Nonelliptic Approximation of a Class
of Partial Differential Equations
with Neumann Boundary Condition

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Abstract. This paper is devoted to the numerical resolution of a class of linear partial differential equations with an inhomogeneous Neumann boundary condition. A first order quadrilateral finite element method is used, together with a one-point integration formula. The resulting scheme is simple and widely used but its theory is not classical, in a sense described as "nonelliptic". An important boundary value theorem is derived, in order to handle the Neumann condition. An error bound shows that the scheme is of order one.

1. Introduction. Let Ω be a bounded domain in the (x, y)-plane with sufficiently smooth boundary Γ. The purpose of this paper is the approximation of a general elliptic equation in Ω with an inhomogeneous Neumann boundary condition:

Continuous Neumann's Problem. Find a function u satisfying

$$(1.1) \quad -\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right)$$

$$+ b_1\frac{\partial u}{\partial x} + b_2\frac{\partial u}{\partial y} + du = f \quad in \ \Omega$$

with the boundary condition

$$(1.2) \quad \left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right)\cos(\vec{n}, x) + \left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right)\cos(\vec{n}, y) = g \quad on \ \Gamma,$$

where

$$(1.3) \quad a_{ij}, b_i \text{ and } d \text{ are given functions in } L^{\infty}(\Omega), f \text{ is}$$

$$\text{given in } L^2(\Omega) \text{ and } g \text{ is given in } L^2(\Gamma),$$

and where $\cos(\vec{n}, x)$ and $\cos(\vec{n}, y)$ denote the direction cosines of the exterior normal to Γ. □

We propose to approximate this problem by an *isoparametric quadrilateral* finite element method of *order one,* coupled with a *one-point* integration formula. This method gives rise to a very simple scheme which, originally, was derived by several authors as a generalization of finite differences on irregular networks. Since then, it

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has been widely and successfully applied in practice to a variety of problems (cf. for instance, Hirt, Amsden & Cook [5]). In a previous work, we have studied its application to a Dirichlet boundary value problem (cf. Girault [4]).

The main originality of our method is that it violates Strang's ellipticity condition (cf. Strang & Fix [10]), and for this reason, we shall call it *nonelliptic*. Because of the nonellipticity, this method *cannot* be studied by the now classical theory of finite element methods and numerical integration (cf. Ciarlet & Raviart [3]). Therefore, its theory must be carried out through an entirely different approach. In order to explain why Strang's condition does not hold, let us recall very briefly the main steps of isoparametric finite element methods.

(1°). The continuous problem is put into an equivalent *variational* form: Find a function u in V satisfying

$$(1.4) \qquad a(u, v) = L(v) \quad \forall v \in V,$$

for a suitable space V and form L, and a suitable *positive definite* form a.

(2°). The domain Ω is subdivided into finite elements which in our case are quadrilaterals; and the variational problem is formulated in terms of a *finite-dimensional* subspace V_h of V, consisting of simple functions defined on each finite element. Thus, problem (1.4) is approximated by: Find a function $\bar{u}_h$ in V_h satisfying

$$(1.5) \qquad a(\bar{u}_h, v_h) = L(v_h) \quad \forall v_h \in V_h.$$

It is now well known that isoparametric finite element methods are very well adapted to the resolution of second order boundary value problems on domains with curved boundaries. However, such methods require the evaluation of a great many integrals which are often difficult, if not impossible, to calculate exactly. As an illustration, in problem (1.1), (1.2), quadrilateral finite elements involve the integration of rational functions over rectangles, and the Neumann boundary condition leads to line integrals taken over curves which practically never coincide with sides of quadrilaterals. None of these integrals can be calculated exactly. Therefore, for all practical purposes, finite element methods are nearly always coupled with one or more adequate integration formulas. Hence, a third step is added to the preceding two.

(3°). The integrals occurring in (1.5) are computed with suitable integration formulas, and (1.5) is approximated by: Find a function u_h in V_h such that

$$(1.6) \qquad a_h(u_h, v_h) = L_h(v_h) \quad \forall v_h \in V_h.$$

Now, if the quadrature formula uses *enough* integration points, then $a_h(v_h, v_h)$ *is uniformly equivalent to* $a(v_h, v_h)$ on V_h and Strang's ellipticity condition is satisfied. Otherwise, if the quadrature formula, *however precise,* has *too few* points then this equivalence is practically never uniform; and, therefore, scheme (1.6) is *nonelliptic.*

Our approximation falls into the last category because a one-point integration formula does not have enough points to make the resulting scheme elliptic. As mentioned above, this is sufficient to break down the existing theory of finite element approximation. The major effort of this paper is to show that, nevertheless, the error in our method is of the order of h. A new theory is introduced, but the variational

framework is maintained. Because of the Neumann condition, this requires, in particular, a careful analysis of the approximation near the boundary.

The next section states precisely what is meant by nonelliptic approximation in general and sets problem (1.1), (1.2) in a variational frame. Sections 3 and 4 are devoted to the approximation of the resulting problem. Section 5 gives an important discrete boundary value theorem. The results of these sections are used, in Section 6, to derive an error estimate for our approximation.

The presentation of this paper differs from that of our previous one (cf. Girault [4]) in that we follow here more closely the finite element formalism. Although the two papers are independent, we have tried, as much as possible, to use the same notation.

2. Theoretical Framework.

2.1. *General Nonelliptic Approximation.* Let V be a real Hilbert space, with norm denoted by $\|\cdot\|$, L a member of the dual space V' of V, and $a(u, v)$ a bilinear, continuous and elliptic form on $V \times V$, that is,

(i) there exists a constant P such that

$$|a(u, v)| \leqslant P\|u\| \, \|v\| \quad \forall u, v \in V,$$

(ii) there exists a constant $\alpha > 0$ such that

$$a(v, v) \geqslant \alpha\|v\|^2 \quad \forall v \in V.$$

We propose to approximate the following problem:

Abstract Continuous Problem. Find u in V satisfying

(2.1) $$a(u, v) = L(v) \quad \forall v \in V. \quad \square$$

Thanks to the above hypotheses, this problem has a unique solution u (cf. Lions [6]). We assume that u belongs to some subspace $\mathcal{V}$ of V; this amounts in fact to making a regularity hypothesis on u.

Let h be a positive parameter intended to tend to zero, and for each h, let V_h be a finite-dimensional subspace of V. Let $v_h \longmapsto \|v_h\|_h$ be a mapping defined on $\mathcal{V} + V_h$ such that $\|\cdot\|_h$ is a norm on V_h. Let $a_h(u_h, v_h)$ be a bilinear form defined on $(\mathcal{V} + V_h) \times (\mathcal{V} + V_h)$ satisfying conditions similar to (i) and (ii):

(2.2) There exists a constant M, independent of h, such that

$$|a_h(u_h, v_h)| \leqslant M\|u_h\|_h\|v_h\|_h \quad \forall u_h \in \mathcal{V} + V_h \text{ and } \forall v_h \in V_h.$$

(2.3) There exists a constant $\beta > 0$, independent of h, such that

$$a_h(v_h, v_h) \geqslant \beta\|v_h\|_h^2 \quad \forall v_h \in V_h.$$

Finally, let L_h be a linear form on $\mathcal{V} + V_h$ satisfying the following condition:

(2.4) There exists a constant μ, independent of h, such that

$$|L_h(v_h)| \leqslant \mu\|v_h\|_h \quad \forall v_h \in V_h.$$

We approximate our continuous problem as follows:

Abstract Approximate Problem. Find u_h in V_h such that

(2.5) $$a_h(u_h, v_h) = L_h(v_h) \quad \forall v_h \in V_h. \quad \square$$

Definition 2.1. We say that the above approximation is *nonelliptic* if form a_h is

not uniformly elliptic (with respect to h) on V_h for the norm $\|\cdot\|$ of V. $\square$

From now on, we shall assume that *approximation* (2.5) *is nonelliptic.* Then, we cannot apply the general theory of internal approximation (cf. Ciarlet & Raviart [3]); but we have the following theorem, which is an easy extension of a result due to Strang (cf. Strang [9]):

THEOREM 2.1. *Let hypotheses* (2.2) *and* (2.3) *be satisfied. Then problem* (2.5) *has a unique solution u_h, and we have the error estimate*:

$$(2.6) \quad \|u - u_h\|_h \leqslant C \left\{ \inf_{v_h \in V_h} \|u - v_h\|_h + \sup_{w_h \in V_h; w_h \neq 0} \frac{1}{\|w_h\|_h} |L_h(w_h) - a_h(u, w_h)| \right\},$$

where u is the solution of problem (2.1) *and C is a constant depending only on M and β.*

Proof. As in the case of the continuous problem (2.1), the existence and uniqueness of u_h are a consequence of the continuity and ellipticity of a_h and the continuity of L_h (cf. Lions [6]), but none of these properties need to be uniform with respect to h.

Let v_h belong to V_h. Hypothesis (2.3) implies that

$$\|u_h - v_h\|_h \leqslant \frac{1}{\beta} \sup_{w_h \in V_h; w_h \neq 0} \frac{1}{\|w_h\|_h} |a_h(u_h - v_h, w_h)|.$$

Now

$$a_h(u_h - v_h, w_h) = a_h(u_h, w_h) - a_h(v_h, w_h)$$
$$= L_h(w_h) - a_h(u, w_h) + a_h(u, w_h) - a_h(v_h, w_h).$$

Therefore, $|a_h(u_h - v_h, w_h)| \leqslant |L_h(w_h) - a_h(u, w_h)| + |a_h(u - v_h, w_h)|$, and thanks to (2.2), we have:

$$\|u_h - v_h\|_h \leqslant \frac{1}{\beta} \sup_{w_h \in V_h; w_h \neq 0} \frac{1}{\|w_h\|_h} \{|L_h(w_h) - a_h(u, w_h)| + M\|u - v_h\|_h \|w_h\|_h\}.$$

Formula (2.6) follows easily from this inequality. $\square$

Remarks 2.1. (1°) The expression $\inf_{v_h \in V_h} \|u - v_h\|_h$ represents the error committed on projecting function u of V on V_h. The second expression, $\sup_{w_h \in V_h} (1/\|w_h\|_h) |L_h(w_h) - a_h(u, w_h)|$, gives a measure of the consistency of (2.5) with the exact problem (2.1).

(2°) Although property (2.4) is not one of the hypotheses of Theorem 2.1, it is included here since it is obviously necessary to the consistency of (2.5) with (2.1). $\square$

2.2. *Variational Formulation.* In order to apply the preceding approximation to problem (1.1), (1.2), we must first write it in a weakly equivalent variational form (cf. Lions [8]):

Variational Neumann Problem. Find u in $H^1(\Omega)$ satisfying

$$(2.1) \qquad\qquad a(u, v) = L(v) \quad \forall v \in H^1(\Omega),$$

where

$$(2.7) \quad a(u, v) = \int_\Omega \left\{ \left(a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y} \right) \frac{\partial v}{\partial x} + \left(a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y} \right) \frac{\partial v}{\partial y} \right.$$
$$\left. + \left(b_1 \frac{\partial u}{\partial x} + b_2 \frac{\partial u}{\partial y} + du \right) v \right\} dx\, dy,$$

$$(2.8) \quad L(v) = \int_\Omega fv\, dx\, dy + \int_\Gamma gv d\Gamma,$$

and

$$(1.3) \qquad a_{ij}, b_i \text{ and } d \in L^\infty(\Omega), f \in L^2(\Omega) \quad \text{and} \quad g \in L^2(\Gamma). \quad \square$$

As usual, $H^m(\Omega)$ denotes the classical Sobolev space of functions v such that v and all its derivatives of order less than or equal to m belong to $L^2(\Omega)$. For example, $H^1(\Omega)$ is a Hilbert space for the norm

$$\|v\|_{H^1(\Omega)} = \left\{ \|v\|^2_{L^2(\Omega)} + \left\| \frac{\partial v}{\partial x} \right\|^2_{L^2(\Omega)} + \left\| \frac{\partial v}{\partial y} \right\|^2_{L^2(\Omega)} \right\}^{1/2}.$$

Let us make the following assumption on the coefficients:

Hypothesis H1 (*ellipticity*). There exists a constant $\gamma > 0$ such that

$$\left. \begin{aligned} & \sum_{i,j=1}^2 a_{ij}(x, y)\xi_i \xi_j \geq \gamma \sum_{i=1}^2 \xi_i^2 \quad \forall (\xi_1, \xi_2) \in R^2 \text{ and} \\ & d(x, y) \geq \gamma \end{aligned} \right\} \quad \text{a.e. in } \Omega.$$

With the above assumptions, it can be shown (cf. Lions [7]) that form L defined by (2.8) belongs to the dual space of $H^1(\Omega)$; if, moreover, $\|b_i\|_{L^\infty(\Omega)}$, $i = 1, 2$, are small enough, then form a defined by (2.7) satisfies (i) and (ii). This can be summarized as follows:

PROPOSITION 2.1. *If Γ is sufficiently smooth, a_{ij}, b_i and $d \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$ and Hypothesis H1 is satisfied, then problem (2.1) (2.7) (2.8) has a unique solution, provided $\|b_i\|_{L^\infty(\Omega)}$ are sufficiently small.* $\square$

We shall take $V = H^2(\Omega) \cap C^1(\bar\Omega)$. Finally, in order to study properly the error estimate (2.6), it will be convenient to strengthen the regularity of the coefficients and boundary. We shall add the following hypotheses:

Hypothesis H2 (*regularity of coefficients*).

$$a_{ij} \in C^2(\bar\Omega) \quad \text{for } i, j = 1, 2, \qquad b_i \in C^1(\bar\Omega) \quad \text{for } i = 1, 2,$$

d and $f \in C^1(\bar\Omega)$ and $g \in C^1(C)$ for each arc C of Γ.

Hypothesis H3 (*regularity of boundary*). Γ consists of a finite number of simple arcs of class C^2. At the angular points of Γ, the angle between the two tangents is greater than an angle θ_0 with $0 < \theta_0 \leq \pi$ (cf. Figure 1).

FIGURE 1

3. Nonelliptic Approximation of Neumann's Problem.

3.1. *The Finite Element Method.* Let $\hat{K}$ denote the unit square, with vertices $\{\hat{a}_i\}_{i=1}^4$, of the reference space $(\hat{x}, \hat{y})$, and let K^* be any plane quadrilateral with vertices $\{a_{i,K}\}_{i=1}^4$ (cf. Figure 2). Let Q_1 denote the space of bilinear polynomials in $\hat{x}$ and $\hat{y}$. Then, there exists one and only one mapping F_K in Q_1^2 such that $F_K(\hat{a}_i) = a_{i,K}$ for $1 \leqslant i \leqslant 4$.
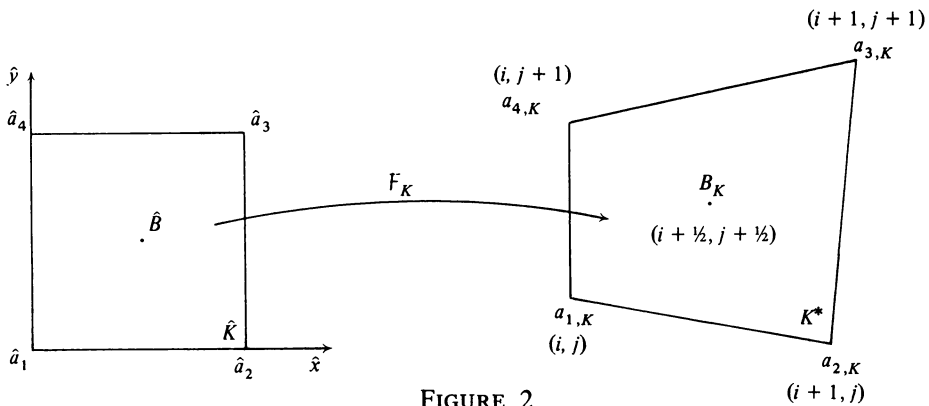

FIGURE 2

Now, let $\mathcal{Q}_h$ be a quadrangulation of Ω made of quadrilaterals or cells K^* with sides smaller than h. Let $\Omega_h^* = \bigcup_{K^* \in \mathcal{Q}_h} \overline{K}^*$ and $\Gamma_h = \partial\Omega_h^*$. We assume that the quadrangulation is *locally regular*, that is (cf. Ciarlet & Raviart [3]):

(3.1) all quadrilaterals K^* are strictly convex;

(3.2) there exists a constant $\sigma_0 > 0$, independent of h, such that

$$h_K' \geqslant \sigma_0 h \quad \text{and} \quad 1 - \gamma_K \geqslant \sigma_0 \quad \forall K^* \in \mathcal{Q}_h,$$

where h_K' denotes the length of the smallest side of K^* and $\gamma_K = \max|\cos\theta_K|$ where θ_K represent the angles of K^*.

Moreover, we assume that:

(3.3) Γ_h is a polygonal line of chords approximating Γ, and all angular points of Γ are vertices of Γ_h;

(3.4) the vertices of $\mathcal{Q}_h$ can be numbered by a pair of indices (i, j).

Hypothesis (3.3) implies that Ω_h^* approximates Ω; more precisely,

(3.5)
$$\sup_{(x,y) \in \Omega_h^*} \text{dist}((x, y), \Omega) = O(h^2).$$

Thus, there exists a compact set $\widetilde{\Omega}$ containing both Ω and Ω_h^*.

Hypothesis (3.1) implies that each F_K is invertible and we define space V_h as follows:

$$V_h = \{v_h \in C^0(\Omega_h^*) | v_{h|K*} = q \circ F_K^{-1} \ \forall q \in Q_1\}.$$

Also, let $V_h' = \{v_h|_{\Gamma_h} | v_h \in V_h\}$.

Finally, for each real function u defined at the vertices of V_h, let $r_h u$ be the function of V_h which takes the same value as u at the vertices of Q_h.

Remark 3.1. The situation here differs slightly from that of Section 2 since in general $V_h \not\subset H^1(\Omega)$. In practice, this "variational crime" is not very important because Ω and Ω_h^* are fairly alike. Theoretically, the difficulty may be overcome by replacing $H^1(\Omega)$ by $H^1(\widetilde{\Omega})$ and extending functions a_{ij}, b_i, d and f to obtain functions, still denoted by a_{ij}, b_i, d and f, belonging, respectively, to $C^2(\widetilde{\Omega})$ and $C^1(\widetilde{\Omega})$. We still denote by V the space $H^2(\widetilde{\Omega}) \cap C^1(\widetilde{\Omega})$, and we extend continuously all functions of V_h to $\widetilde{\Omega}$, so that $V_h \subset H^1(\widetilde{\Omega})$. □

Let us define two integration formulas on K^*:

$$\left.\begin{array}{l} \displaystyle\int_{K*} \varphi(x, y)\,dx\,dy \sim J_K(\hat{B})\varphi(B_K) = F_1(\varphi) \\[4mm] \displaystyle\int_{K*} \varphi(x, y)\,dx\,dy \sim \sum_{i=1}^{4} \frac{1}{4} J_K(\hat{a}_i)\varphi(a_{i,K}) = F_2(\varphi) \end{array}\right\} \quad \forall \varphi \in C^0(K^*),$$

where J_K denotes the Jacobian of F_K, $\hat{B}$ the mean center of $\hat{K}$ and B_K the mean center of K^*. On $\hat{K}$, these formulas are both exact for all polynomials of Q_1. Let K' denote a side of K^*, for instance segment $[a_{p,K}, a_{q,K}]$. Let $a_{pq,K}$ stand for its midpoint and l_K, its length. Formulas F_1 and F_2 induce two formulas, F_1' and F_2', on K':

$$\left.\begin{array}{l} \displaystyle\int_{K'} \varphi(x, y)\,d\sigma \sim l_{K'}\varphi(a_{pq,K}) = F_1'(\varphi) \\[4mm] \displaystyle\int_{K'} \varphi(x, y)\,d\sigma \sim \frac{1}{2} l_{K'} \{\varphi(a_{p,K}) + \varphi(a_{q,K})\} = F_2'(\varphi) \end{array}\right\} \quad \forall \varphi \in C^0(K').$$

Here, F_1' and F_2' are exact for all polynomials of degree one.

Remarks 3.2. (1°) Let S_K^* denote the *area* of K^*. Then $J_K(\hat{B}) = S_K^*$.

(2°) Let $v_h \in V_h$, then

$$v_h(B_K) = \frac{1}{4} \sum_{i=1}^{4} v_h(a_{i,K}) \quad \text{and} \quad v_h(a_{pq,K}) = \frac{1}{2} (v_h(a_{p,K}) + v_h(a_{q,K})).$$

(3°) From (3.1) and (3.2), it follows that there exist two positive constants, c_0 and σ_0, independent of h, such that

$$\inf_{(\hat{x},\hat{y})\in\hat{K}} J_K(\hat{x}, \hat{y}) \geqslant c_0 h^2 \quad \forall K^* \in Q_h, \text{ and}$$

$$l_{K'} \geqslant \sigma_0 h \quad \forall K' \in \Gamma_h.$$

Moreover, it is obvious that there exists a constant C_0, independent of h, such that

$$\sup_{(\hat{x},\hat{y}) \in \hat{K}} J_K(\hat{x},\hat{y}) \leqslant C_0 h^2 \quad \forall K^* \in \mathcal{Q}_h.$$

Thus we have the important bounds

$$(3.6) \qquad c_0 h^2 \leqslant S_K^* \leqslant C_0 h^2 \quad \forall K^* \in \mathcal{Q}_h,$$

$$(3.7) \qquad \sigma_0 h \leqslant l_{K'} \leqslant h \qquad \forall K' \in \Gamma_h. \quad \square$$

We approach Neumann's problem as follows:

Approximate Neumann's Problem. Find u_h in V_h such that

$$(2.5) \qquad a_h(u_h, v_h) = L_h(v_h) \quad \forall v_h \in V_h,$$

where

$$(3.8)\qquad
\begin{aligned}
a_h(u_h, v_h) = \sum_{K^* \in \mathcal{Q}_h} \Bigg\{ F_1 \Bigg[&\left(a_{11} \frac{\partial u_h}{\partial x} + a_{12} \frac{\partial u_h}{\partial y} \right) \frac{\partial v_h}{\partial x} \\
&+ \left(a_{21} \frac{\partial u_h}{\partial x} + a_{22} \frac{\partial u_h}{\partial y} \right) \frac{\partial v_h}{\partial y} \\
&+ \left(b_1 \frac{\partial u_h}{\partial x} + b_2 \frac{\partial u_h}{\partial y} \right) v_h \Bigg] + F_2(du_h v_h) \Bigg\},
\end{aligned}$$

$$(3.9) \qquad L_h(v_h) = \sum_{K^* \in \mathcal{Q}_h} F_1(f v_h) + \sum_{K' \in \Gamma_h} F_1'(g v_h)$$

or

$$(3.9') \qquad L_h(v_h) = \sum_{K^* \in \mathcal{Q}_h} F_2(f v_h) + \sum_{K' \in \Gamma_h} F_1'(g v_h). \quad \square$$

Remarks 3.3. (1°) Strictly speaking, as g is only defined on Γ, formula F_1' can only be used when $a_{pq,K}$ belongs to Γ. However, according to (3.5), there exists at least one point $\tilde{a}_{pq,K}$ on Γ such that $\text{dist}(\tilde{a}_{pq,K}, a_{pq,K}) = O(h^2)$. We then agree to use $g(\tilde{a}_{pq,K})$ in F_1'.

(2°) It will be shown in Section 6 that the use of (3.9') instead of (3.9) does not change the error estimate. $\square$

3.2. *Preliminary Analysis of the Approximation.* Let us define on V_h the scalar product

$$(u_h, v_h)_h = \sum_{K^* \in \mathcal{Q}_h} F_2(u_h v_h)$$

and the norms

$$|u_h|_h = (u_h, u_h)_h^{1/2},$$

$$\|u_h\|_h = \left(\sum_{K^* \in \mathcal{Q}_h} F_1 \left\{ \left(\frac{\partial u_h}{\partial x} \right)^2 + \left(\frac{\partial u_h}{\partial y} \right)^2 \right\} + |u_h|_h^2 \right)^{1/2}.$$

Similarly, we define on V'_h the scalar product

$$(u_h, v_h)_{\Gamma_h} = \sum_{K' \in \Gamma_h} F'_2(u_h v_h)$$

and the associated norm $|u_h|_{\Gamma_h} = (u_h, u_h)^{1/2}_{\Gamma_h}$.

Thanks to the choice of the integration points in F_2 and F'_2, we have the following lemma:

LEMMA 3.1. *The mappings* $u_h \longmapsto |u_h|_h$ *and* $u_h \longmapsto \|u_h\|_h$ *are norms on* V_h. *Similarly*, $u_h \longmapsto |u_h|_{\Gamma_h}$ *is a norm on* V'_h. $\square$

It can be shown by counterexample (cf. Girault [4]) that the norm defined above, $\|\cdot\|_h$, is *not uniformly equivalent* on V_h to norm $\|\cdot\|$; hence our approximation is *non-elliptic*.

The following notation will be useful: let

$$[\varphi, \psi]_h = \sum_{K^* \in \mathcal{Q}_h} F_1(\varphi \psi) \quad \text{for } \varphi, \psi \text{ defined on } \widetilde{\Omega},$$

$$[\varphi, \psi]_{\Gamma_h} = \sum_{K' \in \Gamma_h} F'_1(\varphi \psi) \quad \text{for } \varphi, \psi \text{ defined on } \Gamma_h,$$

$$[u_h]_h = [u_h, u_h]^{1/2}_h \quad \text{and} \quad [u_h]_{\Gamma_h} = [u_h, u_h]^{1/2}_{\Gamma_h}.$$

LEMMA 3.2. *There exists a constant C, independent of h, such that*

(3.10) $$[u_h]_h \leqslant C|u_h|_h \quad \forall u_h \in V_h.$$

Moreover,

(3.11) $$[u_h]_{\Gamma_h} \leqslant |u_h|_{\Gamma_h} \quad \forall u_h \in V'_h.$$

Proof. For (3.10), it suffices to show that, for each K^* in $\mathcal{Q}_h$,

$$F_1(u_h^2) \leqslant C^2 F_2(u_h^2).$$

But

$$F_1(u_h^2) = u_h^2(B_K)S_K^* = \left\{ \frac{1}{4} \sum_{i=1}^{4} u_h(a_{i,K}) \right\}^2 S_K^*$$

$$\leqslant \frac{1}{4} S_K^* \sum_{i=1}^{4} u_h^2(a_{i,K}) \leqslant \frac{C_0}{c_0} F_2(u_h^2).$$

Thus, (3.10) holds with $C = (C_0/c_0)^{1/2}$. The proof of (3.11) is obvious. It is easy to show that $[\cdot]_h$ (respectively $[\cdot]_{\Gamma_h}$) is not a norm on V_h (respectively V'_h). $\square$

The next lemma follows easily from (3.10) and *H2*:

LEMMA 3.3. *There exists a constant M, independent of h, such that*

$$|a_h(u_h, v_h)| \leqslant M\|u_h\|_h \|v_h\|_h \quad \forall u_h \in V_h + V \text{ and } \forall v_h \in V_h. \quad \square$$

LEMMA 3.4. *If* $\|b_i\|_{L^\infty(\widetilde{\Omega})}$ ($i = 1, 2$) *are small enough, there exists a constant* $\beta > 0$, *independent of h, such that*

$$a_h(v_h, v_h) \geqslant \beta \|v_h\|_h^2 \quad \forall v_h \in V_h.$$

Proof.

$$a_h(v_h, v_h) = \sum_{K^* \in \mathcal{Q}_h} \left(\left\{ a_{11}(B_K) \frac{\partial v_h}{\partial x}(B_K) + a_{12}(B_K) \frac{\partial v_h}{\partial y}(B_K) \right\} \frac{\partial v_h}{\partial x}(B_K) \right.$$

$$\left. + \left\{ a_{21}(B_K) \frac{\partial v_h}{\partial x}(B_K) + a_{22}(B_K) \frac{\partial v_h}{\partial y}(B_K) \right\} \frac{\partial v_h}{\partial y}(B_K) \right) S_K^*$$

$$+ \left[b_1 \frac{\partial v_h}{\partial x} + b_2 \frac{\partial v_h}{\partial y}, v_h \right]_h + (dv_h, v_h)_h,$$

$$a_h(v_h, v_h) \geqslant \gamma \sum_{K^* \in \mathcal{Q}_h} \left\{ \frac{\partial v_h^2}{\partial x}(B_K) + \frac{\partial v_h^2}{\partial y}(B_K) \right\} S_K^* + \gamma |v_h|_h^2$$

$$- C \left\{ \|b_1\|_{L^\infty(\widetilde{\Omega})} \left[\frac{\partial v_h}{\partial x} \right]_h + \|b_2\|_{L^\infty(\widetilde{\Omega})} \left[\frac{\partial v_h}{\partial y} \right]_h \right\} |v_h|_h,$$

thanks to *H*1, *H*2 and (3.10). Hence

$$a_h(v_h, v_h) \geqslant \gamma \|v_h\|_h^2 - C(\|b_1\|_{L^\infty(\widetilde{\Omega})} + \|b_2\|_{L^\infty(\widetilde{\Omega})}) \|v_h\|_h^2.$$

If, for instance, $\|b_1\|_{L^\infty(\widetilde{\Omega})} + \|b_2\|_{L^\infty(\widetilde{\Omega})} \leqslant \gamma/2C$, then $a_h(v_h, v_h) \geqslant \gamma \|v_h\|_h^2/2$. □

LEMMA 3.5. *Let $L_h(v_h)$ be defined by (3.9) or (3.9'). There exist two constants λ_1 and λ_2, independent of h, such that*

$$|L_h(v_h)| \leqslant \lambda_1 |v_h|_h + \lambda_2 |v_h|_{\Gamma_h} \quad \forall v_h \in V_h.$$

Proof. Let, for instance, $L_h(v_h)$ be defined by (3.9). Then it follows from (3.10), (3.11) and H2 that

$$|L_h(v_h)| \leqslant \|f\|_{C^0(\widetilde{\Omega})} C(\text{mes } \widetilde{\Omega})^{\frac{1}{2}} |v_h|_h + \|g\|_{C^0(\Gamma)} (\text{mes } \Gamma)^{\frac{1}{2}} |v_h|_{\Gamma_h}. \quad □$$

We have thus shown that a_h satisfies (2.2) and (2.3). However, Lemma 3.5 does not suffice to prove that L_h satisfies (2.4). To achieve this, we must find an estimate for $|v_h|_{\Gamma_h}$ in terms of $\|v_h\|_h$. This is the object of Section 5.

Remark 3.4. To guarantee that a_h is positive definite, it is necessary to integrate the expression $du_h v_h$ by formula F_2. □

3.3. *The Projection Error.* This section gives estimates for $\inf_{v_h \in V_h} \|u - v_h\|_h$. We shall use the following seminorm for u in $H^m(\Omega)$:

$$|u|_{m,2,\Omega} = \left(\sum_{\alpha_1 + \alpha_2 = m} \left\| \frac{\partial^m u}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \right\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.$$

LEMMA 3.6. *Let u be defined on $\widetilde{\Omega}$ and let $r_h u$ be the "projection" of u on V_h, as in subsection 3.1. We assume that $\mathcal{Q}_h$ is locally regular.*

(1°) *If $u \in H^3(\Omega)$, there exists a constant C, independent of u and h, such that,*

$$(3.12) \qquad \|u - r_h u\|_h \leqslant Ch(|u|_{2,2,\widetilde{\Omega}}^2 + h^2 |u|_{3,2,\widetilde{\Omega}}^2)^{\frac{1}{2}}.$$

(2°) *If u belongs only to $V = H^2(\widetilde{\Omega}) \cap C^1(\widetilde{\Omega})$, then*

(3.13) $\|u - r_h u\|_h \leqslant C'h|u|_{2,2,\tilde{\Omega}},$

where C' is another constant independent of u and h.

   Proof. Since $|u - r_h u|_h = 0$, it suffices to consider

$$\left[\frac{\partial}{\partial x}(u - r_h u)\right]_h \quad \text{and} \quad \left[\frac{\partial}{\partial y}(u - r_h u)\right]_h.$$

Let $u \in H^3(\tilde{\Omega})$. Let K^* be a cell of $\mathcal{Q}_h$, let $\hat{u} = u \circ F_K$, and let $F_K^{(1)}$ and $F_K^{(2)}$ denote the components of F_K. Then

(3.14)
$$\left\{\frac{\partial}{\partial x}(u - r_h u)\right\}(B_K)$$
$$= \frac{1}{S_{K^*}}\left\{\frac{\partial}{\partial \hat{x}}(\hat{u} - \widehat{r_h u})\frac{\partial}{\partial \hat{y}}F_K^{(2)} - \frac{\partial}{\partial \hat{y}}(\hat{u} - \widehat{r_h u})\frac{\partial}{\partial \hat{x}}F_K^{(2)}\right\}(\hat{B}),$$

and a similar equality holds for the derivative with respect to y. With the notations of Figure 2,

$$\left(\frac{\partial}{\partial \hat{x}}\widehat{r_h u}\right)(\hat{B}) = \frac{1}{2}\left[(\hat{u}_3 - \hat{u}_4) + (\hat{u}_2 - \hat{u}_1)\right].$$

Thus, the mapping $\hat{u} \longmapsto [\partial(\hat{u} - \widehat{r_h u})/\partial \hat{x}](\hat{B})$ is a linear and continuous form on $H^3(\hat{K})$ which vanishes for polynomials of degree 2. Therefore, by the Bramble and Hilbert lemma (cf. Bramble & Hilbert [1]), there exists a constant c, independent of u and h, such that

$$\left|\left\{\frac{\partial}{\partial \hat{x}}(\hat{u} - \widehat{r_h u})\right\}(\hat{B})\right| \leqslant c|\hat{u}|_{3,2,\hat{K}}.$$

Now, since $\mathcal{Q}_h$ is locally regular,

(3.15) $|\hat{u}|_{3,2,\hat{K}}^2 \leqslant \dfrac{c'}{c_0}h^2(|u|_{2,2,K^*}^2 + h^2|u|_{3,2,K^*}^2),$

where c' is another constant independent of u and h. As there is no loss of generality in having considered derivatives with respect to the first variable, (3.12) follows from these last inequalities, (3.14) and (3.6).

   The proof of (3.13) is much the same as that of (3.12). $\square$

   Hence, if $\mathcal{Q}_h$ is locally regular, then, for fixed u in V:

(3.16) $\displaystyle\inf_{v_h \in V_h} \|u - v_h\|_h \leqslant Ch,$

where C is a constant independent of h.

   Remark 3.5. When K^* is a parallelogram, the coefficient of $|u|_{2,2,K^*}^2$ in the right-hand side of (3.15) vanishes, and if K^* deviates slightly from a parallelogram, this coefficient is small. More precisely, if

(3.17) $\|a_{1,K} + a_{3,K} - a_{2,K} - a_{4,K}\| = O(h^2),$

then

$$|\hat{u}|_{3,2,\hat{K}}^2 \leqslant ch^4(|u|_{2,2,K^*}^2 + |u|_{3,2,K^*}^2),$$

where again c is a constant independent of u and h. If, moreover, $u \in C^3(\tilde{\Omega})$, then $[\partial(u - r_h u)/\partial x_i](B_K) = O(h^2)$, $i = 1, 2$, where x_i denotes either x or y. It is interesting to note that this last property has a converse stated in the next lemma.

LEMMA 3.7. *Let $\mathcal{Q}_h$ be locally regular and let $K^* \in \mathcal{Q}_h$. If, for all functions u in $C^3(\tilde{\Omega})$, $\{\partial(u - r_h u)/\partial x_i\}(B_K) = O(h^2)$, for $i = 1, 2$, then K^* is nearly a parallelogram, that is, its vertices satisfy* (3.17). $\square$

The proof of this lemma is straightforward.

4. Study of the Approximation.

4.1. *Notations Concerning the Quadrangulation.* Let $\dot{O}_h$ denote the set of vertices of $\mathcal{Q}_h$ and $\dot{\Gamma}_h$ the set of vertices of Γ_h. Let $\dot{\Omega}_h = \dot{O}_h - \dot{\Gamma}_h$. According to hypothesis (3.4), we can number the nodes of $\dot{O}_h$ with a pair of indices (i, j). This numbering induces obviously another one on the cells K^* of $\mathcal{Q}_h$ (cf. Figure 2). Let $\dot{\Omega}_h^* = \{$mean center of $K^*|K^* \in \mathcal{Q}_h\}$. If (i, j) is the index of K^*, we agree to denote this cell by $K_{i,j}^*$ and its area by $S_{i,j}^*$. Also, it will be convenient to denote its mean center by (i, j) and by $\varphi_{i,j}$ the value of a function φ at (i, j). Similarly, if (i, j) is the index of a node of $\dot{O}_h$, we agree to denote by $\varphi_{i,j}$ the value of φ at (i, j).

Now, let (i, j) be a node of $\dot{O}_h$, let $R_{i,j} \subset \Omega_h^*$ denote the region consisting of the triangles of Ω_h^* with common vertex (i, j) and let $B_{i,j}$ stand for the boundary of $R_{i,j}$ (cf. Figure 3). Let

$$S_{i,j} = \tfrac{1}{2}(\text{area of } R_{i,j})$$

and, for (i, j) in $\dot{\Gamma}_h$, let $l_{i,j} = \tfrac{1}{2}(\text{length of } B_{i,j} \cap \Gamma_h)$.
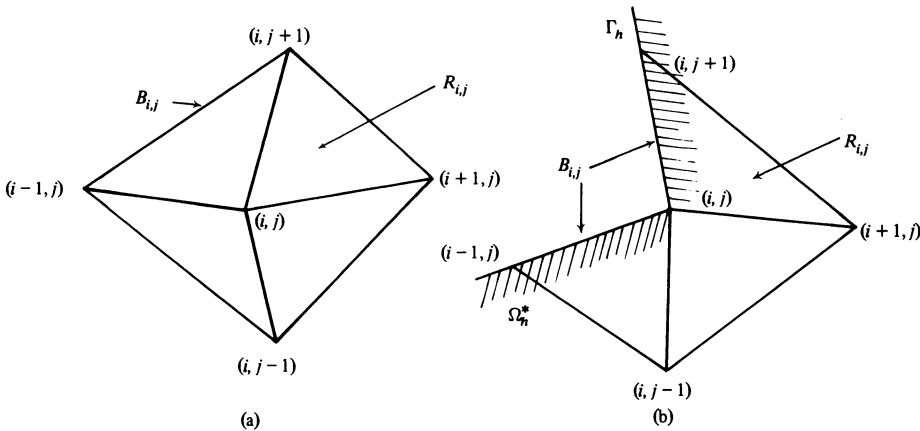

FIGURE 3

With these notations, we can express easily the scalar products $(\cdot, \cdot)_h$ and $(\cdot, \cdot)_{\Gamma_h}$.

LEMMA 4.1.

$$(4.1) \qquad (u_h, v_h)_h = \sum_{(i,j) \in \dot{O}_h} u_{i,j} v_{i,j} S_{i,j}$$

$$(4.2) \qquad (u_h, v_h)_{\Gamma_h} = \sum_{(i,j) \in \dot{\Gamma}_h} u_{i,j} v_{i,j} l_{i,j}$$

$\forall u_h, v_h \in V_h$. $\square$

Remark 4.1. It follows from Remark 3.2 that

(4.3) $$c_0 h^2 \leqslant S_{i,j} \leqslant C_0 h^2 \qquad \forall (i, j) \in \dot{\Omega}_h,$$

(4.3') $$\left. \begin{array}{c} \dfrac{c_0}{4} h^2 \leqslant S_{i,j} \leqslant \dfrac{3}{4} C_0 h^2 \\[2mm] \sigma_0 h \leqslant l_{i,j} \leqslant h \end{array} \right\} \quad \forall (i, j) \in \dot{\Gamma}_h. \quad \square$$

(4.4)

Finally, let $V_h^* \subset L^2(\Omega_h^*)$ denote the space of real step functions which are constant on each cell of Q_h. With each real function φ defined on $\dot{\Omega}_h^*$, we associate the function $r_h^* \varphi$ of V_h^* which takes the same value as φ at the nodes of $\dot{\Omega}_h^*$.

4.2. *Difference Quotients and Averages.*

Definition 4.1. Let φ be a function in V_h^*. We define its discrete derivative with respect to x as follows:

(4.5) $$(\nabla_1^* \varphi)_{i,j} = \frac{1}{2 S_{i,j}} \oint_{B_{i,j}} \varphi(x, y)\, dy \quad \forall (i, j) \in \dot{O}_h,$$

and its discrete derivative with respect to y by:

(4.6) $$(\nabla_2^* \varphi)_{i,j} = -\frac{1}{2 S_{i,j}} \oint_{B_{i,j}} \varphi(x, y)\, dx \quad \forall (i, j) \in \dot{O}_h. \quad \square$$

It can be easily checked that, for (i, j) in $\dot{\Omega}_h$,

(4.7) $$(\nabla_1^* \varphi)_{i,j} = \frac{1}{2 S_{i,j}} \{ \varphi_{i+1/2, j+1/2}(y_{i,j+1} - y_{i+1,j}) + \varphi_{i-1/2, j+1/2}(y_{i-1,j} - y_{i,j+1})$$
$$+ \varphi_{i-1/2, j-1/2}(y_{i,j-1} - y_{i-1,j}) + \varphi_{i+1/2, j-1/2}(y_{i+1,j} - y_{i,j-1}) \},$$

(4.8) $$(\nabla_2^* \varphi)_{i,j} = \frac{1}{2 S_{i,j}} \{ \varphi_{i+1/2, j+1/2}(x_{i+1,j} - x_{i,j+1}) + \varphi_{i-1/2, j+1/2}(x_{i,j+1} - x_{i-1,j})$$
$$+ \varphi_{i-1/2, j-1/2}(x_{i-1,j} - x_{i,j-1}) + \varphi_{i+1/2, j-1/2}(x_{i,j-1} - x_{i+1,j}) \}.$$

These formulas are, of course, different when $(i, j) \in \dot{\Gamma}_h$. As an example, if (i, j) is located as in Figure 3(b), then (4.5) yields:

$$(\nabla_1^* \varphi)_{i,j} = \frac{1}{2 S_{i,j}} \{ \varphi_{i-1/2, j-1/2}(y_{i,j-1} - y_{i,j}) + \varphi_{i+1/2, j-1/2}(y_{i+1,j} - y_{i,j-1})$$
$$+ \varphi_{i+1/2, j+1/2}(y_{i,j} - y_{i+1,j}) \}.$$

The usefulness of these difference quotients appears in the following result:

Proposition 4.1. *Let φ be a function defined on Ω_h^* and v_h a function of V_h. Then the following "discrete Green's formula" holds:*

(4.9) $$\left[\varphi, \frac{\partial v_h}{\partial x_i} \right]_h = -(r_h(\nabla_i^* \varphi), v_h)_h + [r_h^* \varphi \cdot \cos(\vec{n}_h, x_i), v_h]_{\Gamma_h}, \quad i = 1, 2,$$

where x_i denotes either x or y and $\cos(\vec{n}_h, x_i)$ denotes the ith direction cosine of the exterior normal to Γ_h. $\square$

The proof is omitted since it is rather long and technical.

Definition 4.2. Let φ be a function in V_h^*. We define its *average* $\mu^*\varphi$ as follows:

(4.10) $\displaystyle (\mu^*\varphi)_{i,j} = \frac{1}{4S_{i,j}} \sum_{(r,s)\in\dot\Omega_h^*;\, r=i\pm1/2;\, s=j\pm1/2} \varphi_{r,s} S_{r,s}^* \quad \forall(i,j)\in\dot O_h.$ $\square$

This formula yields, when $(i, j) \in \dot\Omega_h$:

(4.11)
$$
(\mu^*\varphi)_{i,j} = \frac{1}{4S_{i,j}} \{ \varphi_{i+1/2,j+1/2} S_{i+1/2,j+1/2}^* + \varphi_{i-1/2,j+1/2} S_{i-1/2,j+1/2}^*
$$
$$
+ \varphi_{i-1/2,j-1/2} S_{i-1/2,j-1/2}^* + \varphi_{i+1/2,j-1/2} S_{i+1/2,j-1/2}^* \}.
$$

If $(i, j) \in \dot\Gamma_h$, as in Figure 3(b) for instance, formula (4.10) becomes:

$$
(\mu^*\varphi)_{i,j} = \frac{1}{4S_{i,j}} \{ \varphi_{i-1/2,j-1/2} S_{i-1/2,j-1/2}^* + \varphi_{i+1/2,j-1/2} S_{i+1/2,j-1/2}^*
$$
$$
+ \varphi_{i+1/2,j+1/2} S_{i+1/2,j+1/2}^* \}.
$$

The averaging operator μ^* is related to V_h in the following way:

LEMMA 4.2. *Let φ be defined on Ω_h^* and let v_h belong to V_h, then*

(4.12) $[\varphi, v_h]_h = (r_h(\mu^*\varphi), v_h)_h.$ $\square$

The proof is much like that of Proposition 4.1.

4.3. *Some Properties of Difference Quotients and Averages. Regularity Conditions on Q_h.* The next lemma follows easily from the regularity hypothesis H3 on Γ:

LEMMA 4.3. *Let $\varphi \in C^1(\widetilde\Omega)$ and $v_h \in V_h'$. Then, for $i = 1, 2$,*

(4.13) $[r_h^*\varphi \cdot \cos(\vec n_h, x_i), v_h]_{\Gamma_h} = [\varphi \cos(\vec n, x_i), v_h]_{\Gamma_h} + O(h)|v_h|_{\Gamma_h}.$ $\square$

Besides the local regularity condition (3.1) (3.2), let us introduce two additional regularity conditions on Q_h.

Definitions 4.3. (1°) We say that quadrangulation Q_h is *globally regular* if, at every node (i, j) of $\dot\Omega_h$, the following holds:

(4.14) $(\nabla_k^* x_l)_{i,j} = \delta_{kl} + O(h) \quad for\ k, l = 1, 2.$

(2°) Q_h is said to be *regular in the mean* if, at each node (i, j) of $\dot\Omega_h$, we have

(4.15)
$$
S_{i,j} = \frac{1}{4} \{ S_{i+1/2,j+1/2}^* + S_{i-1/2,j+1/2}^* + S_{i-1/2,j-1/2}^* + S_{i+1/2,j-1/2}^* \}
$$
$$
+ O(h^3). \quad \square
$$

PROPOSITION 4.2. *Let $\varphi \in C^2(\widetilde\Omega)$. If Q_h is globally regular, the following equality holds at each node (i, j) of $\dot\Omega_h$:*

(4.16) $(\nabla_k^*\varphi)_{i,j} = (\partial\varphi/\partial x_k)_{i,j} + O(h), \quad k = 1, 2.$ $\square$

The proof of this proposition is straightforward (cf. Girault [4]).

LEMMA 4.4. *Let $\varphi \in C^1(\widetilde\Omega)$ and $v_h \in V_h$. If Q_h is regular in the mean, we have*

(4.17) $[\varphi, v_h]_h = (\varphi, v_h)_h + O(h)|v_h|_h + O(h)|v_h|_{\Gamma_h}.$ $\square$

Proof. According to Lemma 4.2, we must study $\mu^*\varphi$. Let $(i, j) \in \dot\Omega_h$ and let us apply (4.11); we get

$$(\mu^*\varphi)_{i,j} = \frac{1}{4S_{i,j}} \{S^*_{i+1/2,j+1/2} + S^*_{i-1/2,j+1/2}$$

$$+ S^*_{i-1/2,j-1/2} + S^*_{i+1/2,j-1/2}\}(\varphi_{i,j} + O(h)).$$

Since $\mathcal{Q}_h$ is regular in the mean, this implies that

$$(\mu^*\varphi)_{i,j} = (\varphi_{i,j} + O(h))(1 + O(h)).$$

Now, let $(i, j) \in \dot{\Gamma}_h$; for instance, let us consider the case of Figure 3(b), then

$$(\mu^*\varphi)_{i,j} = \frac{1}{4S_{i,j}} \{S^*_{i-1/2,j-1/2} + S^*_{i+1/2,j-1/2} + S^*_{i+1/2,j+1/2}\}(\varphi_{i,j} + O(h)).$$

No assumption has been made on the quotient of areas in the neighborhood of Γ_h, but by virtue of (3.6) and (4.3'), we find that

$$\frac{1}{4S_{i,j}} [S^*_{i-1/2,j-1/2} + S^*_{i+1/2,j-1/2} + S^*_{i+1/2,j+1/2}] \leq \frac{C_0}{c_0}.$$

Obviously, this last inequality holds at any point of $\dot{\Gamma}_h$. Hence,

$$(\mu^*\varphi)_{i,j} = \varphi_{i,j} + \rho_{i,j} \quad \forall(i, j) \in \dot{\Gamma}_h,$$

where $|\rho_{i,j}| \leq C$ and C is independent of h and (i, j). Thus, thanks to (4.12) and (4.1),

$$[\varphi, v_h]_h = \sum_{(i,j)\in\dot{O}_h} (\mu^*\varphi)_{i,j} v_{i,j} S_{i,j}$$

$$= \sum_{(i,j)\in\dot{O}_h} \varphi_{i,j} v_{i,j} S_{i,j} + \sum_{(i,j)\in\dot{\Omega}_h} O(h) v_{i,j} S_{i,j} + \sum_{(i,j)\in\dot{\Gamma}_h} \rho_{i,j} v_{i,j} S_{i,j}.$$

Hence $[\varphi, v_h]_h = (\varphi, v_h)_h + O(h)|v_h|_h + R$, where

$$|R| = \left| \sum_{(i,j)\in\dot{\Gamma}_h} \rho_{i,j} v_{i,j} S_{i,j} \right|.$$

Now, by virtue of (4.3') and (4.4),

$$|R| \leq C \cdot \frac{3}{4} \frac{C_0 h}{\sigma_0} \sum_{(i,j)\in\dot{\Gamma}_h} |v_{i,j}| l_{i,j}.$$

Therefore, $R = O(h)|v_h|_{\Gamma_h}$. $\square$

In the sequel, we shall assume that besides satisfying hypotheses (3.1) to (3.4), $\mathcal{Q}_h$ is also globally regular and regular in the mean.

LEMMA 4.5. Let $\varphi \in C^2(\tilde{\Omega})$ and $v_h \in V_h$. Then, for $i = 1, 2,$

$$(4.18) \qquad (r_h(\nabla_i^*\varphi), v_h)_h = \left(\frac{\partial\varphi}{\partial x_i}, v_h\right)_h + O(h)|v_h|_h + O(h)|v_h|_{\Gamma_h}.   \square$$

The proof, which makes use of Proposition 4.2, is similar to that of the preceding lemma.

We derive the next proposition on combining Proposition 4.1 with Lemmas 4.3 and 4.5.

PROPOSITION 4.3. *Let $\varphi \in C^2(\widetilde{\Omega})$ and $v_h \in V_h$. Then, for $i = 1,2$,*

(4.19)
$$\left[\varphi, \frac{\partial v_h}{\partial x_i}\right]_h = -\left(\frac{\partial \varphi}{\partial x_i}, v_h\right)_h + [\varphi \cos(\vec{n}, x_i), v_h]_{\Gamma_h}$$
$$+ O(h)|v_h|_h + O(h)|v_h|_{\Gamma_h}. \quad \square$$

5. Boundary Value Theorem.

5.1. *An Upper Bound.* Let K be a cell of $\mathcal{Q}_h$ with mean center B_K, as in Figure 2. For the sake of simplicity, we denote by φ_i the value of a function φ at the vertex $a_{i,K}$ of K.

LEMMA 5.1. *Let $v_h \in V_h$. With the above notations, we have*

$$v_1 = v_3 + (x_1 - x_3)\frac{\partial v_h}{\partial x}(B_K) + (y_1 - y_3)\frac{\partial v_h}{\partial y}(B_K),$$

$$v_2 = v_4 + (x_2 - x_4)\frac{\partial v_h}{\partial x}(B_K) + (y_2 - y_4)\frac{\partial v_h}{\partial y}(B_K). \quad \square$$

Now, let N be a positive integer and r an integer such that $0 \leqslant r \leqslant N - 1$. Let us introduce a function of V_h, $\theta_{h,r}$, defined on K by:

$$(\theta_{h,r})_2 = (\theta_{h,r})_3 = 1 - \frac{r+1}{N}, \qquad (\theta_{h,r})_1 = (\theta_{h,r})_4 = 1 - \frac{r}{N},$$

and, for $r \geqslant 1$, a function of V_h, $\bar{\theta}_{h,r}$, which coincides with $\theta_{h,r}$ at the vertices of K except at node $a_{1,K}$ (or $a_{4,K}$) where $\bar{\theta}_{h,r} = 1 - (r-1)/N$. This type of function has already been used by Cea (cf. Cea [2]) in the proof of boundary value theorems in the case of finite differences on regular meshes.

LEMMA 5.2. *Let $v_h \in V_h$ and $0 \leqslant r \leqslant N - 1$. Then, for $i = 1, 2$,*

(5.1)
$$\left.\begin{array}{l}\left|\left[\dfrac{\partial}{\partial x_i}(\theta_{h,r}v_h^2)\right](B_K)\right| \\[4mm] \left|\left[\dfrac{\partial}{\partial x_i}(\bar{\theta}_{h,r}v_h^2)\right](B_K)\right|\end{array}\right\} \begin{array}{l} < \dfrac{3}{4Nc_0h}(v_1^2 + v_2^2 + v_3^2 + v_4^2) \\[4mm] + \dfrac{1}{2}(|v_1| + |v_2| + |v_3| + |v_4|)\left|\dfrac{\partial v_h}{\partial x_i}(B_K)\right| \quad (r \neq 1).\end{array}$$

Proof. The lemma follows immediately from the fact that

$$\bar{\theta}_{h,r}(B_K) < \theta_{h,r}(B_K) < 1,$$

$$\left|\frac{\partial}{\partial x_i}\theta_{h,r}(B_K)\right| \leqslant \frac{2}{Nc_0h} \quad \text{and} \quad \left|\frac{\partial}{\partial x_i}\bar{\theta}_{h,r}(B_K)\right| \leqslant \frac{3}{Nc_0h} \quad (r \neq 1). \quad \square$$

Definitions 5.1. (1°) Let M be a vertex of $\mathcal{Q}_h$ and k a positive integer. An *elementary path of length k and origin M* in $\mathcal{Q}_h$ is a broken oriented line, without double point, starting at M, and consisting of k consecutive and distinct diagonals of cells of $\mathcal{Q}_h$. The endpoints of these diagonals are called *vertices* of the path. We agree to number the vertices from 0 to k, beginning with M.

(2°) A *string of length k* in $\mathcal{Q}_h$ is an ordered sequence of k distinct cells of $\mathcal{Q}_h$, $K_1, K_2, \ldots, K_k$, such that K_i and K_{i+1} share a *common side or vertex*, for $1 \leqslant i \leqslant k - 1$. $\square$

Remark 5.1. The set of cells crossed by an elementary path of length k and ordered in a natural fashion by the orientation of the path forms a string of length k. □

LEMMA 5.3. *Let $v_h \in V_h$. Let M be a vertex of Q_h, N a positive integer and P an elementary path of length N and origin M in Q_h. Let $K_0 K_1 \cdots K_{N-1}$ be the string of cells crossed by P and let (i_p, j_p) be the indices of the mean center of K_p. Then,*

$$(5.2) \quad v_h^2(M) \leq h \sum_{p=0}^{N-1} \left\{ \left(\sum_{r=i_p \pm 1/2; s=j_p \pm 1/2} |v_{r,s}| \right) \left(\left| \frac{\partial v_h}{\partial x} \right|_{i_p,j_p} + \left| \frac{\partial v_h}{\partial y} \right|_{i_p,j_p} \right) + \frac{3}{Nc_0 h} \left(\sum_{r=i_p \pm 1/2; s=j_p \pm 1/2} v_{r,s}^2 \right) \right\}.$$

Proof. Let $M_0 = M, M_1, M_2, \ldots, M_N$ denote the vertices of P and let Θ_h be a function of V_h defined on $K_0, K_1, \ldots, K_{N-1}$ by:

$$\Theta_h(M_r) = 1 - r/N \quad \text{and} \quad \left.\begin{array}{c} \\ \end{array}\right\}, \quad 0 \leq r \leq N,$$
$$\Theta_h = \theta_{h,r} \quad \text{or} \quad \Theta_h = \bar{\theta}_{h,r} \quad \text{on } K_r$$

where, for obvious continuity reasons, $\bar{\theta}_{h,r}$ is used instead of $\theta_{h,r}$ when K_r, K_{r-1} and K_{r-2} share a common vertex $(r \geq 2)$.

Let $w_h = v_h^2 \Theta_h$. Then

$$(5.3) \quad w_h(M_0) = v_h^2(M) \quad \text{and} \quad w_h(M_N) = 0.$$

Applying Lemma 5.1 to K_p, for $0 \leq p \leq N-1$, we find:

$$|w_h(M_p)| \leq |w_h(M_{p+1})| + 2h \left(\left| \frac{\partial}{\partial x} w_h \right|_{i_p,j_p} + \left| \frac{\partial}{\partial y} w_h \right|_{i_p,j_p} \right).$$

It can be easily shown by induction that this inequality and (5.3) yield

$$v_h^2(M) \leq 2h \sum_{p=0}^{N-1} \left\{ \left| \frac{\partial}{\partial x} w_h \right|_{i_p,j_p} + \left| \frac{\partial}{\partial y} w_h \right|_{i_p,j_p} \right\}.$$

The desired result is obtained on substituting (5.1) in the last inequality. □

5.2. *Continuity of Boundary Values for Norm $\|\cdot\|_h$ of V_h.*

LEMMA 5.4. *Let C be one of the arcs of Γ. There exist a finite covering cf C, $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_\nu\}$, a set of directions, $\{\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_\nu\}$, and two positive numbers, t and η such that, for $1 \leq i \leq \nu$:*

(1°) *the cylinder of length t borne by Γ_i in direction $\vec{d}_i$ is contained in Ω;*

(2°) *$\Gamma_i \cap \Gamma_{i+1} \neq \emptyset$ $(i \neq \nu)$ and the length of the chord to $\Gamma_i \cap \Gamma_{i+1}$ is greater than η.*

Proof. Let θ_0 be the angle of hypothesis H3 and let us choose an angle λ such that $0 < 2\lambda < \theta_0$. Let $\vec{f} \colon [a, b] \to C$ be the intrinsic parametric representation of C. According to H3, $\vec{f}' \in C^0[a, b]$; hence, there exists a step $\Delta\tau > 0$ such that:

for any τ and τ^* in $[a, b]$, $|\tau - \tau^*| \leqslant \Delta\tau$ implies that the

angle between $\vec{f}'(\tau)$ and $\vec{f}'(\tau^*)$ is smaller than $\frac{1}{2}(\theta_0/2 - \lambda)$.

Let $\Delta\tau \leqslant (b - a)/4$. We shall construct a finite covering of $[a, b]$ with overlapping intervals:

(5.4)
$$\{[a, \tau_1], [\bar{\tau}_1, \tau_2], \ldots, [\bar{\tau}_{\nu-1}, b]\} \quad \text{such that}$$
$$\tau_k - \bar{\tau}_k = \Delta\tau \quad \text{for } 1 \leqslant k \leqslant \nu - 1.$$

A direction $\vec{d}_k$ will be associated with each interval $[\bar{\tau}_{k-1}, \tau_k]$, $1 \leqslant k \leqslant \nu$. Let $\varphi_k(\tau)$ denote the angle between $\vec{d}_k$ and $\vec{f}'(\tau)$ on $[\bar{\tau}_{k-1}, \tau_k]$. The covering and set of directions will be chosen so that

(5.5)
$$\lambda < \varphi_k(\tau) < \pi - \lambda \quad \text{on } [\bar{\tau}_{k-1}, \tau_k] \text{ for } 1 \leqslant k \leqslant \nu.$$

Let the first direction $\vec{d}_1$ (respectively, last direction $\vec{d}_\nu$) be that of the interior bisector to the angle between the tangents at the intersection of arc C and its preceding arc (respectively, following arc). Then

$$\frac{\theta_0}{2} \leqslant \varphi_1(a) \leqslant \pi - \frac{\theta_0}{2} \quad \text{and} \quad \frac{\theta_0}{2} \leqslant \varphi_\nu(b) \leqslant \pi - \frac{\theta_0}{2}.$$

Therefore,

$$\lambda < \varphi_1(a) < \pi - \lambda \quad \text{and} \quad \lambda < \varphi_1(b) < \pi - \lambda.$$

The choice of $\Delta\tau$ implies that the first inequality holds on an interval $[a, \tau_1']$ and the second on an interval $[b', b]$, where:

$$\tau_1' - a \geqslant 2\Delta\tau \quad \text{and} \quad b - b' \geqslant 2\Delta\tau.$$

- If $\tau_1' < b' + \Delta\tau$, we take $\tau_1 = \tau_1'$, $\bar{\tau}_1 = \tau_1 - \Delta\tau$, and we choose for $\vec{d}_2$ the direction of the interior normal to C at $\bar{\tau}_1$. Then (5.5) certainly holds at $\bar{\tau}_1$ (since $\varphi_2(\bar{\tau}_1) = \pi/2$), and we can repeat the process with starting point $\bar{\tau}_1$ and direction $\vec{d}_2$. Since the length of each interval is greater than $2\Delta\tau$, we reach $b' + \Delta\tau$ after a finite number of steps.
- If $\tau_k' \geqslant b' + \Delta\tau$, we end the process by taking $\nu = k + 1$, $\tau_{\nu-1} = \max(b' + \Delta\tau, a + 2\Delta\tau)$ and $\bar{\tau}_{\nu-1} = \tau_{\nu-1} - \Delta\tau$.

By construction, this covering satisfies (5.4) and (5.5).

Now, let Γ_k denote the image by $\vec{f}$ of $[\bar{\tau}_{k-1}, \tau_k]$, $1 \leqslant k \leqslant \nu$. Then, thanks to (5.5) and the choice of direction $\vec{d}_k$, there exists a number $t_k > 0$ such that the cylinder of length t_k borne by Γ_k in direction $\vec{d}_k$ is contained in Ω. Moreover, (5.4) implies that $\Gamma_k \cap \Gamma_{k+1}$ is an arc of C with chord of positive length, say η_k, for $1 \leqslant k \leqslant \nu - 1$. Since ν is finite, the lemma is proved with

$$t = \min_{1 \leqslant k \leqslant \nu} t_k \quad \text{and} \quad \eta = \min_{1 \leqslant k \leqslant \nu - 1} \eta_k. \quad \square$$

Remarks 5.2. (1°) ν, t and η depend upon C, but the angle λ does not and may be chosen for the entire boundary Γ.

(2°) If C is reasonably smooth, ν is small (cf. Figure 4). In particular, $\nu = 2$ when C is a straight line. $\square$
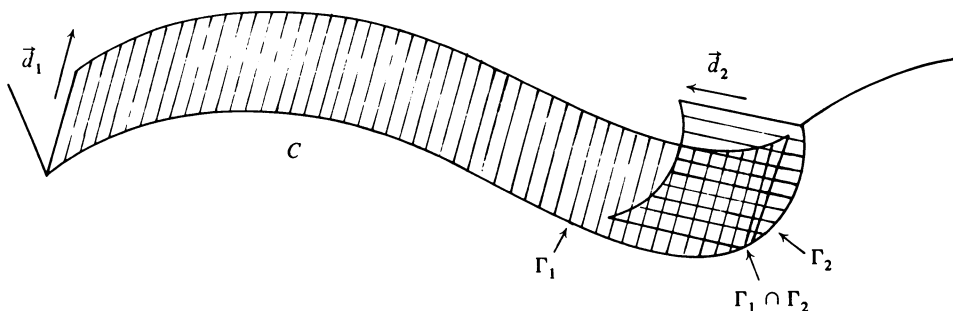
FIGURE 4

Since Γ consists of a finite number of arcs, we derive the next theorem immediately from Lemma 5.4.

THEOREM 5.1. *There exist a finite covering of* Γ, $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\}$, *a set of directions,* $\{\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_n\}$, *and two positive numbers* l *and* h_0 *such that, for* $1 \leqslant i \leqslant n$:

($1°$) *the cylinder of length* l *borne by* Γ_i *in direction* $\vec{d}_i$ *is contained in* Ω;

($2°$) $\Gamma_i \cap \Gamma_{i+1} \neq \varnothing$ *and, except at the angular points of* Γ, *the length of the chord to* $\Gamma_i \cap \Gamma_{i+1}$ *is greater than* h_0. $\square$

COROLLARY 5.1. *Let* h *be small enough. Then, for each pair of adjacent nodes,* M *and* P, *of* $\dot{\Gamma}_h$, *there exists a direction* $\vec{d}$ *belonging to the set* $\{\vec{d}_1, \ldots, \vec{d}_n\}$ *such that the parallelogram borne by* $\overline{MP}$, *with length* $l/2$ *in direction* $\vec{d}$, *is contained in* Ω_h^*. $\square$

This corollary follows easily from Theorem 5.1 and the fact that Ω_h^* approximates Ω.

From now on, we shall assume that h is sufficiently small for Corollary 5.1 to hold. Note that, for most domains, this is not a severe restriction.

LEMMA 5.5. *Let* M *and* P *be two adjacent nodes of* $\dot{\Gamma}_h$ *such that (for example)* $\overrightarrow{MP}$ *is oriented in the positive direction of integration. Let* T_M *denote the parallelogram of Corollary* 5.1. *There exists an elementary path* P_M *of length* N *and origin* M *such that:*

- *each cell of the string associated with* P_M *cuts* T_M;

(5.6)   • $Nh > l/4$.

Proof. We only sketch the proof. It proceeds in two steps:

($1°$) We construct a string $K_0 K_1 \cdots K_p$ of cells such that each K_i cuts T_M, K_i and K_{i+1} share a common side, $\overrightarrow{PM}$ is a side of K_0 and finally, the distance from M to the center of K_p is greater than $l/2$. Denote by K_p' the side of K_p opposite to the common side of K_p and K_{p-1}.

($2°$) We construct an elementary path P_M with origin M, terminal at one of the extremities of K_p', and crossing a substring of $K_0 K_1 \cdots K_p$. Let N be the length of P_M. By construction, each cell crossed by P_M cuts T_M. Finally, since the length of each diagonal is smaller than $2h$, we have $Nh > l/4$. $\square$

THEOREM 5.2. *There exists a constant* C, *independent of* h, *such that*

(5.7) $|v_h|_{\Gamma_h} \leqslant C \|v_h\|_h \quad \forall v_h \in V_h.$

Proof. According to (4.2), $|v_h|^2_{\Gamma_h} = \Sigma_{M \in \dot{\Gamma}_h} v^2_h(M) l_M$.

Let M be a node of $\dot{\Gamma}_h$. Let T_M be the parallelogram and P_M a path of origin M satisfying the properties of the preceding lemma. Let N be the length of P_M. Applying Lemma 5.3 and inequality (5.6), we get:

$$v^2_h(M) l_M \leqslant h^2 \sum_{p=0}^{N-1} \left\{ \left(\sum_{r=i_p \pm 1/2 ; s=j_p \pm 1/2} |v_{r,s}| \right) \left(\left| \frac{\partial v_h}{\partial x} \right|_{i_p,j_p} + \left| \frac{\partial v_h}{\partial y} \right|_{i_p,j_p} \right) \right.$$
$$\left. + \frac{12}{c_0 l} \left(\sum_{r=i_p \pm 1/2 ; s=j_p \pm 1/2} v^2_{r,s} \right) \right\},$$

$$v^2_h(M) l_M \leqslant \frac{2}{c^2_0} \left[\left(\sum_{p=0}^{N-1} v^2_{i_p+1/2, j_p+1/2} S_{i_p+1/2, j_p+1/2} \right)^{1/2} \right.$$

$$+ \left(\sum_{p=0}^{N-1} v^2_{i_p-1/2, j_p+1/2} S_{i_p-1/2, j_p+1/2} \right)^{1/2}$$

$$+ \left(\sum_{p=0}^{N-1} v^2_{i_p-1/2, j_p-1/2} S_{i_p-1/2, j_p-1/2} \right)^{1/2}$$

$$+ \left(\sum_{p=0}^{N-1} v^2_{i_p+1/2, j_p-1/2} S_{i_p+1/2, j_p-1/2} \right)^{1/2} \right] \cdots$$

$$\cdots \times \left[\left(\sum_{p=0}^{N-1} \left| \frac{\partial v_h}{\partial x} \right|^2_{i_p,j_p} S_{i_p,j_p} \right)^{1/2} + \left(\sum_{p=0}^{N-1} \left| \frac{\partial v_h}{\partial y} \right|^2_{i_p,j_p} S_{i_p,j_p} \right)^{1/2} \right]$$

$$+ \frac{48}{c^2_0 l} \left[\sum_{p=0}^{N-1} \left(\sum_{r=i_p \pm 1/2 ; s=j_p \pm 1/2} v^2_{r,s} S_{r,s} \right) \right].$$

In order to derive an upper bound for $|v_h|^2_{\Gamma_h}$, the last inequality must be summed over all M of $\dot{\Gamma}_h$. In doing so, one must evaluate the maximum number of paths, corresponding to the different nodes of $\dot{\Gamma}_h$, that can cross a given cell. Now, to each M, there corresponds a different path P_M and a different parallelogram T_M such that each cell crossed by P_M cuts T_M. Hence, it suffices to evaluate the maximum number of parallelograms that can be cut by a given cell. The width of a cell is always smaller than $2h$, and, according to (4.4) and (5.5), the width of a parallelogram T_M is always greater than $\sigma_0 h \sin \lambda$. Therefore, a cell can intersect at most $2/\sigma_0 \sin \lambda$ parallelograms in one direction. As there are n directions, it follows that a cell can be crossed at most by $2n/\sigma_0 \sin \lambda$ different paths. This constant is independent of h. Therefore,

$$|v_h|^2_{\Gamma_h} \leqslant \frac{8n}{c^2_0 \sigma_0 \sin \lambda} \left[2 \sum_{M \in \dot{O}_h} v^2_h(M) S_M + \sum_{K \in \mathcal{Q}_h} \left\{ \left(\frac{\partial v_h}{\partial x} (B_K) \right)^2 + \left(\frac{\partial v_h}{\partial y} (B_K) \right)^2 \right\} S_K \right.$$

$$\left. + \frac{48}{l} \sum_{M \in \dot{O}_h} v^2_h(M) S_M \right].$$

The theorem is proved since all constants of the right-hand side are independent of h and v_h. $\square$

Remark 5.3. From Theorem 5.2 and Lemma 3.5 we derive property (2.4) of L_h:

$$|L_h(v_h)| \leqslant \mu\|v_h\|_h \quad \forall v_h \in V_h,$$

where μ is a constant independent of h. $\square$

6. Error Estimate and Conclusion.

6.1. *Order of the Approximation*. The results of the preceding paragraphs are summed up in the next theorem which shows that our approximation is of order one.

THEOREM 6.1. *Under the following hypotheses on the continuous problem* (1.1) (1.2):

(1°)
$$\begin{cases} \textit{ellipticity of coefficients: } H1, \\[2mm] \|b_i\|_{L^\infty(\Omega)} \textit{ sufficiently small,} \end{cases}$$

(2°)
$$\begin{cases} \textit{regularity of coefficients: } H2, \\[2mm] \textit{regularity of boundary: } H3, \\[2mm] \textit{regularity of solution: } u \in C^3(\bar{\Omega}), \end{cases}$$

and the following hypotheses on the quadrangulation Q_h:

(3°)
$$\begin{cases} \textit{local regularity: } (3.1) \textit{ and } (3.2), \\[2mm] \textit{global regularity: } (4.14), \\[2mm] \textit{mean regularity: } (4.15), \end{cases}$$

(4°)   *good approximation of boundary*: (3.3),

(5°)   *cartesian numbering of nodes*: (3.4),

there exists a constant C, independent of h, such that, for all h sufficiently small,

$$(6.1) \qquad \qquad \|u - u_h\|_h \leqslant Ch,$$

where u_h *denotes the solution of the approximate problem* (2.5)(3.8)(3.9) *or* (2.5) (3.8)(3.9').

Proof. The hypotheses of Theorem 2.1 have been checked in Section 3; therefore, inequality (2.6) holds:

$$(2.6) \quad \|u - u_h\|_h \leqslant C_1 \left\{ \inf_{v_h \in V_h} \|u - v_h\|_h + \sup_{w_h \in V_h} \frac{1}{\|w_h\|_h} |L_h(w_n) - a_h(u, w_h)| \right\},$$

where C_1 is a constant independent of h.

An upper bound for the projection error has been given in Section 3:

(3.16)
$$\inf_{v_h \in V_h} \|u - v_h\|_h \leqslant C_2 h,$$

where C_2 is a constant independent of h.

Therefore, there remains to establish an upper bound for

$$\frac{1}{\|w_h\|_h} |L_h(w_h) - a_h(u, w_h)|, \quad \text{for any } w_h \neq 0 \text{ in } V_h.$$

Now,

$$a_h(u, w_h) = \left[a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y}, \frac{\partial w_h}{\partial x} \right]_h + \left[a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y}, \frac{\partial w_h}{\partial y} \right]_h$$
$$+ \left[b_1 \frac{\partial u}{\partial x} + b_2 \frac{\partial u}{\partial y}, w_h \right]_h + (du, w_h)_h.$$

Combining Proposition 4.3 and Theorem 5.2, we get,

$$a_h(u, w_h) = -\left(\frac{\partial}{\partial x} \left(a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y} \right), w_h \right)_h \cdots$$
$$\cdots + \left[\left(a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y} \right) \cos(\vec{n}, x), w_h \right]_{\Gamma_h}$$
$$- \left(\frac{\partial}{\partial y} \left(a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y} \right), w_h \right)_h$$
$$+ \left[\left(a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y} \right) \cos(\vec{n}, y), w_h \right]_{\Gamma_h} + O(h)\|w_h\|_h \cdots$$
$$\cdots + \left[b_1 \frac{\partial u}{\partial x} + b_2 \frac{\partial u}{\partial y}, w_h \right]_h + (du, w_h)_h, \quad \text{for sufficiently small } h.$$

Let L_h be defined by (3.9). Then

$$L_h(w_h) = [f, w_h]_h + [g, w_h]_{\Gamma_h}.$$

According to (1.1) and (1.2), this yields,

$$L_h(w_h) = -\left[\frac{\partial}{\partial x} \left(a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left(a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y} \right), w_h \right]_h$$
$$+ \left[b_1 \frac{\partial u}{\partial x} + b_2 \frac{\partial u}{\partial y} + du, w_h \right]_h$$
$$+ \left[\left(a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y} \right) \cos(\vec{n}, x), w_h \right]_{\Gamma_h}$$
$$+ \left[\left(a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y} \right) \cos(\vec{n}, y), w_h \right]_{\Gamma_h}.$$

Combining Lemma 4.4 and Theorem 5.2, we get, for h small enough,

$$L_h(w_h) = -\left(\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial y} + a_{22}\frac{\partial u}{\partial y}\right), w_h\right)_h$$

$$+ (du, w_h)_h + O(h)\|w_h\|_h + \left[b_1\frac{\partial u}{\partial x} + b_2\frac{\partial u}{\partial y}, w_h\right]_h \cdots$$

$$\cdots + \left[\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right)\cos(\vec{n}, x), w_h\right]_{\Gamma_h}$$

$$+ \left[\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right)\cos(\vec{n}, y), w_h\right]_{\Gamma_h}.$$

Therefore, for any function w_h in V_h and h sufficiently small,

$$L_h(w_h) = a_h(u, w_h) + O(h)\|w_h\|_h,$$

where the quantity $O(h)$ is independent of w_h.

Hence, if L_h is defined by (3.9) and h is sufficiently small,

$$\sup_{w_h\in V_h; w_h\neq 0} \frac{1}{\|w_h\|_h} |L_h(w_h) - a_h(u, w_h)| \leqslant C_3 h,$$

where C_3 is a constant independent of h.

Similarly, it can be shown that the same inequality holds, with a different constant, when L_h is defined by (3.9′). Hence, the theorem is proved. □

When the solution and coefficients do not satisfy the regularity hypotheses included in n°2, we cannot show the error estimate (6.1). However, if we replace these by:

$$u \in H^1(\Omega), \quad a_{ij}, b_i, d \text{ and } f \in C^0(\bar{\Omega}) \quad and \quad g \in C^0(\Gamma),$$

it can be shown, by a finite-difference approach, that

$$\lim_{h\to 0} \left(u_h, \frac{\partial u_h}{\partial x}, \frac{\partial u_h}{\partial y}, u_h\Big|_{\Gamma_h}\right) = \left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, u\Big|_\Gamma\right)$$

in appropriate discrete L^2 norms.

All the regularity hypotheses on Q_h have already been encountered and discussed in our previous paper (cf. Girault [4]). Nevertheless, the following points are worth stressing. The two conditions, global regularity and mean regularity, do not occur in the classical finite element theory. They are an inevitable consequence of the non-ellipticity of our approximation. In other words, failure to satisfy Strang's ellipticity condition must be compensated by additional regularity on the quadrangulation. It is interesting to note that these two hypotheses are satisfied when the cells of Q_h are nearly parallelograms, but this condition is not necessary.

6.2. *Conclusion.* From a practical point of view, the method we have studied in this paper is, by far, the simplest that can be derived in using quadrilateral finite elements: it leads to a nine-point scheme whose coefficients can be computed directly by a simple formula. Provided the quadrangulation is not too distorted, this method is of order one, a precision which is adequate in many applications. The important point of this paper has been to show that such a method could handle problems involving *surface integrals* without loss of accuracy.

The study of nonelliptic approximation is now being extended to nonlinear problems.

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