 1)v(x + h, t)/2 + (1 - \rho^2)v(x, t) + \rho(\rho + 1)v(x - h, t)/2,$$

the symbol is

$$G(\xi) = 1 - i\rho \sin \xi - \rho^2(1 - \cos \xi),$$

so that for $\rho < 1$, Eq. (1.7) becomes

$$u_t + u_x = -(h^2/6)(1 - \rho^2)u_{xxx} - (h^3/8)\rho(1 - \rho^2)u_{xxxx}.$$

Various authors have based stability analysis of (1.1) on the question of well-posedness of the Cauchy problem for (1.7), (1.8), or some similar equation; see, for example, the papers of Hirt [3], McGuire and Morris [4], and Janenko and Šokin [5]. Because the response of difference schemes to step-function initial data is so important, Chin [6] and Lerat and Peyret [7], [8] have used (1.7) or (1.8) to approximate special difference schemes with step-function data. Though at first glance it may appear that (1.7) or (1.8) approximates (1.1) well only for smooth data, numerical experiments and Chin's proof [6] of a special case indicate that there is good agreement even in the case of step discontinuities.

Our main result shows that for general dissipative schemes, there is good agreement between (1.7) or (1.8) and (1.1) for step-function initial data.

THEOREM. *Let the difference scheme* (1.1) *be dissipative, and let the symbol satisfy* (1.5) *with* $1 < r < s$, *r odd, s even,* β_j $(r \leqslant j < s)$ *real,* $\beta_r > 0$, *and* $\operatorname{Re} \gamma > 0$. *Thus, the order of accuracy is* $r - 1$ *and the order of dissipation is s. Let u be the solution of* (1.7) *with initial data,*

$$\begin{aligned} u(x, 0) &= v(x, 0) = -\tfrac{1}{2} \qquad (x > 0), \\ u(0, 0) &= v(0, 0) = 0, \\ u(x, 0) &= v(x, 0) = \tfrac{1}{2} \qquad (x < 0). \end{aligned} \tag{1.9}$$

Then there exist positive constants C_1, C_2, ω_0, κ_1, *and* κ_2 *such that at grid points,* x, $t = n\Delta t$, *we have the error bounds,*

$$|u(x, t) - v(x, t)| \leqslant C_1 h^{-2/r} \qquad (|x - t|\rho \leqslant tn^{-(r-1)/r}), \tag{1.10}$$

$$\begin{aligned} |u(x, t) - v(x, t)| &\leqslant C_2 g(n, \rho(-1 + x/t)) \exp\{-n\kappa_1(\rho(-1 + x/t))^{r/(r-1)}\}, \\ &\qquad (tn^{-(r-1)/r} \leqslant (x - t) \leqslant \omega_0 t), \end{aligned} \tag{1.11}$$

$$\begin{aligned} |u(x, t) - v(x, t)| &\leqslant C_2 g(n, \rho(1 - x/t)) \exp\{-n\kappa_2(\rho(1 - x/t))^{s/(r-1)}\} \\ &\qquad (-\omega_0 t \leqslant \rho(x - t) \leqslant -tn^{-(r-1)/r}), \end{aligned} \tag{1.12}$$

where

$$g(n, \omega) = n^{-1/2}\omega^{(4-r)/(2r-2)} \qquad (0 < \omega \leqslant n^{-(r-1)/(s-1)}),$$

$$g(n, \omega) = n^{1/2}\omega^{(2s+2-r)/(2r-2)} \qquad (\omega > n^{-(r-1)/(s-1)}).$$

In the case when G satisfies (1.4), *so that the order of accuracy is* $s - 1$, *then* (1.10)–(1.12) *are valid if u is the solution of* (1.8) *and if each r is replaced by s. In the unlikely situation when* (1.5) *holds with even r, estimates* (1.10) *and* (1.12) *are still valid, but there are oscillations ahead of the front as well as behind it, and* (1.11) *is replaced by*

$$|u(x, t) - v(x, t)| \leqslant C_2 g(n, \rho(-1 + x/t))\exp\{-n\kappa_2(\rho(-1 + x/t))^{s/(r-1)}\}$$

$$(tn^{-(r-1)/r} \leqslant \rho(x - t) \leqslant \omega_0 t).$$

Finally, if (1.5) *holds with odd r and* $\beta_r < 0$, *there are oscillations ahead of the front only, and the exponential factors in the right-hand sides of* (1.11) *and* (1.12) *are exchanged.*

The proof is given in Section 3 and is based on some saddle-point estimates given in Section 2.

For the sake of comparison, we present the behavior of the solution of (1.1) with step-function initial data (1.9) under the condition that (1.5) holds with r odd, $\beta_r > 0$. It follows from the work of Brenner and Thomée [9], Hedstrom [10, Theorem 5.3], and Serdjukova [11] that

$$|v(x, t)| \leqslant C_3 \qquad (\rho|x - t| \leqslant tn^{-(r-1)/r}),$$

$$|v(x, t) + 1/2| \leqslant C_4 n^{-1/2}(\rho(-1 + x/t))^{-r/(2r-2)}\exp\{-n\kappa_1(\rho(-1 + x/t))^{r/(r-1)}\}$$

$$(1.13) \qquad (tn^{-(r-1)/r} \leqslant \rho(x - t) \leqslant \omega_0 t),$$

$$|v(x, t) - 1/2| \leqslant C_4 n^{-1/2}(\rho(1 - x/t))^{-r/(2r-2)}\exp\{-n\kappa_2(\rho(1 - x/t))^{s/(r-1)}\}$$

$$(-\omega_0 t \leqslant \rho(x - t) \leqslant -tn^{-(r-1)/r}).$$

In fact, if $\beta_{s+1} \neq 0$, these bounds and the bounds in the theorem are sharp, and the exponential factors are identical; only an oscillatory factor $\cos(\alpha + n\psi(\rho|1 - x/t|))$ is left out. Hence, we see that for $|\rho(x - t)| \leqslant tn^{-(r-1)/(s+1)}$, u mimics v better than v mimics the solution of $u_t + u_x = 0$, but the reverse is true otherwise. It does not matter, though, that u does not mimic v so well for $|\rho(x - t)| > tn^{-(r-1)/(s+1)}$, because the exponential factors are then very small.

We remark that there are equations other than (1.7) that one could consider. It is clear that the first dispersive term, $(\beta_r/\rho)(h/i)^{r-1}\partial^r u/\partial x^r$, should be kept; and it follows from the role of s in (1.13) that we want the first dissipative term, $-(-1)^{s/2}(\gamma/\rho)h^{s-1}\partial^s u/\partial x^s$. Thus, we should consider the equation,

$$u_t + u_x = (\beta_r/\rho)(h/i)^{r-1}\partial^r u/\partial x^r - (-1)^{s/2}(\gamma/\rho)h^{s-1}\partial^s u/\partial x^s.$$

Our methods may be easily applied to this equation. They show that if there is a dispersive coefficient $\beta_j \neq 0$ with $r < j < s$ and if q is the smallest such index j, then (1.10)–(1.12) still hold but with $g(n, \omega)$ replaced by $g_1(n, \omega)$, where

$$g_1(n, \omega) = n^{-1/2}\omega^{(4-r)/(2r-2)} \qquad (0 < \omega \leqslant n^{-(r-1)/(q-2)}),$$
$$g_1(n, \omega) = n^{1/2}\omega^{(2q-r)/(2r-2)} \qquad (\omega > n^{-(r-1)/(q-2)}).$$

Hence, in this case u approximates v better than v approximates the solution of $u_t + u_x = 0$ only for $\rho|x - t| < tn^{-(r-1)/q}$, and since the oscillatory region behind the wave [see (1.13)] extends back to $\rho(x - t) = -tn^{-(r-1)/s}$, there is a significant region in which the approximation is not very good. It is for this reason that we use Eq. (1.7). Such considerations, letting $s \longrightarrow \infty$, indicate that it is not a good idea to try to use a partial differential equation to mimic the oscillations of v if $|G(\xi)| \equiv 1$.

This paper was written to answer a question raised by C. K. Chu during the discussion of a paper of Lerat and Peyret at the Fourth International Conference on Numerical Methods and Fluid Dynamics at Boulder, Colorado, in June 1974.

II. Estimates of an Integral. We take the most interesting case, namely, we assume that the symbol satisfies (1.5) with r odd and $\beta_r > 0$. The other cases may be analyzed in the same way. We begin by making saddle-point estimates of an integral.

LEMMA. *Let ϕ and f be analytic functions in the disc $|\xi| < \pi$. Let ϕ have Maclaurin expansion,*

$$\phi(\xi) = i\sum_{r}^{s-1} \beta_j\xi^j - \gamma\xi^s + \dots, \tag{2.1}$$

where $1 < r < s$, r is odd, s is even, $\beta_r > 0$, each β_j $(r \leqslant j < s)$ is real, and $\operatorname{Re}\gamma > 0$. *Let $|f(\xi)| \leqslant M$ for $|\xi| \leqslant \pi$. For integers $n, p \geqslant 0$ and for real parameter ω define the integral,*

$$A_{n,p}(\omega) \doteq \int_{-\pi}^{\pi} \xi^p f(\xi)\exp\{n(\phi(\xi) - i\omega\xi)\}\, d\xi.$$

Then there exist positive constants $C_5(p)$, $C_6(p)$, ω_0, κ_1, and κ_2 such that

$$|A_{n,p}(\omega)| \leqslant MC_5(p)n^{-(p+1)/r} \qquad (|\omega| \leqslant n^{-(r-1)/r}), \tag{2.2}$$

$$|A_{n,p}(\omega)| \leqslant MC_6(p)n^{-1/2}|\omega|^{(2p+2-r)/(2r-2)}\exp\{-n\kappa_1|\omega|^{r/(r-1)}\} \qquad (-\omega_0 \leqslant \omega \leqslant -n^{-(r-1)/r}), \tag{2.3}$$

$$|A_{n,p}(\omega)| \leqslant MC_6(p)n^{-1/2}|\omega|^{(2p+2-r)/(2r-2)}\exp\{-n\kappa_2\omega^{s/(r-1)}\} \qquad (n^{-(r-1)/r} \leqslant \omega \leqslant \omega_0). \tag{2.4}$$

Proof. We change the path of integration in the complex ξ-plane to make it pass through two saddle points of the function $\phi(\xi) - i\omega\xi$, that is, through solutions of the equation, $\phi'(\xi) - i\omega = 0$. As $|\omega| \longrightarrow 0$, there are $r - 1$ saddle points of $\phi(\xi) - i\omega\xi$ given by

$$\xi_0 = (\omega/(r\beta_r))^{1/(r-1)} + O(|\omega|^{2/(r-1)}), \tag{2.5}$$

one for each branch of the $(r - 1)$st root.

Consider any fixed branch of the $(r-1)$st root. The function $\phi(\xi) - i\omega\xi$ has Taylor series about ξ_0 of the form,

$$\phi(\xi) - i\omega\xi = b_0(\omega) + \sum_2^\infty b_j(\omega)(\xi - \xi_0)^j. \tag{2.6}$$

It follows from (2.1) and (2.5) that as $|\omega| \to 0$, we have

$$b_2(\omega) = ir(r-1)(\beta_r/2)(\omega/(r\beta_r))^{(r-2)/(r-1)} + O(|\omega|), \tag{2.7}$$

$$|b_j(\omega)| = O(|\omega|^{(r-j)/(r-1)}) \qquad (j = 3, 4, \ldots, r-1), \tag{2.8}$$

$$b_r(\omega) = i\beta_r + O(|\omega|^{1/(r-1)}). \tag{2.9}$$

If $|\omega|$ is small enough so that $|\xi_0| < \pi - 1$, the radius of convergence of the series (2.6) is at least one. Hence, there is a constant C_7 such that

$$|b_j(\omega)| \leq C_7 \qquad (j = r+1, r+2, \ldots). \tag{2.10}$$

Besides the constant term $b_0(\omega)$, the most important terms in (2.6) are $b_2(\omega)(\xi - \xi_0)^2$ and $b_r(\omega)(\xi - \xi_0)^r$. We see from (2.7) and (2.9) that we can choose a contour Γ_0 through ξ_0 on which

$$\mathrm{Re}(b_2(\omega)(\xi - \xi_0)^2 + b_r(\omega)(\xi - \xi_0)^r) \leq -\kappa_3 |\omega|^{(r-2)/(r-1)} |\xi - \xi_0|^2 - \kappa_4 |\xi - \xi_0|^r$$

for $|\omega| < \omega_0$ and for some positive constants ω_0, κ_3, and κ_4. It follows from (2.8) and (2.10) that on Γ_1, the part of Γ_0 on which $|\xi - \xi_0| \leq C_8 |\xi_0|$, we have

$$\mathrm{Re}(\phi(\xi) - i\omega\xi) \leq \mathrm{Re}\, b_0(\omega) - \kappa_3 |\omega|^{(r-2)/(r-1)} |\xi - \xi_0|^2 - \kappa_4 |\xi - \xi_0|^r \tag{2.11}$$

for $|\omega| \leq \omega_0$ and for smaller positive constants ω_0, κ_3, and κ_4. We see from (2.11) that

$$\begin{aligned} &\left| \int_{\Gamma_1} \xi^p f(\xi) \exp\{n(\phi(\xi) - i\omega\xi)\}\, d\xi \right| \\ &\quad \leq 2^p M \int_{\Gamma_1} (|\xi_0|^p + |\xi - \xi_0|^p) \exp\{n(\mathrm{Re}\, b_0(\omega) - \kappa_4 |\xi - \xi_0|^r)\}\, d\xi \\ &\quad \leq M C_9 n^{-1/r} (|\omega|^{p/(r-1)} + n^{-p/r}) \exp\{n\, \mathrm{Re}\, b_0(\omega)\} \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} &\left| \int_{\Gamma_1} \xi^p f(\xi) \exp\{n(\phi(\xi) - i\omega\xi)\}\, d\xi \right| \\ &\quad \leq 2^p M \int_{\Gamma_1} (|\xi_0|^p + |\xi - \xi_0|^p) \\ &\qquad \cdot \exp\{n(\mathrm{Re}\, b_0(\omega) - \kappa_3 |\omega|^{(r-2)/(r-1)} |\xi - \xi_0|^2)\}\, |d\xi| \\ &\quad \leq M C_{10} n^{-1/2} |\omega|^{-(r-2)/(2r-2)} (|\omega|^{p/(r-1)} + n^{-p/2} |\omega|^{-p(r-2)/(2r-2)}) \\ &\qquad \cdot \exp\{n\, \mathrm{Re}\, b_0(\omega)\}. \end{aligned} \tag{2.13}$$

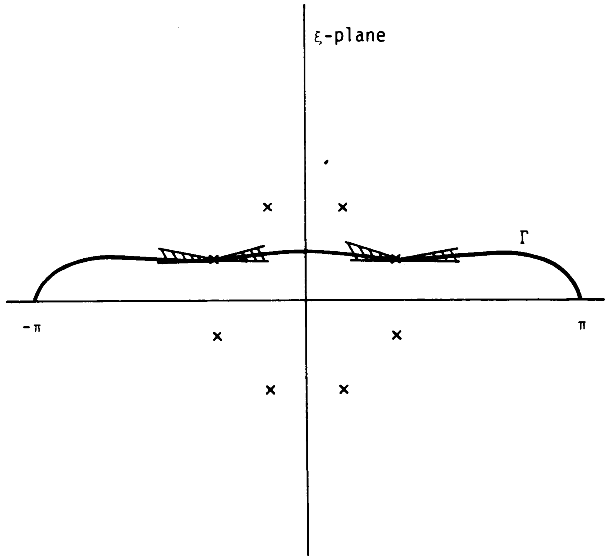

FIGURE 1

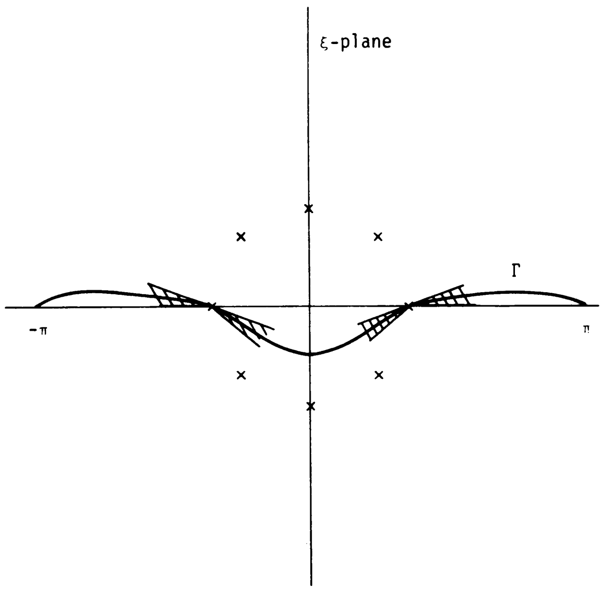

FIGURE 2

It is clear that we should use (2.12) if $|\omega| \leqslant n^{-(r-1)/r}$, and (2.13) if $n^{-(r-1)/r} < |\omega| < \omega_0$.

We want to use a contour joining $-\pi$ to π while going over the lowest possible saddle points, those with Re $b_0(\omega)$ as small as possible. If $\omega < 0$, we choose the path of integration Γ as in Fig. 1, passing through the saddle points that are closest to the real axis in the upper half-plane, and these are the highest points on the path. The case $r = 9$ and $\omega < 0$ is illustrated in Fig. 1, and the saddle points are denoted by crosses. Near the saddle points we choose Γ so that (2.11) holds. At these two saddle points we conclude from (2.5) that

(2.14) $$\operatorname{Re} b_0(\omega) = \operatorname{Re}(\phi(\xi_0) - i\omega\xi_0) \leqslant -\kappa_1 |\omega|^{r/(r-1)}.$$

The estimate (2.3) now follows from (2.13) and (2.14), and inequality (2.2) for $\omega \leqslant 0$ follows from (2.12) and (2.14).

If $\omega > 0$, the best path Γ to take from $-\pi$ to π has as its summits the two saddle points (2.5) nearest to the real axis. We choose Γ as in Fig. 2 so that (2.11) holds near the saddle points. It follows from (2.1) and (2.5) that for these two saddle points,

$$\operatorname{Im} \xi_0 = -(\operatorname{Re} \gamma s/(r-1))(r\beta_r)^{-(s-1)/(r-1)}\omega^{(s-r)/(r-1)}(1 + O(\omega^{1/(r-1)}))$$

as $|\omega| \longrightarrow 0$. Consequently, we see that for $0 \leqslant \omega \leqslant \omega_0$, there exists a positive number κ_2 such that

(2.15) $$\operatorname{Re} b_0(\omega) = \operatorname{Re}(\phi(\xi_0) - i\omega\xi_0) \leqslant -\kappa_2 \omega^{s/(r-1)}.$$

We now obtain (2.4) from (2.13) and (2.15), and we obtain (2.2) for $\omega > 0$ from (2.12) and (2.15). This proves the Lemma.

III. Proof of the Theorem. We again concentrate on the case when (1.5) holds with r odd and $\beta_r > 0$; the other cases are proved similarly. The Fourier transform of the step-function initial data (1.9) is given by $\hat{v}(\xi, 0) = i/\xi$, so that (1.3) becomes

(3.1) $$\hat{v}(\xi, n\Delta t) = iG^n(h\xi)/\xi,$$

while the solution of (1.7) with the same initial data has Fourier transform,

(3.2) $$\hat{u}(\xi, t) = (i/\xi)\exp\{t(-i\xi + \phi(h\xi)/(\rho h))\},$$

where

$$\phi(\xi) = i\sum_{r}^{s-1} \beta_j \xi^j - \gamma\xi^s.$$

The difference $w = u - v$ at time $t = n\Delta t$ is given by the inverse Fourier transform,

(3.3) $$w(x, t) = 1/(2\pi)\int_{-\infty}^{\infty} (\hat{u}(\xi, t) - \hat{v}(\xi, t))e^{ix\xi}\, d\xi.$$

We split the integral (3.3) into three parts: the central part,

$$I_1 = 1/(2\pi)\int_{-\pi/h}^{\pi/h} (\hat{u}(\xi, t) - \hat{v}(\xi, t))e^{ix\xi}\, d\xi,$$

the tail for the differential equation,

$$I_2 = 1/(2\pi)\int_{|\xi|\geqslant\pi/h} \hat{u}(\xi, t)e^{ix\xi}\, d\xi,$$

and the tail for the difference scheme,

$$I_3 = -1/(2\pi)\int_{|\xi|\geqslant\pi/h} \hat{v}(\xi, t)e^{ix\xi}\, d\xi.$$

We estimate I_1 first. It follows from (1.6) and (3.2) that

$$I_1 = t/(2\pi)\int_{-\pi/h}^{\pi/h} h^s\xi^s f(h\xi)\exp\{i(x-t)\xi + n\phi(h\xi)\}\, d\xi$$

for some function $f(\zeta)$, which is bounded and analytic on the disc $|\zeta| \leqslant \pi$, $|f(\zeta)| \leqslant M$. The substitution $\zeta = h\xi$ transforms the integral into

$$I_1 = t/(2\pi h)\int_{-\pi}^{\pi} \zeta^s f(\zeta)\exp\{n(i\rho(-1 + x/t)\zeta + \phi(\zeta))\}\, d\zeta.$$

An application of the Lemma with $\omega = \rho(1 - x/t)$ and $p = s$ shows that because

$$t/h = n\rho, \tag{3.4}$$

we have

$$|I_1| \leqslant MC_5(s)\rho/(2\pi)n^{-(s+1-r)/r} \qquad (\rho|x - t| \leqslant tn^{-(r-1)/r}), \tag{3.5}$$

$$\begin{gathered}|I_1| \leqslant MC_6(s)\rho/(2\pi)n^{1/2}|\rho(1 - x/t)|^{(2s+2-r)/(2r-2)}\exp\{-n\kappa_1|\rho(1 - x/t)|^{r/(r-1)}\}\\ (tn^{-(r-1)/r} \leqslant \rho(x - t) \leqslant \omega_0 t),\end{gathered} \tag{3.6}$$

$$\begin{gathered}|I_1| \leqslant MC_6(s)\rho/(2\pi)n^{1/2}(\rho(1 - x/t))^{(2s+2-r)/(2r-2)}\exp\{-n\kappa_2(\rho(1 - x/t))^{s/(r-1)}\}\\ (-\omega_0 t \leqslant \rho(x - t) \leqslant -tn^{-(r-1)/r}).\end{gathered} \tag{3.7}$$

We now estimate I_2. By the change of variable $\zeta = h\xi$, it follows from (3.2) that

$$I_2 = i/(2\pi)\int_{|\zeta|\geqslant\pi} \exp\{n(i\rho(-1 + x/t)\zeta + \phi(\zeta))\}\, d\zeta/\zeta.$$

Consequently, we have

$$|I_2| \leqslant 1/\pi\int_{\pi}^{\infty} \exp\{-n\gamma\zeta^s\}\, d\zeta/\zeta \leqslant C_{11}n^{-1/s}\exp\{-n\gamma\pi^s\}. \tag{3.8}$$

In the integral I_3 we also make the substitution $\zeta = h\xi$ and obtain the representation,

$$I_3 = -i/(2\pi)\int_{|\zeta|\geqslant\pi} G^n(\zeta)\, e^{ix\zeta/h} d\zeta/\zeta.$$

It is clear from (1.2) that G is periodic with period 2π, so that at a grid point x we may rewrite I_3 as

$$I_3 = -i/\pi\sum_{j=1}^{\infty}\int_{-\pi}^{\pi} G^n(\zeta)\zeta/(\zeta^2 - 4j^2\pi^2)e^{ix\zeta/h}\, d\zeta.$$

Here, we have used the fact that $\exp\{-2\pi ijx/h\} = 1$ at a grid point x. We now use the Lemma with $p = 1$ and with $M = 1/((4j^2 - 1)\pi^2)$ to show that

$$|I_3| \leqslant C_5(1)n^{-2/r} \qquad (\rho|x - t| \leqslant tn^{-(r-1)/r}), \tag{3.9}$$

$$\begin{gathered}|I_3| \leqslant C_6(1)(\rho(-1 + x/t))^{(4-r)/(2r-2)}\exp\{-n\kappa_1(\rho(-1 + x/t))^{r/(r-1)}\}n^{-1/2}\\ (tn^{-(r-1)/r} \leqslant \rho(x - t) \leqslant \omega_0 t),\end{gathered} \tag{3.10}$$

$$\begin{gathered}|I_3| \leqslant C_6(1)(\rho(1 - x/t))^{(4-r)/(2r-2)}\exp\{-n\kappa_2(\rho(1 - x/t))^{s/(r-1)}\}n^{-1/2}\\ (-\omega_0 t \leqslant \rho(x - t) \leqslant -tn^{-(r-1)/r}).\end{gathered} \tag{3.11}$$

The theorem now follows from estimates (3.5)–(3.11). Note that for $\rho|x - t| \dot{<} tn^{-(r-1)/(s-1)}$, the largest contribution comes from the tail I_3, while for $tn^{-(r-1)/(s-1)} \leqslant \rho|x - t| \leqslant \omega_0 t$, the largest contribution comes from the central term I_1.

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