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Complex Roots of $\sin z = az$, $\cos z = az$, and $\cosh z = az$

By Henry E. Fettis

Abstract. Values of the first five complex roots of the equations $\sin z = az$, $\cos z = az$ and $\cosh z = az$ are given to 10S, for $a = 10, 5, 2, 1.6, 1.2, 1(.1).3$, and selected values below.

Introduction. If "a" is a real parameter, the equations

$$\sin z = az, \tag{1}$$

$$\cos z = az, \tag{2}$$

$$\cosh z = az, \tag{3}$$

all have a finite number of real roots and an infinite number of pairs of conjugate complex ones. Since Eq. (1) is unchanged if z is replaced by $-z$, the complex roots of (1) are symmetrically situated in each of the four quadrants of the complex z-plane. Replacing z by $-z$ in Eqs. (2) and (3) is equivalent to replacing "a" by "$-a$". Thus the roots of these equations which lie in the first and fourth quadrants for positive "a" are identical to those in the second and fourth quadrants for "a" negative.

The first ten complex roots of Eq. (1) for $a = \pm 1$ were first given by Fadle [2] to 5D and later by Hillman and Salzer [1] to 10D, by Mittelman and Hillman [3] to 7D and by Robbins and Smith [4] to 6D. The most extensive tabulation is that of Ling [7], who computed the first 100 roots of Eqs. (1), (2) and (3) to 11D. Tabulations of the roots of Eq. (1) for values of "a" other than ± 1 are few and difficult to obtain. Notable are those of Ricci [5] which appear in *La Rivesta di Ingegneria*, No. 2 (Milan), and those of Coghlan and Smith [6] which have been deposited in the UMT file of the present journal. Another table by Sidney Johnson which gives only the real roots is mentioned in the UMT section of *Mathematical Tables and Other Aids to Computation* (v. 4, no. 31, July, 1950) but is virtually unattainable.

As far as the author has been able to determine, no tables of the roots of either (2) or (3) other than for $a = \pm 1$ have been published. Miller [8] gives six roots of Eq. (2) for $a = 1$, lying in the first quadrant, but only three of these, namely the second, fourth and sixth are correct, due to an oversight which introduced three extraneous roots belonging in actuality to the equation $\cos z = \bar{z}$. Later Silberstein [9] gave three roots of this equation lying in the second and third quadrants which were overlooked by Miller, but failed to note those in the first and fourth quadrants. Both Miller's and Silberstein's results may be found in [11, p. 96].

The most complete tabulation of the roots of Eqs. (1), (2) and (3) for $a = \pm 1$ is that of Ling [7].

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1. The Equation $\sin z = az$. It is evident that $z = 0$ is always a simple root of this equation and that for $a = 1$, it is a threefold root. It is also clear from graphical considerations that for $a > 1$, no other real roots exist, so that for this range of a, all zeros of Eq. (1) occur in complex conjugate pairs, lying symmetrically in the four quadrants of the z-plane. For this last reason, future discussions will be restricted to roots in the first quadrant.

The original complex equation (1) can be rewritten as two real equations:

$$\begin{cases} \sin x \cosh y = ax, & \text{(1a)} \\ \cos x \sinh y = ay, & \text{(1b)} \end{cases}$$

where $z = x + iy$. If $x = 0$, Eq. (1a) is automatically satisfied, while Eq. (1b) will also be satisfied provided

$$\sinh y = ay. \tag{4}$$

For any $a > 1$, this equation has a single pair of nonzero real roots which are equal and opposite in sign, and which coincide with the root $z = 0$ when $a = 1$. Thus Eq. (1) has, for any $a > 1$, a pair of roots located symmetrically on the imaginary axis, in addition to the complex roots noted earlier.

As a is decreased below unity, the triple root becomes again a simple zero, and two real roots appear (equal and opposite in sign) lying in the interval $0 < x < \pi$, all subsequent roots remaining complex. As a continues to decrease, a value is reached where the imaginary part of the first complex pair vanishes. This value of a is determined by

$$a = \cos \alpha, \tag{5}$$

where α is the smallest positive root of the equation

$$\theta = \tan \theta. \tag{6}$$

The nonzero real part of this double root has the value

$$x = \cos^{-1} a + 2\pi, \tag{7}$$

where the principal value of the inverse cosine is understood. As a is decreased further, two additional real roots appear, lying in the interval $2\pi \leqslant x \leqslant 3\pi$, the remaining roots being complex. This continues until a attains the value corresponding to the second root of Eq. (6) for which $\cos \alpha > 0$ (the third root of this equation in order of increasing magnitude) at which value there will be another double root. This continues indefinitely, with more real roots being added as each successive zero of Eq. (6) with $\cos \alpha > 0$ is reached, until finally, for $a = 0$, all roots are real. The real parts of the double roots are given by

$$x = \cos^{-1} a + 2n\pi, \quad n = 1, 2, \ldots . \tag{8}$$

For negative values of a, all roots are complex, until a reaches the value determined by the first root of Eq. (6) for which $\cos \alpha < 0$ (the second root of (6) in order of increasing magnitude). As before, for this value of a, the first pair of complex roots coincide at a double root determined by

$$x = \cos^{-1}|a| + \pi; \tag{9}$$

and, as previously, there is an infinite sequence of such double roots, given by

$$x = \cos^{-1}|a| + (2n-1)\pi, \qquad n = 1, 2, \ldots . \tag{10}$$

2. **The Equation** $\cos z = az$. The roots of this equation in the right half-plane are identical to those of the same equation in the left half-plane with "a" replaced by "$-a$". Thus, again, it suffices to restrict the discussion to positive x. For $a > 0$, there is always one real root, lying in the interval $0 < x < \pi/2$. The remaining ones are all complex until a reaches the value $-\sin\alpha$, where α is the first root of

$$\cot\theta + \theta = 0 \tag{11}$$

for which $\sin\alpha$ is negative. Other double roots are found at the higher roots of (11) for which $\sin\alpha < 0$, and are given by

$$x = -\sin^{-1}|a| + 2n\pi, \qquad n = 1, 2, \ldots . \tag{12}$$

As each double root is passed, a pair of real roots is added, until, for $a = 0$, all roots are real.

Double roots for negative "a" are found when "a" takes on the values corresponding to the roots of (11) for which $\sin\alpha > 0$. The appropriate values of x in this case are

$$x = -\sin^{-1}|a| + (2n-1)\pi, \qquad n = 1, 2, \ldots . \tag{13}$$

3. **The Equation** $\cosh z = az$. For $a > 0$, this equation has one double root which occurs when "a" has the value $\sinh\alpha$, α being the positive root of

$$\coth\theta = \theta. \tag{14}$$

For values of "a" above this value, there are two real roots of (3), the remaining ones being complex. For $a < 0$, all of the roots are complex. Also, because of symmetry, the roots in the left half-plane for $a > 0$ are identical to those in the right half-plane when $a < 0$.

4. **Method of Computation.** The roots of Eq. (4) may be found, e.g., by successive Newtonian approximations:

$$y_{k+1} = \frac{y_k \cosh y_k - \sinh y_k}{\cosh y_k - a}. \tag{15}$$

For the complex roots, Eqs. (1a) and (1b) are solved by successive approximations as follows:

$$y_{k+1} = \cosh^{-1}\left[\frac{a(n\pi + \theta_k)}{\sin\theta_k}\right], \qquad n = 1, 2, \ldots , \tag{16}$$

where

$$\theta_k = \cos^{-1}\left[\frac{ay_k}{\sinh y_k}\right], \qquad 0 \leqslant \theta_k \leqslant \pi/2. \tag{17}$$

In the above, even values of n gives the roots for $a > 0$ and odd values the roots for $a < 0$. As an initial assumption, the value

$$y_0 = \cosh^{-1}[a(n + \tfrac{1}{2})\pi] \tag{18}$$

may be used. However, a somewhat better starting value may be obtained by linear interpolation of $\cosh y$ between the already computed results for $a = 1$ as given in [7] and the value of "a" for which $y = 0$, as given by Eq. (5). When convergence has been obtained for y, the corresponding value of x is given by

$$x = \theta + n\pi. \tag{19}$$

For Eq. (2), a similar sequence is formed:

$$y_{k+1} = \cosh^{-1}\left[\frac{a(n\pi - \theta_k)}{\cos\theta_k}\right], \qquad n = 1, 2, \ldots, \tag{20}$$

$$\theta_k = \sin^{-1}\left[\frac{ay_k}{\sinh y_k}\right]. \tag{21}$$

As before, either the value

$$y_0 = \cosh^{-1}(n\pi a) \tag{22}$$

or that obtained by linear interpolation of ($\cosh y$) may be used as an initial assumption. The final value of x is found by

$$x = n\pi - \theta. \tag{23}$$

The corresponding sequences for Eq. (3) are

$$x_{k+1} = \sinh^{-1}\left[\frac{a(n\pi + \theta_k)}{\sin\theta_k}\right], \tag{24}$$

$$\theta_k = \cos^{-1}\left[\frac{ax_k}{\cosh x_k}\right], \qquad 0 < \theta_k < \pi/2, \tag{25}$$

with

$$x_0 = \sinh^{-1}[a(n + \tfrac{1}{2})\pi] \tag{26}$$

and, after convergence,

$$y = n\pi + \theta. \tag{27}$$

5. Numerical Results. Tables of numerical values of the roots of Eqs. (1), (2) and (3) are to be found in the microfiche supplement of this issue. Table 1 gives the values of the parameter "a" for which these equations have double roots, together with the real parts of the corresponding roots. In computing these, use was made of the appropriate roots of Eqs. (6), (11) and (14) as given in [10].

Table 2 contains the values of the pure imaginary roots of Eq. (1), for selected values of "a" between 1 and 10.

Tables 3 through 22 give the complex roots of Eqs. (1) and (2) for the following values of "a":

$$10, 5, 2, 1.6, 1.2, 1.1; 1.0(-.1).3$$

and for selected values below the last value in the neighborhood of the double root, which in each table constitutes the final entry. The final value of "a" (indicated by * in the tables) is given to only 3S, since more accurate values are contained in Table 1.

Tables 23 through 32 give the complex roots of Eq. (3) for $a = 10, 5, 2, 1.6, 1.2, 1.1(-.1).1, .05.0$, with the exception of the first root for which "a" ranges from 0 to 1.5, and terminates with the upper limiting value 1.508879561... for which the imaginary part of the complex pair vanishes.

The calculations were made by the author on a Hewlett-Packard HP-35 pocket calculator which displays only 10S. For this reason the terminal digits are subject to verification.

Tables 3 through 32 also contain the quantities $\cosh y$ (Eqs. (1) and (2)) and $\sinh x$ (Eq. (3)) which are very nearly linear functions of "a".

1885 California, Apt. 62
Mountain View, California 94041

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