

# Mathematics of Computation

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MAY 1974

*'Very important mathematics  
is being spoken here'*

— Paul R. Halmos

MATHEMATICAL DEVELOPMENTS  
ARISING FROM  
HILBERT PROBLEMS

EDITED BY  
FELIX E. BROWDER

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