REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

The numbers in brackets are assigned according to the revised indexing system printed in Volume 28, Number 128, October 1974, pages 1191–1194.

39 [2.05, 2.10, 6.15].—L. M. DELVES & J. WALSH, Editors, *Numerical Solution of Integral Equations,* Clarendon Press, Oxford, 1974, 335 pp., 24 cm. Price $14.50.

The book consists of a collection of papers, by various authors, presented at the University of Manchester Summer School in July 1973, dealing with methods and principles in the numerical solution of integral and integro-differential equations.

The material is divided into three parts. Part 1, consisting of the first five chapters, gives a brief overview of the mathematical tools necessary in the numerical analysis of integral equations. The topics discussed are: the theory of linear integral equations, numerical integration, linear algebra, functional analysis, and approximation theory.

Part 2, Chapters 6–18, deals with the actual numerical methods for solving various integral equations, including Fredholm equations of first and second kinds, Volterra equations of first and second kinds, various ordinary and partial integro-differential equations, and nonlinear equations and systems. This is a reasonably complete and up-to-date survey of known numerical techniques. Both theoretical and practical aspects are considered.

Part 3, Chapters 19–25, is a selection of various applications, such as potential and flow problems, water waves, and diffraction and scattering. These are examples of the usefulness of integral equation techniques in the solution of a variety of problems from engineering and physics.

The papers in this volume are expository and written in a rather informal style. The material is generally presented in outline form, without proofs or undue rigor, emphasizing principles rather than technical or algorithmic detail. For more specific information the reader will have to consult the cited references; most papers fortunately include a good bibliography. The apparent aim of the book is to provide the reader with a quick and painless introduction to the use of integral equations in practical applications. In this aim it has succeeded quite well.

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40 [7.00].—R. B. DINGLE, *Asymptotic Expansions: Their Derivation and Interpretation,* Academic Press, New York and London, 1973, xv + 521 pp., 24 cm. Price $38.00.

This is a valuable book on asymptotics. The author acknowledges he is a theoretical physicist; but when it comes to the subject of asymptotics, his use of theory as it relates to mathematical rigor in establishing that certain series are asymptotic, as the concept is usually employed, plays a minor role. The tempo is set in the opening sentence of what is called the prologue. "Throughout this book, the designation 'asymptotic series' will be reserved for those series in which for large values of the variable at all phases the terms first progressively decrease in magnitude, then reach a minimum and thereafter increase." More on this point, he later states that the "exposition will be heuristic and descriptive rather than rigorously doctrinaire." The author cites examples of such asymptotic series dating from the times of Stirling, Euler and Maclaurin. He notes the Poincaré definition for an asymptotic power series and criticizes it because of certain deficiencies which can arise in the pragmatic use of such series.

For convenience, let x be real and positive,

$$F(x) = f(x) + g(x), \qquad f(x) = \sum_{k=0}^{n-1} a_k x^{-k} + R_n(x).$$

According to Poincaré, if M_n is free of x,

$$|R_n(x)| \leqslant M_n x^{-n}, \quad \text{i.e. } R_n(x) = O(x^{-n}) \quad \text{as } x \longrightarrow \infty,$$

then the infinite series $\Sigma_{k=0}^{\infty} a_k x^k$ is asymptotic to $f(x)$ and one writes

$$f(x) \sim \sum_{k=0}^{\infty} a_k x^{-k}.$$

If $g(x)$, for example, is of the form

$$g(x) = e^{-\lambda x} h(x), \qquad \lambda > 0, \qquad h(x) = \sum_{k=0}^{n-1} h_k x^{-k} + O(x^{-n}), \qquad x \longrightarrow \infty,$$

then the series $\Sigma_{k=0}^{\infty} a_k x^{-k}$ is also asymptotic to $F(x)$. Hence a series can be asymptotic to more than one function. In practice, we desire to use the series for finite values of x. Here $F(x)$ cannot always be efficiently approximated by $f(x)$ as the contribution of $g(x)$ might be significant.

The Poincaré definition is a theoretical guide and serves to characterize the dominant part of the contribution to $F(x)$ as $x \longrightarrow \infty$. The definition can also apply to the function $e^{\lambda x}[F(x) - f(x)]$. I am sure Poincaré recognized the disparity between the definition of an asymptotic series and its use in numerical evaluation. The above discussion shows that to safely use asymptotic series, one should know the complete representation for $F(x)$; and if $F(x)$ is approximated by $f(x)$ which in turn is approximated by $\Sigma_{k=0}^{n-1} a_n x^{-k}$, then one should have bounds for the errors committed by truncation. The theory states only that $x^n |R_n(x)|$ be bounded as $x \longrightarrow \infty$. It does not say how sharp the bound must be.

The author's criticism of the Poincaré definition and his interpretation of the state of affairs as being vague and severely limited as to accuracy and range of applicability is not fair. Thus in his work the formal results obtained are not proved asymptotic in the Poincaré sense (or in a more generalized sense not noted here), and there is little on error bounds.

In his research which began around 1955, the author concentrated on deriving the 'complete asymptotic representation' of functions including the exponentially small terms, if any. A considerable part of the present volume is based on this research. Most examples treated are of hypergeometric type which are a special case of Meijer's G-function. In a series of papers dating from 1946, Meijer* derives the "complete asymptotic representation" of the G-function. I find it strange that this work is totally ignored except for mention of a single Meijer paper in the references on p. 55. Numerous other important references are omitted.

A brief outline of the work follows. Chapter 1 is a "behavioral survey" of asymptotics. Chapter 2 is concerned with the derivation of asymptotic series from converging series. Chapter 3 deals with the conversion of power series into integral representations.

*C. S. MEIJER, "On the G-function. I–VIII," *Nederl. Akad. Wetensch. Proc. Ser. A*, v. 49, 1946, pp. 227–237, 344–356, 457–469, 632–641, 765–772, 936–943, 1063–1072, 1165–1175. A rather complete summary of this work is given in Y. L. LUKE, *The Special Functions and Their Approximations*, Vol. 1, Academic Press, New York, 1969. See also *Math. Comp.*, v. 26, 1972, pp. 297–299.

Chapters 4–11 treat the derivation of asymptotic expansions, both uniform and nonuniform, from various types of integral representations. Derivation of asymptotic series, both uniform and nonuniform, from homogeneous and inhomogeneous differential equations is taken up in Chapters 12–20. In Chapters 21–26, there is presented the theory of terminants, a topic related to the idea of converging factors. In this connection, it is known that in many cases approximations based on finite sections of asymptotic series can be weighted to produce converging series which are far more efficient than using a finite section of the asymptotic series up to and including the smallest term augmented by the portion furnished by the converging factor method. This aspect is not considered.

In summary, the volume contains a wealth of information. Though much of it is formal and error estimates are wanting, the tome is valuable for its many approximations and ideas.

Y. L. L.

41 [13.40].—MARTIN GREENBERGER, JULIUS ARONOFSKY, JAMES L. MCKENNEY & WILLIAM F. MASSY, *Networks for Research and Education: Sharing Computer and Information Resources Nationwide,* The M.I.T. Press, Cambridge, 1974, xv + 418 pp., 24 cm. Price $12.50.

This book presents the papers, discussions and analyses of three working seminars and twelve workshop reports held in late 1972 and early 1973 sponsored by NSF and conducted by EDUCOM. The seminars were designed to help identify the central issues in building and operating networks on a national basis. The term networks was used to designate "the more general set of activities and arrangements whereby computers and communications are used for extensive resource sharing by a large number of separate, independent organizations". The main area chosen was networking for research and education on the national level.

The seminar themes included:

1. User characteristics and needs—discussions on the way in which computing and information are used.

2. Organizational matters—discussions on topics of network management, institutional relations, user organizations and regional computing systems.

3. Operations and funding—discussions on computers and communications, software systems and operating procedures, applications development and user services and network economics and funding.

In an overview, the editors discuss highlights of the issues covered and present conclusions and recommendations of the seminars they feel were identified during the discussions. The conclusions presented are worth restating here:

1. Computer networking must be acknowledged as an important new mode for obtaining information and computation. It is a real alternative that needs to be given serious attention in current planning and decision making.

2. The major problems to be overcome in applying networks to research and education are political, organizational, and economic in nature rather than technological.

3. Networking does not in and of itself offer a solution to current deficiencies. What it does offer is a promising vehicle with which to bring about important changes in user practices, institutional procedures, and government policy that can lead to effective solutions.

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42 [2.00, 3].—ALSTON S. HOUSEHOLDER. *Principles of Numerical Analysis,* Dover, New York, 1974, x + 274 pp., 21 cm. Price $4.00 (paperbound).

This a most welcome and slightly corrected reissue of the 1953 edition, originally published by McGraw-Hill. The thorough mathematical developments and the scholarly bibliographic discussions continue to make this work an indispensable classic.

The eight chapter headings are: 1. The art of computation, 2. Matrices and linear equations, 3. Nonlinear equations and systems, 4. The proper values and vectors of a matrix, 5. Interpolation, 6. More general methods of approximation, 7. Numerical integration and differentiation, 8. The Monte Carlo method.

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43 [9.00].—ALLAN M. KIRCH, *Elementary Number Theory: A Computer Approach,* Intext Educational Publishers, New York, 1974, xi + 339 pp., 25 cm. Price $11.75.

Elementary Number Theory: A Computer Approach, by Allan M. Kirch is an attempt to build a text around some reasonable mix of the two subjects. The avowed objective is to present certain aspects of elementary number theory together with related computer applications. This is carried out in terms of twenty-eight chapters, which are called "problems"; plus three appendices, the last of which is a "quick course in Basic Fortran IV".

The idea of attempting a work of this kind is certainly a valid one. Moreover, the nature of its objectives requires a rather detailed frame of reference which includes:

(i) for what group of students is the text intended,
(ii) what is meant by "elementary number theory",
(iii) what is meant by a "computer approach",
(iv) at what level and with what implied pedagogical technique is the material presented.

Having brought this book into existence, the author evidences a clear point of view concerning each of the above.

With regard to (i) the author presents the book as "a text for a beginning number theory course for students with good backgrounds in high school algebra" or as "a computer supplement to a more advanced treatment". The equivocation inherent in this becomes clear as one reads the book. The first eighteen problems touch on the basic properties of divisibility, greatest common divisor and least common multiple, the definition of prime numbers, unique factorization, number bases, and linear congruences. In this portion of the book the relevant theorems are either proved or included among the exercises (solutions at the end of the book). Here two criticisms might be made. First, the material is so thinly spread out that the subject hardly seems like a "theory". Secondly, the proofs tend to be very terse and awkward. In many instances the source of a critical theorem lies far away in the text. For example, the Unique Factorization Theorem is stated and proved in Problem 14, whereas it depends on material of Problems 4 and 5. In fact, the main step in the proof of the Unique Factorization Theorem is given as an exercise, which appears *after* the theorem. From Problem 19 to the end of the book proofs per se tend to be of less importance; and the reader is often referred to other texts. The existence of primitive roots modulo a prime power, and the quadratic reciprocity law, are typical casualties of this policy.

There is little doubt as to the author's conception of "computer approach". It involves a series of examples of gradually increasing complexity which illustrates various aspects of Fortran IV programming. This is carried out carefully, extensively, and enthusiastically. There is a clear impression that this is the main focus of the book; and that the number theory is a convenient excuse. Questions of optimal program design are equated to questions of programming technique rather than of methodology. This last is consistent with the very elementary level of the mathematics, and cannot be faulted.

The aforementioned level of the number theoretical material is quite basic, and certainly within the scope of an average undergraduate class. However, because of the tight mathematical presentation, the number theory itself would require considerable teacher supplementation. On the other hand, various attractive examples are provided which motivate at several levels. These include chapters on determining prime factors, calender analysis, and palindromic numbers; all of these being pointed principally towards related programming problems.

An overall summary view of the content of this book would reveal a modest amount of number theory together with a much larger amount of Fortran programming, both rather compactly presented. Whereas the amount of number theory technique which emerges is quite limited, the programming aspects fare better due to the detailed specimen programs.

Reactions to a noble experiment such as this one are of necessity subjective. This reviewer finds that the proposed marriage of number theory and Fortran IV leaves the former in a somewhat henpecked state. However, connubial matters of this sort are not that cut and dried. The author does succeed in exciting one's interest in the possibility of such a union and a search for an optimal basis of compatibility. Giving a course from this text might very well serve as a good starting point for such a quest.

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44 [4.00].— ROBERT E. O'MALLEY, JR., *Introduction to Singular Perturbations,* Academic Press, New York, 1974, viii + 206 pp., 25 cm. Price $16.50.

The term "singular perturbations" in this book refers to the study of ordinary differential equations which are modified by adding a small term of higher order of differentiation. The equation $\epsilon y'' + y' + y = 0$, which is a singular perturbation of the first order equation $y' + y = 0$, is a trivial illustration of this concept.

Until the late 1930's this type of problem was almost completely ignored by mathematicians, although the phenomena met in the boundary layer theory of Fluid Dynamics were known to be mathematically described by such differential equations. Since then, the theory of singular perturbations has grown into a substantial field of study, which has attracted numerous mathematicians in many countries. It has been recognized that questions of this sort often have surprising and fascinating mathematical answers and that singular perturbation aspects explain more physical phenomena than anybody would have foreseen forty years ago. The author of this book is one of the leading contributors to the progress in singular perturbation theory in the last ten years.

The book is written by a mathematician in the spirit of mathematics; but it is also, perhaps primarily, intended for users of mathematics in physics, engineering, biology and economics. The mathematical prerequisites are therefore held elementary, essentially at the undergraduate level, and many recondite matters of existence and of asymptotic smallness are omitted, with references to the appropriate literature. In the choice of subjects and of methods, the author has been influenced by his personal preferences and experience. Wherever possible, a unifying principle of uniform approximations that are obtained as the sum of two series is used, the first of which is itself a solution of the differential equation, while the second one involves a "stretched variable" and describes the asymptotic behavior of the true solution in the narrow intervals of rapid transition, typical in these theories.

After an elementary introductory chapter and a brief review of regular perturbation methods, the author develops a thorough theory of linear boundary problems. This is followed by an equally complete account of nonlinear initial value problems, which includes a section on differential-difference equations with small delay. Such delay problems, while not strictly singular perturbations in the sense of the original definition, have so many analogous features that their inclusion is justified. The next chapter, on nonlinear boundary value problems, described an intrinsically more difficult and less completely explored subject and is, therefore, less general than the preceding one. The remaining three chapters are devoted, respectively, to special types of optimal control problems, a boundary value problem with multiple solutions that arises in chemical reactor theory, and some turning point problems. Here and elsewhere in the book, applications are mentioned for motivation, but are never discussed as such.

The writer's style is lively and natural, and the techniques are well motivated. The author's facility with complicated algebraic manipulations may have led him to describe some of these calculations with less detail than this reviewer would have liked. An attentive reader will, however, be able to fill in these gaps himself. In the chapter on the regulator problem (Chapter 6), a certain amount of previous experience with such problems is expected from the reader. But these are minor objections.

The book is an up-to-date, attractive introduction to a subject whose importance for the applications is bound to grow.

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45 [5.10.3].—J. ROBINSON, *Integrated Theory of Finite Element Methods*, John Wiley & Sons, London, 1973, xxi + 428 pp., 24 cm. Price $29.50.

First of all, the title of the book is somewhat a misnomer. It does not contain an integrated account of finite elements nor is it a book on the theory of finite element methods. It is a book principally aimed at an engineering audience, which deals primarily with the formulative aspects of finite element methods applied to wide classes of problems in the analysis of elastic structures. While the introductory chapter does outline in broad strokes a review of some of the elementary properties of finite element methods, the book is certainly more readable to those with some experience with the method. The book deals with linear, static, finite-dimensional systems, and matrix notation is used throughout. It contains very little on computational methods, the numerical aspects of finite elements, equation solving, convergence, or on errors inherent in the method.

The first chapter of the book contains a brief account of the basic force method, the displacement method and the closely related direct stiffness method, and three "combined" methods, one of which involves the use of Lagrange multipliers. The principal aim of this chapter is to show various ways in which systems of equations governing large collections of elements can be properly assembled, for various choices of the dependent variables. It does not dwell on how the matrix relations governing individual elements are obtained.

The book contains a considerable amount of material on the force method, a subject in which its author has invested a number of years of work and made some contributions. To date, virtually all mathematical work on finite element methods has been concerned with so-called stiffness techniques; and those searching for new research problems in finite element methods might find plenty connected with the force method described in this book. Historically, the force method did not receive serious attention in the finite element literature because of difficulties in formulating flexibility matrices for large complicated structural systems. In more recent years, it has been recognized that the force method really involved a discretization of the dual problem in linear elasticity and, as such, can be attacked quite generally if one introduces the notion of stress functions or displacement potentials. This point of view is described in some detail in Chapter 2 of the book, and appropriate stress functions for second-order three-dimensional, and fourth-order two-dimensional problems are presented. Chapter 3 contains a collection of simple examples wherein the ideas outlined in Chapter 2 are used to produce various structural matrices. Chapter 3 is devoted to "strain elements" while Chapter 4 is aimed at the dual problem and describes "stress elements".

Chapter 5 deals with "inconsistent elements" by which the author means elements for which spurious rigid motions or equilibrating stress fields are contained. The author claims that these can be used effectively, but his arguments are not convincing. Chapter 6 is devoted to a readable account of conventional isoparametric elements and contains a number of examples.

Chapter 7 of the book is devoted to "isoparametric stress elements" and is quite an interesting chapter. Here the notions of isoparametrics are applied to the dual problem by representing stress functions as combinations of shape functions used in the coordinate transformations. This is apparently an original idea, and it provides an interesting family of new finite elements whose properties have not been explored mathematically to date.

That the generation of flexibility matrices is not as straightforward as the conventional matrix approach becomes clear in reading Chapters 8, 9, and 10 of the book, in which a great deal of algebra must be used to produce the proper matrices. However, there are some advantages to these methods for problems of singularities as is pointed out in Chapter 11 which is devoted to "cracked finite elements".

The book is unconventional, and it is unlikely that it will be used widely as a textbook for engineers interested in finite element methods, or as a reference for those interested in computational methods. Nevertheless, it does contain a number of new ideas which are worthy of consideration, not only by the practitioner but also by the theoretician.

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46 [12.05.1].—MARVIN SCHAEFER, *A Mathematical Theory of Global Program Optimization,* Prentice-Hall, Inc., Englewood Cliffs, N. J., 1973, xvii + 198 pp., 23 cm. Price $12.50.

The title of Schaefer's book would lead the reader to expect a unified mathematical theory of the subject area. The book, however, is something quite different; it is an assortment of optimization techniques mostly couched in mathematical notation. It is composed of two principal parts and three appendices. Part I, "Theoretical Foundations", is concerned with graph-theoretic methods. Part II, "Applications to Global Program Analysis", discusses a number of specific techniques, using some of the results in Part I.

The graph-theoretic methods developed in Part I are, for the most part, quite specific to the requirements of optimization. The main results given in this part are the partitioning of a program graph into intervals (regions with a single entry node); numbering algorithms that display the interval structure of a program graph (the Basic Numbering Algorithm and the Strict Numbering Algorithm); node splitting techniques for transforming irreducible graphs into reducible ones; and applications of the graph connectivity matrix. (A graph is reducible if its sequence of derived graphs leads to a single node. The first element of this sequence is the original graph; successive elements are obtained by treating each interval in the current element as a single node of the next element.)

In the first chapter of Part II, a method is given for determining those variables whose values are available at exit from each node of the program (data flow analysis). The next chapter discusses constant subsumption, common subexpression suppression, and code motion. In constant subsumption, calculations involving constants that might have to be done many times at run time are replaced by single calculations performed at compile time. In common subexpression suppression, multiple occurrences of the calculation of the value of an expression are replaced by a single occurrence of the calculation in the case where the different occurrences are known to produce the same result. In code motion, identical calculations appearing in several different paths are moved to a common point on the paths.

The chapter on loop optimization considers two techniques: invariant expression removal and reduction in strength of operators. An invariant expression of a loop is an expression whose value is known not to change during the execution of the loop; the calculation of the value of such expressions is profitably moved outside the loop. The "reduction in strength" optimization consists of replacing multiplication by addition and, less frequently, exponentiation by multiplication.

A short chapter on safety and execution frequency analysis follows. Some of the problems of insuring that program behavior remains invariant under optimization are discussed here. The remaining chapters contain brief discussions of subroutine linkages, register allocation, and the elimination of dead (unused) variables. The three appendices present, respectively, a collection of selected algorithms in APL; the organization of a specific optimizing compiler; and the problems of partial recompilation in an interactive environment.

Schaefer's book is deeply flawed in two respects. First, it is shot through with minor errors as well as sloppy proofs and explanations. The examples are far far too numerous to list exhaustively, but some samples can be given. On page 15, line 5, the minimal element should be a maximal one. On page 17, the "min" operation does not yield a unique result. On page 70, algorithm 7.12, step 2b, the paths should be restricted to lie within $T(i)$. On page 85, Figure 8.1(b), the "α" in the middle should be an "a".

A second serious flaw lies in the obscurity of the presentation. A particularly noxious example of this is Algorithm 6.19 for node splitting, on page 46, which is heavily encrusted with triply subscripted and superscripted variables. This algorithm is almost impossible to comprehend, and there is no explanation to accompany it; furthermore, it appears to contain errors. Other examples appear in the chapter on vertex ordering algorithms. Three desirable properties of such algorithms are stated at the beginning of the chapter; but nowhere is it either stated or proved that exactly these properties hold for either or both of the two algorithms presented. Furthermore, it is not apparent whether or not the Strict Numbering Algorithm is a special case of the Basic Numbering Algorithm, i.e., whether the order obtained from the SNA could have been obtained from the BNA with an appropriate series of arbitrary choices. Yet another example is the introduction, without explanation, of the variables x_h^M and x_h^m on page 89. It is only after reading several pages on that one becomes aware of the intent of these variables. In a slightly different vein, Appendix II starts with the following paragraph:

> This Appendix describes the architecture of a specific optimizing compiler in which the order of the various analyses is based on pragmatic considerations of the algorithmic source language. A modification of the order of the analyses might be more appropriate for languages of a different type.

However, nowhere is it stated what language is being compiled by this compiler, nor are any references given which might enable the reader to identify either the language or the compiler. Since the author states that the language is a major consideration in the design, the omission seems inexcusable.

It should also be noted that much of the material in Part II is given informally in English, in the form of general heuristics for the writer of an optimizing compiler. While this material is useful, it does not bear the slightest resemblance to a mathematical theory.

In summary, this book can be useful as a compendium of techniques, but it does not develop any unifying principles. For a mathematician, it is unlikely to be of much interest. For a compiler writer, it can serve as a useful source of ideas. On account of its muddy explanations and careless editing, however, it just is not worth the effort of a close reading.

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47 [13.15].— ANDREW B. TEMPLEMAN, Editor, *Engineering Optimization*, Vol. 1, No. 1 (1974); Gordon & Breach, New York, 1974, 69 pp., 26 cm. Price $15.00 for individuals, $52.00 for institutions.

Recognition of the common interest of differing branches of engineering in optimization has prompted the founding of this interesting journal. Although not excluding the publication of "technique" papers which concentrate on algorithms for solving narrowly-stated or algebraically-described optimization problems, the main emphasis will be on the formulation of optimization models and experiences in solving them using existing techniques. Perusal of the first issue shows a breadth of concern from the purely practical ("The cost of obtaining the final design is as significant as the optimality of the final design") to such theoretical/practical considerations as the sensitivity of the optimal design to errors in bounding moments given by plate finite element analysis. The journal should be particularly valuable as a source of case studies and examples of optimization for classroom use. A list of the papers in the first issue is: "The application of optimization techniques in the professional practice," "The optimum design of concrete structures," "Optimality conditions for trusses with nonzero minimum cross-sections," "Heuristic approaches to road network optimization," "The CIRIA optimization study of sewage treatment," "Michell framework for uniform load between fixed supports," and a particularly fine introductory article by the editor, A. B. Templeman, "*Engineering Optimization*—scope and aims."

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48 [9.20].—SOL WEINTRAUB, *Computer Program for Compact Prime List,* Microfiche supplement, this issue.

This FORTRAN program computes primes beginning at N and lists them in compact form as shown in [1].

The user enters the number N, where N ranges from zero to a limit set by the user's computer. For 32-bit machines the maximum N is $\sim 2^{30}$, or approximately 1.1 billion. N should be a multiple of 50000.

The program calculates primes for 20 intervals of 50000, i.e., an interval of 10^6, and lists the primes in compact form at the rate of an interval of 50000 per page.

The primes are generated by a modified sieve. The number N is divided by odd numbers j and the remainders r_j are noted. The *composite* numbers are then the numbers of the form $N - r_j + k_j$, where $k = 1, 2, 3, \ldots$. The entire run takes a few seconds of computer time.

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1. S. WEINTRAUB, "A compact prime listing," *Math. Comp.*, v. 28, 1974, pp. 855–857.

49 [4.00].—RALPH A. WILLOUGHBY, Editor, *Stiff Differential Systems,* Plenum Press, New York, 1974, x + 323 pp., 25 cm. Price $25.00.

This proceedings of an International Symposium on Stiff Differential Systems held in October, 1973, consists of nineteen papers on theoretical and practical aspects of stiff methods and their applications. In addition to a unified bibliography with dates from 1885 to 1973, there is an extremely valuable subject index which refers not only to the symposium papers but also to the bibliography.

Stability is the principal subject of four of the papers. Tests for A-stability and for stiff stability of composite multistep methods are given by Bickart and Rubin. Van Veldhuisen uses new concepts of consistency and stability to analyse global errors when the step size is large. Gear, Tu and Watanabe give sufficient conditions for changing both order and step size without affecting the stability of Adams methods. Also, Brayton shows that an A-stable multistep formula together with "passive" interpolation yields stable difference equations for certain difference-differential systems.

Five papers deal with new methods. Enright presents his stiffly stable second derivative methods and two modifications directed at large systems. Liniger and Gagnebin give an explicit construction for a $(2k - 2)$-parameter family of second order A-stable methods. The implicit midpoint method with smoothing and extrapolation is the basis for Lindberg's stiff system solver, and Dahlquist's paper provides the theoretical foundation for this approach. Two unconventional classes of methods are introduced by Lambert: a linear multistep method with variable matrix coefficients and an explicit nonlinear method based on local rational extrapolation.

Other program packages besides those of Enright and Lindberg are described. The special package by Edsberg for simulating chemical reaction kinetics automatically constructs an o.d.e. system from given reaction equations, generates subroutines for both the derivative and the Jacobian and uses Lindberg's solver. Gourlay and Watson indicate some of the practical difficulties they encountered while adding a version of Gear's stiff solver to the IBM "Continuous System Modeling Program", and Hull discusses validation and comparison of stiff program packages.

Stiff systems obtained by semidiscretization of parabolic partial differential equations are treated in two papers. Chang, Hindmarsh and Madsen study the simulation of chemical kinetics transport in the stratosphere, while Loeb and Schiesser study how the eigenvalues and stiffness ratio vary with the number of points in the spatial difference grid for a model problem.

Asymptotic approximation appears in several of the articles. Lapidus, Aiken and Liu survey the occurrence of stiff physical and chemical systems and show certain relationships among pseudo-steady-state approximation, singular perturbations, and stiff systems. Kreiss' paper deals with solutions for singular perturbations of two-point boundary value problems.

Stetter analyses and extends some novel ideas of Zadunaisky on global error estimation using nonstandard local error estimates. Hachtel and Mark combine polynomial prediction and truncation error control with a Davidenko-type parameter stepping method for nonlinear algebraic equations. Bulirsch and Branca briefly discuss computation in real-time-control situations.

As one can see from the topics mentioned here, this book is not for the novice. However, for the mature reader, it is an excellent guide to the literature on and introduction to the many difficult aspects of stiff equations.

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50 [9.00, 9.20].—SAMUEL YATES, *Prime Period Lengths,* 104 Brentwood Drive, Mt. Laurel, N. J., 1975, ii + 131 pp. Price $10.00 (paperbound).

This is a privately printed and bound version of the author's UMT previously reviewed in [1], which one should see for additional description. The new version achieves a reduction in size by a factor of 8 by printing four reduced-size pages of the previous table on each side of a page. The main content, as before, is a list of the 105000 primes $p \leqslant 1370471$ (excluding $p = 2$ and 5) versus the period P of the decimal expansion of the reciprocal $1/p$. Of course, P is also the order of 10 $\pmod p$ and what the author calls "full-period primes" (those with $P = p - 1$) are the primes having 10 as a primitive root.

The preface indicates that the main purpose of this publication is to enable investigators to study questions of distribution and to formulate appropriate conjectures. That being the case, it is surprising that the author does not include derived tables of such distributions, e. g., a table of the distribution of the "full-period primes." The reviewer agrees that there are interesting distribution problems here. He must admit, though, that he has also used the table for a much simpler purpose, namely, as a convenient list of primes $\leqslant 1370471$.

The one-page introduction contains a passage of such ambiguity that it must be quoted in full:

"Asymptotically speaking,

a. The period lengths of all primes are distributed evenly among the sixteen possible residue classes (mod 40).

b. The period lengths of half of all the primes congruent to 13 or 37 (mod 40) are even, and half are odd.

c. The period lengths of five sixths of all primes congruent to 1 or 9 (mod 40) are even, and one sixth are odd.

d. The period lengths of two thirds of all primes are even, and one third are odd.

e. If we divide all primes into three categories—full-period, odd period length, and non-full-period with even period length—the ratio of totals to each other in the given order, is 9:8:7.

These assertions are among others in articles by Samuel Yates and by Daniel Shanks in The Journal of Recreational Mathematics, beginning in 1969."

Since the five propositions are all called "assertions" it is not clear here which are true and which are false, and since no more exact references are given the reader would have a problem in determining which of the five "assertions" are due to which of the two named authors.

The facts are these: Proposition (a) is simply a special case of de la Vallée Poussin's famous theorem (1896) concerning primes in an arithmetic progression [2]. Proposition (d) was conjectured by Krishnamurthy [3] and proven by me in [4]. In the proof, (b) and (c) are preliminary results. Assertion (e), I am happy to report, is solely due to Yates. There is no reason to think that it is true and much reason to think that it is false.

If $\nu_{10}(x)$ is the number of primes $\leqslant x$ having 10 as a primitive root, then Assertion (e) would follow from proposition (d) if, and only if,

$$r_{10}(x) = \nu_{10}(x)/\pi(x) \longrightarrow 3/8 = 0.375 \tag{1}$$

as $x \longrightarrow \infty$. But there is every reason to believe that (1) is false. Heuristically, the correct asymptote is almost certainly the somewhat smaller Artin's constant:

$$A = \prod_{p=2}^{\infty} \left(1 - \frac{1}{p(p-1)}\right) = 0.3739558, \tag{2}$$

and Hooley [5] has proven this by assuming certain Riemann hypotheses.

The *empirical* case for (1) might seem more favorable at first. Up to Yates' limit $x = 1370471$, $r_{10}(x)$ is usually $> A$ (see [1] for an exception), and $r_{10}(x)$ is even $> 3/8$ throughout much of this range. The closing quotation is $r_{10}(1370471) = 0.37568$. However, it has long been known [6] that the convergence to A should be mostly from above. The first factor in (2), from $p = 2$, equals 1/2 and represents the fraction of primes having 10 as a quadratic nonresidue. While 1/2 is the correct asymptotic proportion of such primes, for finite x the fraction is usually [7] slightly more than 1/2 and, therefore, $r_{10}(x)$ will usually exceed A. In any case, in the following UMT, by Baillie [8], $\nu_{10}(x)$ (and other $\nu_a(x)$) are extended out to $x = 33 \cdot 10^6$. One finds this: after $x = 2.1 \cdot 10^6$, $r_{10}(x)$ remains < 0.37500; after $3.2 \cdot 10^6$, $r_{10}(x) < 0.37475$; after $9.8 \cdot 10^6$, $r_{10}(x) < 0.37450$ and after $14.1 \cdot 10^6$, $r_{10}(x) < 0.37425$.

While that proves nothing about $r_{10}(x) \longrightarrow A$, it does show that there is no case whatsoever for (1) and therefore no case whatsoever for Assertion (e). The correct proportions are almost surely not 9:8:7 but rather the less elegant 8.97494:8:7.02506.

D. S.

1. SAMUEL YATES, "Prime period lengths," UMT 10, *Math. Comp.*, v. 27, 1973, p. 216.

2. Ch. de la VALLÉE POUSSIN, "Recherches analytiques sur la théorie des nombres premiers, deuxième partie," *Ann. Soc. Sci. Bruxelles*, v. 20, part 2, 1896, pp. 281–362.

3. E. V. KRISHNAMURTHY, "An observation concerning the decimal periods of prime reciprocals," *J. Recreational Math.*, v. 4, 1969, pp. 212–213.

4. DANIEL SHANKS, "Proof of Krishnamurthy's conjecture," *J. Recreational Math.*, v. 6, 1973, pp. 78–79.

5. CHRISTOPHER HOOLEY, "On Artin's conjecture," *Crelle's J.*, v. 225, 1967, pp. 209–220.

6. DANIEL SHANKS, *Solved and Unsolved Problems in Number Theory*, Spartan, Washington, D.C., 1962, p. 83.

7. DANIEL SHANKS, "Quadratic residues and the distribution of primes," *MTAC*, v. 13, 1959, pp. 272–284. See Table 6.

8. ROBERT BAILLIE, *Data on Artin's Conjecture*, UMT 51, *Math. Comp.*, v. 29, 1975, pp. 1164–1165.

51 [9.00].—ROBERT BAILLIE, *Data on Artin's Conjecture*, Computer-Based Education Research Laboratory, University of Illinois, Urbana, Illinois, 1975, xlvi + about 280 pp. of computer output followed by 108 pp. of hard copy summaries deposited in the UMT file.

This is by far the most extensive data known to me that supports Artin's primitive root conjecture as it was modified by the Lehmers [1] and by Heilbronn, cf. [2, §23.2]. Let $\nu_a(x)$ be the number of primes $\leqslant x$ having a as a primitive root and let $\pi(x)$ be the number of primes $\leqslant x$. For $a = \pm 2, \pm 3, -4, \pm 5, \pm 6, \pm 7, \pm 8, -9, \pm 10, \pm 11, \pm 12, \pm 13$ the UMT deposited here strongly supports the conjecture

$$(1) \qquad \nu_a(x) \sim f_a A \pi(x)$$

where A is given in (2) of the previous review, and $f_a = 1$ for these a except in these cases: $f_{-3} = f_{-12} = 6/5, f_5 = 20/19, f_{-7} = 42/41, f_{-8} = f_8 = 3/5, f_{-11} = 110/109$ and $f_{13} = 156/155$. (The original Artin conjecture had an error in that it set all of these $f_a = 1$ except for $a = \pm 8$; cf. [6] of the previous review.)

The main table is in two parts: the first is for the ten positive values of a, and then, at this reviewer's suggestion, the twelve negative a were also computed. For $x = 25 \cdot 10^3(25 \cdot 10^3)33 \cdot 10^6$ there are listed here x, $\pi(x)$, the counts $\nu_a(x)$, the approximations $\langle f_a A\pi(x)\rangle$ rounded to the nearest integer, the empirical ratio $\nu_a(x)/\pi(x)$ to 8D and the ratio-differences $[\langle f_a A\pi(x)\rangle - \nu_a(x)]/\pi(x)$ also to 8D. But note that the rounding of $f_a A\pi(x)$ in the last quantity affects it considerably; the use of unrounded approximations would change these 8D differences substantially. These extensive tables required seventy-one hours of idle CPU time on a CDC Cyber-73.

The foregoing data was then summarized in four different ways on a PLATO IV terminal that produced the hard copies included here. Each of the thirty-three pages of Part I of these summaries lists the data for a single value of $x = n \cdot 10^6$ ($n = 1$ to 33) versus all twenty-two values of a. It lists $\nu_a(x)$, $\langle f_a A\pi(x)\rangle$ and their difference

$$(2) \qquad d_a(x) = \nu_a(x) - \langle f_a A\pi(x)\rangle.$$

Each of the twenty-two pages of Part II lists the data for a single a versus the thirty-three values of x. Part III is a plot of the noncumulative differences in each interval of 10^6 while Part IV plots the cumulative differences $d_a(x)$ versus x with a fixed. Part IV also lists the maximal $d_a(x)$ that occurs up to $33 \cdot 10^6$ and the distribution of the thirty-three values of $d_a(x)$ according to the sign of $d_a(x)$.

For most a, $d_a(x)$ is usually > 0. See the previous review for a discussion of the case $a = 10$. In Part IV, $d_{10}(x) > 0$ thirty-three times and < 0 never. But at $x = 150000$ in the original data there is a short interval when $d_{10}(x) < 0$; specifically, $\nu_{10}(150000) = 5167$ and $\langle f_{10} A\pi(x)\rangle = 5179$. For $a = -8, -10$ and -13, negative $d_a(x)$ predominate.

For all twenty-two a and all thirty-three $x = n \cdot 10^6$ I find the empirical law

$$(3) \qquad |d_a(x)| < \sqrt{f_a A\pi(x)},$$

i.e., the difference is less than the square-root of the expected value. On the other hand,

$$|d_a(x)| > \tfrac{1}{2}\sqrt{f_a A\pi(x)}$$

occurs frequently, so if (3) is true for all $x >$ some small x_0, then the right side of (3) would be close to the best possible bound. A weakened version of (3), namely,

$$(4) \qquad d_a(x) = O\left(\frac{x}{\log x}\right)^{1/2+\epsilon}$$

would not only imply (1) but also that the error term is smaller than has been previously suggested. Hooley's theory ([5] of the previous review) deduces the much larger error

$$O\left(\frac{x}{\log^2 x}\log\log x\right)$$

from his hypotheses. Perhaps someone should review his calculation to see if a smaller error term is not obtainable.

For earlier, far less extensive tables see [1]–[3] and the references cited there. The program used in these computations is also deposited.

D. S.

1. D. H. LEHMER & EMMA LEHMER, "Heuristics anyone?," *Studies in Mathematical Analysis and Related Topics*, Stanford Univ. Press, Stanford, Calif., 1962, pp. 202–210.

2. A. E. WESTERN & J. C. P. MILLER, *Indices and Primitive Roots*, Cambridge Univ. Press, 1968. Reviewed in *Math. Comp.*, v. 23, 1969, pp. 683–685.

3. J. C. P. MILLER, *Primitive Root Counts*, UMT 54, *Math. Comp.*, v. 26, 1972, p. 1024.