MATHEMATICS OF COMPUTATION, VOLUME 30, NUMBER 136
OCTOBER 1976, PAGES 739–746

A Necessary Condition for A-Stability of Multistep Multiderivative Methods

By Rolf Jeltsch

Abstract. The region of absolute stability of multistep multiderivative methods is studied in a neighborhood of the origin. This leads to a necessary condition for A-stability. For methods where $\rho(\zeta)/(\zeta - 1)$ has no roots of modulus 1 this condition can be checked very easily. For Hermite interpolatory and Adams type methods a necessary condition for A-stability is found which depends only on the error order and the number of derivatives used at (x_{n+k}, y_{n+k}).

1. Introduction and Results. A multistep method using higher derivatives for solving the initial value problem $y' = f(x, y), y(a) = \eta$ is given by

$$(1) \quad \sum_{i=0}^{k} \alpha_i y_{n+i} - \sum_{j=1}^{l} h^j \sum_{i=0}^{k} \beta_{ji} f^{(j)}(x_{n+i}, y_{n+i}) = 0, \qquad n = 0, 1, 2, \ldots .$$

α_i, β_{ji} are real constants, $\alpha_k \neq 0$, $\Sigma_{i=0}^k |\beta_{li}| \neq 0$, $|\alpha_0| + \Sigma_{j=1}^l |\beta_{j0}| \neq 0$, $x_n = a + nh$, $h > 0$, and

$$f^{(1)}(x, y) = f(x, y);$$

$$f^{(j+1)}(x, y) = \frac{\partial f^{(j)}(x, y)}{\partial x} + f(x, y) \frac{\partial f^{(j)}(x, y)}{\partial y}, \qquad j = 1, 2, \ldots, l - 1.$$

It is well known that the method has order p if

$$(2) \qquad \rho(e^z) - \sum_{j=1}^{l} z^j \sigma_j(e^z) = \sum_{j=p+1}^{\infty} C_j z^j, \qquad C_{p+1} \neq 0,$$

where $\rho(\zeta)$ and $\sigma_j(\zeta)$ are the polynomials

$$\rho(\zeta) = \sum_{i=0}^{k} \alpha_i \zeta^i, \qquad \sigma_j(\zeta) = \sum_{i=0}^{k} \beta_{ji} \zeta^i, \qquad j = 1, 2, \ldots, l.$$

We shall always assume that the polynomials ρ and σ_j, $j = 1, 2, \ldots, l$, have no common factor. The method is convergent if and only if $p \geqslant 1$ and $\rho(\zeta)$ is a simple von Neumann polynomial; that is, if ζ is a root of $\rho(\zeta)$, then $|\zeta| \leqslant 1$; and if $|\zeta| = 1$, then it is a simple root (see R. Jeltsch [8]).

If the multistep method (1) is applied to the test equation $y' = \mu y$, $y(0) = 1$, μ complex, then (1) is a linear recurrence relation with constant coefficients. The corresponding characteristic equation is

---

Received June 23, 1975.

AMS (MOS) subject classifications (1970). Primary 65L05.

Key words and phrases. Linear k-step methods using higher derivatives, behavior of the region of absolute stability at the origin, necessary condition for A-stability, Hermite interpolatory multistep multiderivative methods, Adams-type multistep multiderivative methods.

$$(3) \qquad \rho(\zeta) - \sum_{j=1}^{l} z^j \sigma_j(\zeta) = 0, \qquad z = \mu h.$$

For each z, (3) has k roots $\zeta_i(z)$, $i = 1, 2, \ldots, k$. The set $A = \{z \mid |\zeta_i(z)| < 1, i = 1, 2, \ldots, k\}$ is called the region of absolute stability. Let $\partial A = \bar{A} - A$, where $\bar{A}$ is the closure of A. A method is called A-stable if A contains the whole left-hand plane $\operatorname{Re} z < 0$.

In several articles the boundary ∂A of A has been plotted in order to determine whether a method is A-stable or not, see Brown [1], Enright [4], Jeltsch [7]. However, if all growth parameters λ_j, given by (4), are positive, then ∂A will be extremely close to the imaginary axis for z close to 0. Roundoff errors may defeat the attempt to determine whether ∂A is in a neighborhood of $z = 0$ in $H^+ = \{z \in \mathbf{C} \mid \operatorname{Re} z > 0\}$ or in $H^- = \{z \in \mathbf{C} \mid \operatorname{Re} z < 0\}$. Our results fill this gap. In particular, we shall find a necessary condition for A-stability. It should be noted that a method which violates this condition may still behave numerically almost like an A-stable method even though it is not A-stable. In Section 2 this necessary condition for A-stability is applied to Hermite interpolatory and Adams-type multistep multiderivative methods; and it is found that these cannot be A-stable if the error order p is equal to $2l_k + 1$ modulo 4, where

$$l_k = \begin{cases} 0 & \text{if } \displaystyle\sum_{j=1}^{l} |\beta_{jk}| = 0, \\[2ex] t & \text{if } \displaystyle\sum_{j=t+1}^{l} |\beta_{jk}| = 0 \text{ and } \beta_{tk} \neq 0. \end{cases}$$

The proofs are given in Section 3.

Let ζ_j, $j = 1, 2, \ldots, s$, be the roots of $\rho(\zeta)$ with modulus 1. Let us introduce the growth parameters

$$(4) \qquad \lambda_j = \sigma_1(\zeta_j)/\zeta_j \rho'(\zeta_j), \qquad j = 1, 2, \ldots, s,$$

and

$$(5) \quad \mu_j = \frac{1}{\zeta_j \rho'(\zeta_j)} \left(\sigma_2(\zeta_j) + \zeta_j \lambda_j \sigma_1'(\zeta_j) - \frac{1}{2} \zeta_j^2 \lambda_j^2 \rho''(\zeta_j) \right), \qquad j = 1, 2, \ldots, s.$$

Furthermore, let the method have order $p \geqslant 1$. Then we define recursively

$$(6) \qquad c_j = \left(C_j - \sum_{i=1}^{j-p-1} c_{j-i} s_i \right) \Big/ s_0, \qquad j = p+1, p+2, \ldots, 2p,$$

where $s_0, s_1, \ldots, s_{p-1}$ are given by

$$(7) \qquad \sum_{j=1}^{l} j z^{j-1} \sigma_j(e^z) = \sum_{i=0}^{p-1} s_i z^i + O(z^p).$$

For the disk $\{z \in \mathbf{C} \mid |z| < R\}$ we shall use the symbol $D(R)$.

THEOREM 1. *Let the multistep method of form* (1) *be convergent, of order* $p \geqslant 1$ *and let* $\rho(\zeta)$ *have* s *roots of modulus* 1, ζ_i, $i = 1, 2, \ldots, s$, *with* $\zeta_1 = 1$. *Let* λ_i *be real and positive*, $i = 1, 2, \ldots, s$; *and define*

$$\delta = \begin{cases} 1, & s = 1, \\ \min\limits_{j=2,3,\ldots,s} \{2 \operatorname{Re} \mu_j - \lambda_j^2\}, & s \geqslant 2, \end{cases}$$

where λ_j *and* μ_j *are given by* (4) *and* (5), *respectively. Assume that one of the conditions* (I), (II_1)–(II_4) *holds, where*

(I) $\delta < 0$.

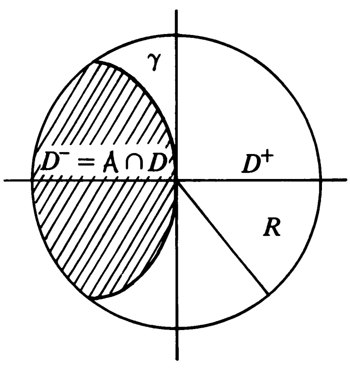
(II_1) $\delta > 0$, *p odd*, $c_{p+1}(-1)^{(p+1)/2} > 0$.

(II_2) $\delta > 0$, *p even*, $c_{p+2q}(-1)^{(p/2)+q} > 0$, $c_{p+2j} = 0, j = 1, 2, \ldots, q-1$, *for some* $q \leqslant p/2$.

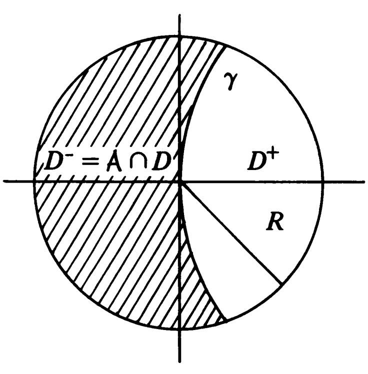
(II_3) $\delta > 0$, *p odd*, $c_{p+1}(-1)^{(p+1)/2} < 0$.

The numbers c_j, $j = p+1, p+2, \ldots, 2p$, *are given by* (6). *Then there exists a disk* $D = D(R)$, $R > 0$, *such that* $\gamma = \partial A \cap D$ *is a continuously differentiable curve which intersects the real axis and the imaginary axis only at* $z = 0$. *The imaginary axis is tangent to* γ *at* $z = 0$. γ *divides* D *in two simply connected regions* $D^- = A \cap D$ *and* $D^+ = D - D^-$, *see Figure* 1. *Moreover, each of the conditions* (I), (II_3), (II_4) *implies that* $D^- \subset H^-$ *while each of the conditions* (II_1), (II_2) *implies that* $D^+ - \{0\} \subset H^+$.

FIGURE 1. *Absolute stability region in a neighborhood of the origin*



(a) *if one of the conditions* (I), (II_3), (II_4) *holds*



(b) *if one of the conditions* (II_1), (II_2) *holds*

Remarks. 1. Using (2) and (7), one finds the explicit formulas

$$C_n = \frac{1}{n!} \sum_{m=0}^{k} \alpha_m m^n - \sum_{j=1}^{\min\{n,l\}} \frac{1}{(n-j)!} \sum_{m=0}^{k} \beta_{jm} m^{n-j}, \tag{8}$$
$$n = p+1, p+2, \ldots,$$

and

$$s_n = \sum_{j=1}^{\min\{n+1,l\}} \frac{j}{(n+1-j)!} \sum_{m=0}^{k} \beta_{jm} m^{n+1-j}, \quad n = 0, 1, 2, \ldots, p-1. \tag{9}$$

Moreover, from (6), (7) and (2) follows

$$c_{p+1} = C_{p+1}/\rho'(1) \neq 0 \tag{10}$$

and

$$c_{p+2} = \left(C_{p+2} - \frac{C_{p+1}}{\rho'(1)}\,(\sigma_1'(1) + 2\sigma_2(1))\right)\Big/\rho'(1). \tag{11}$$

2. Let $s = 1$. If p is odd, then Theorem 1 describes ∂A close to $z = 0$ in all cases since $c_{p+1} \neq 0$. The methods with p even and $c_{p+2j} = 0, j = 1, 2, \ldots, p/2$, are not covered by Theorem 1. However, there are only a few methods with this property since one has the following result by Griepentrog [6]. There exists no k-step method of form (1) with $k \geqslant 2$ and $s = 1$ for which ∂A lies exactly on the imaginary axis in a neighborhood of $z = 0$. Moreover, a one-step method of form (1) with $p \geqslant 1$ has ∂A on the imaginary axis in a neighborhood of $z = 0$ if and only if $\beta_{j1} = (-1)^{j+1}\beta_{j0}, j = 1, 2, \ldots, l$.

THEOREM 2. *It is necessary for a method to be A-stable that all growth parameters are real and nonnegative,* $\delta \geqslant 0$ *and either* (III) *or* (IV) *holds, where* δ *is defined as in Theorem* 1 *and*

(III) *p odd,* $c_{p+1}(-1)^{(p+1)/2} > 0$.

(IV) *p even, either* $c_{p+2j} = 0, j = 1, 2, \ldots, p/2$, *or* $c_{p+2j}(-1)^{(p/2)+q} > 0$, $c_{p+2j} = 0, j = 1, 2, \ldots, q-1$, *for some* $q \leqslant p/2$.

Remark. This necessary condition for A-stability is very easy to check for $s = 1$. If p is odd, only c_{p+1} has to be calculated. If p is even one finds for most methods that $c_{p+2} \neq 0$; and hence, only c_{p+2} has to be calculated. The following lemma simplifies the problem of determining the sign of c_{p+1}.

LEMMA. *Let the multistep method using higher derivatives be convergent, then* $\operatorname{sign}\rho'(1) = \operatorname{sign}\alpha_k$.

Proof. Since the method is convergent, all roots of $\rho(\zeta)$ and $\rho'(\zeta)$ lie in the unit disk and hence the lemma holds.

2. Application to Hermite Interpolatory and Adams Type Methods.

Definition 1. A linear multistep method using higher derivatives of the form

$$\sum_{i=0}^{k} \alpha_i y_{n+i} - \sum_{i=0}^{k}\sum_{j=1}^{l_i} h^j \beta_{ji} f^{(j)}(x_{n+i}, y_{n+i}) = 0$$

is called Hermite interpolatory if the error order p is at least $\Sigma_{i=0}^{k} l_i + k - 1$.

In Jeltsch [9] the following theorem is proved.

THEOREM 3. *Let a set of nonnegative integers* $l_0, l_1, \ldots, l_k$ *with* $\max_{i=0,1,\ldots,k} l_i = l > 0$ *be given. Then there exists a unique Hermite interpolatory multistep method with the given* l_i, $\alpha_k \neq 0$ *and* $\beta_{l_k k} \neq 0$. *The error order is* $p = \Sigma_{i=0}^{k} l_i + k - 1$ *and one has*

$$\operatorname{sign} C_{p+1} = (-1)^{l_k} \operatorname{sign}\alpha_k.$$

A similar result can be established for Adams-type methods which are defined as follows.

Definition 2. A linear multistep method using higher derivatives is said to be of Adams type if it is of the form

$$y_{n+k} - y_{n+k-1} - \sum_{i=0}^{k} \sum_{j=1}^{l_i} h^j \beta_{ji} f^{(j)}(x_{n+i}, y_{n+i}) = 0,$$

and its error order p is at least $\Sigma_{i=0}^{k} l_i$.

THEOREM 4. *Let a set of nonnegative integers $l_0, l_1, \ldots, l_k$ with $\max l_i = l > 0$ be given. Then there exists a unique Adams-type multistep method with the given l_i and $\beta_{l_k k} \neq 0$. The error order is $p = \Sigma_{i=0}^{k} l_i$ and one has*

$$\operatorname{sign} C_{p+1} = (-1)^{l_k} \operatorname{sign} \alpha_k.$$

Using Theorems 2, 3, 4 and the Lemma, one then finds immediately the

THEOREM 5. *A convergent linear multistep method using higher derivatives which is of Adams type or Hermite interpolatory cannot be A-stable if the error order p satisfies*

$$p = 2l_k + 1 \mod 4.$$

Example 1. Brown's methods are interpolatory with

$$l_0 = l_1 = \cdots = l_{k-1} = 0, \qquad l_k = l.$$

Hence, the methods are not A-stable if $p = 2l + 1 \mod 4$. Especially the methods with $k = 4, l = 2, p = 5$; $k = 5, l = 3, p = 7$ and $k = 6, l = 4, p = 9$ are not A-stable. The method with $p = 10, k = 7$ and $l = 4$ is not covered by Theorem 5. However, A. H. Sipilä has computed C_{11} and C_{12} using rational arithmetic and it was found that

$$c_{12} = (C_{11}/3\rho'(1))\,(-4.653007\ldots).$$

Hence, by our Lemma and Theorem 3, one has $c_{12}(-1)^{p/2+1} < 0$. Hence, by Theorem 2 this method is not A-stable. Note that in Brown [1] the plots of ∂A lead to the wrong conclusion that these methods are A-stable.

Example 2. Consider the linear one-step methods using higher derivatives which are based on the (r, l) entry of the Padé table of $\exp(x)$, see Jeltsch [8] or Ehle [3, p. 89]. These methods have order $p = r + l$ and are interpolatory. It is known, see Ehle [3], that the methods are A-stable for $r = l, l - 1, l - 2$. From Theorem 5 it follows that the methods are not A-stable for $r = l - 3$. This result has been found by Ehle [3].

Example 3. Enright's second derivative methods are of Adams type with $l_0 = l_1 = \cdots = l_{k-1} = 1$ and $l_k = 2$, with order $p = k + 2$, see Enright [4]. Using the Lemma and Theorems 1 and 4, one finds that for $k = 3 \mod 4$ the region of absolute stability behaves at the origin as given in Figure 1a and for $k = 5 \mod 4$ as given in Figure 1b.

3. Proof of the Results.

Proof of Theorem 1. The algebraic function $\zeta(z)$ which satisfies (3) has k branches $\zeta_j(z)$ with $\zeta_j(0) = \zeta_j, j = 1, 2, \ldots, k$. Since $|\zeta_j(0)| < 1$ for $j = s + 1, s + 2, \ldots, k$ there exists a $D(R_1), R_1 > 0$ such that $|\zeta_j(z)| < 1$ for all $z \in D(R_1)$,

$j = s + 1, s + 2, \ldots, k$. $\zeta_j(0), j = 1, 2, \ldots, s$, are simple zeros of $\rho(\zeta)$; and hence, there exists a disk $D(R_2)$, $0 < R_2 < R_1$, such that the branches $\zeta_j(z)$ are analytic in $D(R_2)$. By the method of undetermined coefficients one finds

$$\zeta_j(z) = \zeta_j(0)(1 + \lambda_j z + \mu_j z^2 + O(z^3)), \qquad j = 1, 2, \ldots, s; \tag{12}$$

and hence,

$$\left.\frac{d\zeta_j(z)}{dz}\right|_{z=0} = \zeta_j(0)\lambda_j \neq 0, \qquad j = 1, 2, \ldots, s. \tag{13}$$

Hence, there exists an R_3, $0 < R_3 < R_2$, such that the mapping $\zeta_j(z): z \longrightarrow \zeta = \zeta_j(z)$ is one to one on $z \in D(R_3)$. Moreover, R_3 can be chosen so small that the curves $\gamma^{(j)} = \{z \in D(R_3) \mid |\zeta_j(z)| = 1\}$ are continuously differentiable. Clearly, $\{0\} \in \gamma^{(j)}$ and from (13) it follows that the imaginary axis is tangent to $\gamma^{(j)}$ at $z = 0$. If $i \neq j$, then either $\gamma^{(j)} \cap \gamma^{(i)}$ is a finite set or $\gamma^{(j)} \cap \gamma^{(i)}$ is a continuous curve which contains $z = 0$. Hence, there exists $\widetilde{R}$, $0 < \widetilde{R} < R_3$, such that either $\gamma_j \cap \gamma_i = \{0\}$ or $\gamma_j \equiv \gamma_i$ and $\gamma_j \cap [-\widetilde{R}, \widetilde{R}] = \{0\}$ for $j = 1, 2, \ldots, s$, where $\gamma_j = D(\widetilde{R}) \cap \gamma^{(j)}$. Each γ_j separates $D(\widetilde{R})$ in the two sets $D_j^- = \{z \in D(\widetilde{R}) \mid |\zeta_j(z)| < 1\}$ and $D_j^+ = \{z \in D(\widetilde{R}) \mid |\zeta_j(z)| \geqslant 1\}$. Clearly, $(-\widetilde{R}, 0) \subset D_j^-, j = 1, 2, \ldots, s$. We distinguish now two cases:

(i) Consider $\zeta_j(z), j = 2, 3, \ldots, s$. With $z = iy, y \in (-\widetilde{R}, \widetilde{R})$, one finds from (12)

$$\begin{aligned} |\zeta_j(iy)| &= |1 - y^2 \mathrm{Re}\,\mu_j + i(\lambda_j y - y^2 \mathrm{Im}\,\mu_j) + O(y^3)| \\ &= \mathrm{sqrt}(1 - y^2(2\mathrm{Re}\,\mu_j - \lambda_j^2) + O(y^3)). \end{aligned} \tag{14}$$

(ii) Consider $\zeta_1(z)$. It is well known, see, e.g. Gear [5] that $\zeta_1(z) - e^z = O(z^{p+1})$. Since $\zeta_1(z)$ is analytic at the origin, we can write

$$\zeta_1(z) = e^z \left(1 - \sum_{j=p+1}^{2p} c_j z^j + O(z^{2p+1})\right). \tag{15}$$

If one substitutes (15) in (3) and uses (2), one finds easily that $c_j, j = p + 1, p + 2, \ldots, 2p$, are determined by (6) and (7). Note that c_j is a real number. Let p be odd. Then $c_{p+1} i^{p+1} = c_{p+1}(-1)^{(p+1)/2}$ is real and nonzero. Hence we find for $z = iy$, y real,

$$\begin{aligned} |\zeta_1(iy)| &= |e^{iy}|\,|1 - c_{p+1} i^{p+1} y^{p+1} + O(y^{p+2})| \\ &= \mathrm{sqrt}(1 - 2c_{p+1}(-1)^{(p+1)/2} y^{p+1} + O(y^{p+2})) \quad \text{for } p \text{ odd}. \end{aligned} \tag{16}$$

Let p be even. Then $c_{p+2j} i^{p+2j} = c_{p+2j}(-1)^{(p/2)+j}$ is real for $j = 1, 2, \ldots, p/2$. Hence, we find for $z = iy$, y real

$$\begin{aligned} |\zeta_1(iy)| = |e^{iy}| \Bigg| 1 &- \sum_{j=1}^{p/2} c_{p+2j} i^{p+2j} y^{p+2j} \\ &- i \sum_{j=0}^{(p/2)-1} c_{p+2j+1} i^{p+2j} y^{p+2j+1} + O(y^{2p+1}) \Bigg| \\ = \mathrm{sqrt}&\left(1 - 2\sum_{j=1}^{p/2} c_{p+2j}(-1)^{(p/2)+j} y^{p+2j} + O(y^{2p+1})\right) \quad \text{for } p \text{ even}. \end{aligned} \tag{17}$$

Assume now that condition (I) holds. Then it follows from (14) that there exists R, $0 < R < \widetilde{R}$, such that $|\zeta_j(iy)| > 1$ for y real, $0 < |y| < R$ for at least one $j \in \{2, 3, \ldots, s\}$. Therefore, $D^- = A \cap D(R) = \bigcap_{j=1}^{s} D_j^- \cap D(R) \subset H^-$.

If (II_1), (II_2), respectively, hold then by (16), (17) and (14) there exists R, $0 < R < \widetilde{R}$, such that $|\zeta_1(iy)| < 1$ and $|\zeta_j(iy)| < 1$, $j = 2, 3, \ldots, s$, for y real, $0 < |y| < R$. Therefore, $D^+ = \bigcup_{j=1}^{s} D_j^+ \cap D(R)$ satisfies $D^+ - \{0\} \subset H^+$ since $D_j^+ \cap D(R) - \{0\} \subset H^+$. Similarly, one finds that (II_3), (II_4) imply $D^- \subset H^-$. This completes the proof of Theorem 1.

Proof of Theorem 2. Let $\lambda_j = de^{i\phi}$, $d > 0$, $\phi \in (0, 2\pi)$. Clearly,

$$\psi = \frac{3\pi}{2} - \frac{\phi}{2} \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \quad \text{and} \quad \psi + \phi \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right).$$

Hence, using (12), one finds

$$|\zeta_j(re^{i\psi})| = |1 + rde^{i(\phi+\psi)} + O(r^2)| > 1$$

for all $r > 0$, r sufficiently small. Therefore, the method is not A-stable. Let $\lambda_j \geqslant 0$, $j = 1, 2, \ldots, s$, and $\delta < 0$. From (14) follows immediately that the method is not A-stable. Similarly, using (16) and (17) one finds that (III), (IV) are necessary for A-stability. This establishes Theorem 2.

Proof of Theorem 4. In Jeltsch [9] it is shown that to given nonnegative integers $l_0, l_1, \ldots, l_k$ with $\max l_i = l > 0$ there exists a unique Adams-type method with the given l_i, $\beta_{l_k k} \neq 0$ and that the error order $p = \Sigma_{i=0}^{k} l_i$. Hence, it remains to show that

$$\operatorname{sign} C_{p+1} = (-1)^{l_k} \operatorname{sign} \alpha_k. \tag{18}$$

To show this we construct the method explicitly. Let $P(x)$ be the interpolation polynomial of degree $\Sigma_{i=0}^{k} l_i - 1$ which satisfies

$$P^{(j-1)}(x_i) = y_{n+i}^{(j)} = f^{(j)}(x_{n+i}, y_{n+i}), \qquad j = 1, 2, \ldots, l_i,\ i = 0, 1, 2, \ldots, k.$$

The multistep method is obtained by setting

$$y_{n+k} - y_{n+k-1} = \int_{x_{n+k-1}}^{x_{n+k}} P(x)\, dx. \tag{19}$$

To find the error order and C_{p+1} we apply the method given by (19) to a sufficiently smooth function $y(x)$. Clearly,

$$y'(x) - P(x) = f^*(x) \prod_{i=0}^{k} (x - x_{n+i})^{l_i}, \tag{20}$$

where $f^*(x)$ is the generalized divided difference of the function $y'(x)$ on the set

$$S = \{x, \underbrace{x_n, x_n, \ldots, x_n}_{l_0\text{-times}}, \underbrace{x_{n+1}, x_{n+1}, \ldots, x_{n+1}}_{l_1\text{-times}}, x_{n+2}, \ldots, \underbrace{x_{n+k}, x_{n+k}, \ldots, x_{n+k}}_{l_k\text{-times}}\},$$

see e.g. Conte and de Boor [2, p. 223]. Hence,

$$\int_{x_{n+k-1}}^{x_{n+k}} (y'(x) - P(x))\,dx = \int_{x_{n+k-1}}^{x_{n+k}} f^*(x) \prod_{i=0}^{k} (x - x_{n+i})^{l_i}\,dx$$

$$= f^*(\zeta) \int_{x_{n+k-1}}^{x_{n+k}} \prod_{i=0}^{k} (x - x_{n+i})^{l_i}\,dx,$$

since the factor $\Pi_{i=0}^{k}(x - x_{n+i})^{l_i}$ does not change sign in the interval $[x_{n+k-1}, x_{n+k}]$; and hence, the second mean value theorem of the integral calculus can be applied, $\zeta \in [x_{n+k-1}, x_{n+k}]$. But $f^*(\zeta) = 1/((p+1)!)\,y^{(p+1)}(\eta)$, where $\eta \in [x_n, x_{n+k}]$; and hence,

$$\int_{x_{n+k-1}}^{x_{n+k}} (y'(x) - P(x))\,dx = Kh^{p+1}y^{(p+1)}(\eta), \tag{21}$$

where

$$K = \frac{1}{(p+1)!} \int_0^1 \prod_{i=0}^{k} (s + k - 1 - i)^{l_i}\,ds. \tag{22}$$

Using (21), it is easy to see that the method given by (19) is of error order p and that $C_{p+1} = K$. From (22) follows that $\operatorname{sign} C_{p+1} = (-1)^{l_k}$. The proof of Theorem 4 is complete since there exists exactly one Adams-type method.

Acknowledgement. Part of this work has been done while the author attended a research seminar at the University of Victoria. It is a pleasure to thank the participants of the seminar, especially A. H. Sipilä and R. Skeel, for stimulating discussions. Moreover I would like to thank A. H. Sipilä for computing the C_j's for Brown's methods on the computer of the University of Waterloo.

Department of Mathematics
University of Kentucky
Lexington, Kentucky 40506

1. R. L. BROWN, "Multi-derivative numerical methods for the solution of stiff ordinary differential equations," Dept. of Comput. Sci., Univ. of Illinois, Report UIUCDCS-R-74-672, 1974.

2. S. D. CONTE & C. de BOOR, *Elementary Numerical Analysis: An Algorithmic Approach*, McGraw-Hill, New York, 1972.

3. B. L. EHLE, *On Padé Approximations to the Exponential Function and A-stable methods for the Numerical Solution of Initial Value Problems*, Res. Rep. CSRR 2010, Dept. of Appl. Anal. and Comput. Sci., Univ. of Waterloo, Canada, 1969.

4. W. H. ENRIGHT, "Second derivative multistep methods for stiff ordinary differential equations," *SIAM J. Numer. Anal.*, v. 11, 1974, pp. 321–331. MR **50** # 3574.

5. C. W. GEAR, *Numerical Initial Value Problems in Ordinary Differential Equations*, Prentice-Hall, Englewood Cliffs, N. J., 1971. MR **47** # 4447.

6. E. GRIEPENTROG, "Mehrschrittverfahren zur numerischen Integration von gewöhnlichen Differentialgleichungssystemen und asymptotische Exaktheit," *Wiss. Z. Humboldt-Univ. Berlin Math.-Natur. Reihe*, v. 19, 1970, pp. 637–653. MR **47** # 9833.

7. R. JELTSCH, "Stiff stability and its relation to A_0- and $A(0)$-stability," *SIAM J. Numer. Anal.*, v. 19, 1976, pp. 8–17.

8. R. JELTSCH, "Multistep methods using higher derivatives and damping at infinity," *Math. Comp.* (To appear.)

9. R. JELTSCH, "Multistep multiderivative methods and Hermite Birkhoff interpolation," *Proc. 5th Manitoba Conference on Numerical Mathematics and Computing*, October 1–4, 1975, Winnipeg.