MATHEMATICS OF COMPUTATION, VOLUME 31, NUMBER 138
APRIL 1977, PAGES 608–611

Quasi-Amicable Numbers

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Abstract. If $m = \sigma(n) - n - 1$ and $n = \sigma(m) - m - 1$, the integers m and n are said to be *quasi-amicable* numbers. This paper is devoted to a study of such numbers.

Let $\sigma(N)$ denote the sum of the positive divisors of the integer N (where $N > 1$), and let

$$(1) \qquad L(N) = \sigma(N) - N - 1$$

so that $L(N)$ is the sum of the "nontrivial" divisors of N. A qt-cycle ("q" for quasi) is a t-tuple of distinct positive integers $(m_1, m_2, \ldots, m_t)$ such that $m_i = L(m_{i-1})$ for $i \neq 1$ and $m_1 = L(m_t)$. A $q1$-cycle is usually called a quasi-perfect number; and we shall call $q2$-cycles quasi-amicable numbers. (In both [2] and [5] $q2$-cycles are referred to as "reduced" amicable pairs. Garcia, however, calls them "números casi amigos" (see the editorial note in [5]).) No quasi-perfect numbers have been found as yet; and if one exists, it exceeds 10^{20} (see [1]). They have been studied by Cattaneo [3], Abbott-Aull-Brown-Suryanarayana [1], and Jerrard-Temperley [4].

In [5] Lal and Forbes list the nine quasi-amicable pairs with smallest member less than 10^5. Beck and Najar [2] have continued the search as far as 10^6 and found six more quasi-amicable pairs. Using the CDC 6400 at the Temple University Computing Center, a search was made for *all* quasi-amicable pairs with smallest member less than 10^7. Forty-six pairs were found including two (526575–544784 and 573560–817479) with smallest member between 10^5 and 10^6 which apparently were missed by Beck and Najar [2]. These are listed at the end of this paper. (For the sake of convenience and completeness the pairs given in [2] and [5] are included here.)

It will be noticed that each pair in our list is of opposite parity, leading us to inquire: Are there any quasi-amicable pairs of the *same* parity? Now, the positive integers m and n are quasi-amicable if and only if $L(m) = n$ and $L(n) = m$ so that from (1), we have

$$(2) \qquad \sigma(m) = \sigma(n) = m + n + 1.$$

Therefore, since $\sigma(N)$ is odd if and only if $N = S^2$ or $N = 2S^2$, we see that a necessary condition for m and n to be quasi-amicable numbers of the same parity is that each be a square or twice a square. Making use of this fact a search was made for quasi-amicable pairs of the same parity in the range $[10^7, 10^{10}]$. None was found so

Received March 8, 1976; revised October 18, 1976.
AMS (MOS) subject classifications (1970). Primary 10A20.

that we have:

PROPOSITION 1. *If $m < n$ and (m, n) is a quasi-amicable pair having the same parity, then $m > 10^{10}$.*

If m and n are *relatively prime* quasi-amicable numbers then, using (2), we have $\sigma(mn)/mn = \sigma(m)\sigma(n)/mn = (m + n + 1)^2/mn > (m + n)^2/mn = 2 + m/n + n/m > 4$. Since, if p is a prime, $\sigma(p^\alpha)/p^\alpha < p/(p - 1)$ and since $x/(x - 1)$ is a decreasing function, it follows that if mn has less than four prime factors then $\sigma(mn)/mn < (2/1)(3/2)(5/4) < 4$. If mn is odd and mn has fewer than twenty-one prime factors, then

$$\sigma(mn)/mn < (3/2)(5/4)(7/6) \cdots (71/70)(73/72) < 4.$$

We have proved:

PROPOSITION 2. *Let m and n be relatively prime quasi-amicable numbers. Then mn has at least four different prime factors. If, also, m and n have the same parity (so that mn is odd), mn has at least twenty-one different prime factors.*

Now, if $(m, n) = 1$ and mn is odd then, as noted earlier, m and n are squares so that from Proposition 2 we have $mn \geqslant (3 \cdot 5 \cdot 7 \cdots 73 \cdot 79)^2 > 25 \cdot 10^{59}$. If $n > m$ and $n < 2.5m$, then $2.5m^2 > mn > 25 \cdot 10^{59}$ and $m > 10^{30}$. If $n > 2.5m$, then $\sigma(m)/m = (m + n + 1)/m > 3.5$. m has at least thirteen prime factors since $(3/2)(5/4) \cdots (37/36)(41/40) < 3.5$; and $m > (3 \cdot 5 \cdot 7 \cdots 41 \cdot 43)^2 > 10^{30}$. We have established:

PROPOSITION 3. *If m and n are relatively prime quasi-amicable numbers of the same parity, then m and n each exceeds 10^{30}.*

We note that no number in our list of quasi-amicable numbers is a prime power. *If* p^α and n are quasi-amicable numbers then, of course, $\alpha > 1$. If $p = 2$, then $\sigma(n)$ $(= \sigma(p^\alpha))$ is odd so that n is of the form S^2 or $2S^2$. From (2) $2^{\alpha+1} - 1 = 2^\alpha + n + 1$ so that $n = 2(2^{\alpha-1} - 1)$. Therefore, $S^2 = 2^{\alpha-1} - 1$. But this is impossible since $S^2 \equiv 1 \pmod 8$ and $2^{\alpha-1} - 1 \equiv -1 \pmod 8$. (For $\alpha > 3$ since neither 2^2 nor 2^3 is a member of a quasi-amicable pair.) If p is odd and $2|\alpha$, then $\sigma(n)$ *and* n are both odd and $n = S^2$. From (2) $1 + p + \cdots + p^\alpha = p^\alpha + n + 1$ so that $n = p(1 + p + \cdots + p^{\alpha-2})$. Thus, $p \parallel n$ which contradicts $n = S^2$. Since we now know that p and α are both odd, it follows that $\sigma(p^\alpha)$ *and* n are both even. If $\alpha = 3$, $n = p(1 + p)$ so that

$$\begin{aligned}\sigma(n) &= (1 + p)\sigma(1 + p) < (1 + p)(1 + 2 + 3 + \cdots + (p + 1)) \\ &= (1 + p)(1 + p)(2 + p)/2.\end{aligned}$$

But $\sigma(n) = \sigma(p^3) = (1 + p)(1 + p^2)$. It follows that $2 + 2p^2 < 2 + 3p + p^2$ so that $p < 3$. This contradiction completes the proof of

PROPOSITION 4. *If (p^α, n) is a quasi-amicable pair, then p is an odd prime, α is an odd number greater than 3, and n is even.*

COROLLARY 4.1. *It is not possible for both members of a quasi-amicable pair to be prime powers.*

Quasi-Amicable Pairs

$48 = 2^4 \cdot 3$	$75 = 3 \cdot 5^2$
$140 = 2^2 \cdot 5 \cdot 7$	$195 = 3 \cdot 5 \cdot 13$
$1050 = 2 \cdot 3 \cdot 5^2 \cdot 7$	$1925 = 5^2 \cdot 7 \cdot 11$
$1575 = 3^2 \cdot 5^2 \cdot 7$	$1638 = 2^4 \cdot 103$
$2024 = 2^3 \cdot 11 \cdot 23$	$2295 = 3^3 \cdot 5 \cdot 17$
$5775 = 3 \cdot 5^2 \cdot 7 \cdot 11$	$6128 = 2^4 \cdot 383$
$8892 = 2^2 \cdot 3^2 \cdot 13 \cdot 19$	$16587 = 3^2 \cdot 19 \cdot 97$
$9504 = 2^5 \cdot 3^3 \cdot 11$	$20735 = 5 \cdot 11 \cdot 13 \cdot 29$
$62744 = 2^3 \cdot 11 \cdot 23 \cdot 31$	$75495 = 3 \cdot 5 \cdot 7 \cdot 719$
$186615 = 3^2 \cdot 5 \cdot 11 \cdot 13 \cdot 29$	$206504 = 2^3 \cdot 83 \cdot 311$
$196664 = 2^3 \cdot 13 \cdot 31 \cdot 61$	$219975 = 3 \cdot 5^2 \cdot 7 \cdot 419$
$199760 = 2^4 \cdot 5 \cdot 11 \cdot 227$	$309135 = 3 \cdot 5 \cdot 37 \cdot 557$
$266000 = 2^4 \cdot 5^3 \cdot 7 \cdot 19$	$507759 = 3 \cdot 7 \cdot 24179$
$312620 = 2^2 \cdot 5 \cdot 7^2 \cdot 11 \cdot 29$	$549219 = 3 \cdot 11^2 \cdot 17 \cdot 89$
$526575 = 3 \cdot 5^2 \cdot 7 \cdot 17 \cdot 59$	$544784 = 2^4 \cdot 79 \cdot 431$
$573560 = 2^3 \cdot 5 \cdot 13 \cdot 1103$	$817479 = 3^3 \cdot 13 \cdot 17 \cdot 137$
$587460 = 2^2 \cdot 3 \cdot 5 \cdot 9791$	$1057595 = 5 \cdot 7 \cdot 11 \cdot 41 \cdot 67$
$1000824 = 2^3 \cdot 3 \cdot 11 \cdot 17 \cdot 223$	$1902215 = 5 \cdot 7 \cdot 17 \cdot 23 \cdot 139$
$1081184 = 2^5 \cdot 13 \cdot 23 \cdot 113$	$1331967 = 3 \cdot 7^2 \cdot 13 \cdot 17 \cdot 41$
$1139144 = 2^3 \cdot 23 \cdot 41 \cdot 151$	$1159095 = 3 \cdot 5 \cdot 7^2 \cdot 19 \cdot 83$
$1140020 = 2^2 \cdot 5 \cdot 7 \cdot 17 \cdot 479$	$1763019 = 3^3 \cdot 17 \cdot 23 \cdot 167$
$1173704 = 2^3 \cdot 7 \cdot 20959$	$1341495 = 3^3 \cdot 5 \cdot 19 \cdot 523$
$1208504 = 2^3 \cdot 11 \cdot 31 \cdot 443$	$1348935 = 3 \cdot 5 \cdot 7 \cdot 29 \cdot 443$
$1233056 = 2^5 \cdot 11 \cdot 31 \cdot 113$	$1524831 = 3 \cdot 7^2 \cdot 11 \cdot 23 \cdot 41$
$1236536 = 2^3 \cdot 7 \cdot 71 \cdot 311$	$1459143 = 3^2 \cdot 7 \cdot 19 \cdot 23 \cdot 53$
$1279950 = 2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 23 \cdot 53$	$2576945 = 5 \cdot 7 \cdot 17 \cdot 61 \cdot 71$
$1921185 = 3^3 \cdot 5 \cdot 7 \cdot 19 \cdot 107$	$2226014 = 2 \cdot 7 \cdot 17 \cdot 47 \cdot 199$
$2036420 = 2^2 \cdot 5 \cdot 19 \cdot 23 \cdot 233$	$2681019 = 3^5 \cdot 11 \cdot 17 \cdot 59$
$2102750 = 2 \cdot 5^3 \cdot 13 \cdot 647$	$2142945 = 3^2 \cdot 5 \cdot 7 \cdot 6803$
$2140215 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 17 \cdot 109$	$2421704 = 2^3 \cdot 263 \cdot 1151$
$2171240 = 2^3 \cdot 5 \cdot 17 \cdot 31 \cdot 103$	$3220119 = 3^2 \cdot 7 \cdot 79 \cdot 647$
$2198504 = 2^3 \cdot 7 \cdot 11 \cdot 43 \cdot 83$	$3123735 = 3 \cdot 5 \cdot 29 \cdot 43 \cdot 167$
$2312024 = 2^3 \cdot 11 \cdot 13 \cdot 43 \cdot 47$	$3010215 = 3 \cdot 5 \cdot 13 \cdot 43 \cdot 359$
$2580864 = 2^7 \cdot 3 \cdot 11 \cdot 13 \cdot 47$	$5644415 = 5 \cdot 7 \cdot 29 \cdot 67 \cdot 83$
$2958500 = 2^2 \cdot 5^3 \cdot 61 \cdot 97$	$3676491 = 3^2 \cdot 7 \cdot 13 \cdot 67^2$
$4012184 = 2^3 \cdot 11 \cdot 127 \cdot 359$	$4282215 = 3 \cdot 5 \cdot 7 \cdot 17 \cdot 2399$
$4311024 = 2^4 \cdot 3 \cdot 19 \cdot 29 \cdot 163$	$7890575 = 5^2 \cdot 7 \cdot 11 \cdot 4099$

$5088650 = 2 \cdot 5^2 \cdot 7^2 \cdot 31 \cdot 67$	$6446325 = 3 \cdot 5^2 \cdot 23 \cdot 37 \cdot 101$
$5416820 = 2^3 \cdot 5 \cdot 43 \cdot 47 \cdot 67$	$7509159 = 3^3 \cdot 7 \cdot 67 \cdot 593$
$6081680 = 2^4 \cdot 5 \cdot 11 \cdot 6911$	$9345903 = 3 \cdot 7 \cdot 17 \cdot 47 \cdot 557$
$6618080 = 2^5 \cdot 5 \cdot 7 \cdot 19 \cdot 311$	$12251679 = 3 \cdot 11 \cdot 17 \cdot 21839$
$7460004 = 2^2 \cdot 3 \cdot 23 \cdot 151 \cdot 179$	$10925915 = 5 \cdot 7 \cdot 11 \cdot 13 \cdot 37 \cdot 59$
$7875450 = 2 \cdot 3^2 \cdot 5^2 \cdot 11 \cdot 37 \cdot 43$	$16381925 = 5^2 \cdot 7^2 \cdot 43 \cdot 311$
$8713880 = 2^3 \cdot 5 \cdot 7 \cdot 31121$	$13693959 = 3^2 \cdot 17 \cdot 37 \cdot 41 \cdot 59$
$8829792 = 2^5 \cdot 3^2 \cdot 23 \cdot 31 \cdot 43$	$18845855 = 5 \cdot 7 \cdot 23 \cdot 41 \cdot 571$
$9247095 = 3^3 \cdot 5 \cdot 11 \cdot 13 \cdot 479$	$10106504 = 2^3 \cdot 47 \cdot 26879$

Acknowledgement. The authors were informed almost simultaneously and independently by the referee and Professor R. Najar of the work of Lal and Forbes [5] on "quasi-aliquot" sequences and quasi-amicable numbers. An earlier version of the present paper contained many of the results to be found in [5], and we are grateful for being informed of this duplication of effort.

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