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Two Linear Programming Algorithms for the Linear Discrete L_1 Norm Problem

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Abstract. Computational studies by several authors have indicated that linear programming is currently the most efficient procedure for obtaining L_1 norm estimates for a discrete linear problem. However, there are several linear programming algorithms, and the "best" approach may depend on the problem's structure (e.g., sparsity, triangularity, stability). In this paper we shall compare two published simplex algorithms, one referred to as primal and the other referred to as dual, and show that they are conceptually equivalent.

1. Introduction. The linear discrete approximation problem in the L_1 norm can be stated as follows: Given points $(f_i, c_{i1}, c_{i2}, \ldots, c_{im})$, $i = 1, 2, \ldots, n$, in $m + 1$ Euclidean space, determine a value for $a = (a_1, a_2, \ldots, a_m)$ which will

$$\text{minimize} \sum_{i=1}^{n} |f_i - c_{i1}a_1 - c_{i2}a_2 - \cdots - c_{im}a_m|. \tag{1}$$

It is a well-known result that (1) can be written as the following linear programming problem.

$$\text{Minimize} \sum_{i=1}^{n} (P_i + N_i), \tag{2}$$

subject to

$$c_{i1}a_1 + c_{i2}a_2 + \cdots + c_{im}a_m + P_i - N_i = f_i,$$
$$P_i \geqslant 0 \text{ and } N_i \geqslant 0, \quad i = 1, 2, \ldots, n,$$

where P_i and N_i are the positive and negative deviations of the ith observation, respectively. Since this relationship between (1) and (2) was demonstrated by Charnes, Cooper and Ferguson [8], it has generally been agreed that linear programming is computationally the most efficient method for obtaining an optimal a value. Wagner [15] showed that the linear programming dual of (2) can be solved with simple upper bounding techniques which requires a working basis of size m by m. This would be as opposed to the n by n basis if (1) were to be solved with a standard primal simplex algorithm. Generally, m is substantially less than n and, thus, Wagner's approach was considered the most efficient for some time. However, Barrodale and Roberts [5] demonstrated how the structure of (2) could be utilized to solve it directly with a special purpose primal algorithm. This algorithm maintains a basis of size m by m (or the

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rank of (c_{ij})) and also combines several standard pivots into one. In other words, instead of passing from an extreme point to an adjacent extreme point, an iteration may pass through several extreme points. This approach has been shown [5], [9] to be computationally superior to solving the dual directly.

Abdelmalek [2] also presents a special purpose linear programming algorithm to solve (1). He employs a transformation on the dual of (2) and solves it with a modification of the dual simplex algorithm for bounded variables [14].

The dual linear program of (2) is:

$$\text{(3)} \qquad \text{Maximize } \sum_{k=1}^{n} f_k v_k$$

subject to

$$\sum_{k=1}^{n} v_k c_{kj} = 0, \quad j = 1, 2, \ldots, m, \; -1 \leqslant v_k \leqslant 1, \; k = 1, \ldots, n.$$

The dual problem that is considered by Abdelmalek is the bounded variable linear program determined by the transformation $b_k = v_k + 1$:

$$\text{(4)} \qquad \text{Maximize } \sum_{k=1}^{n} f_k(b_k - 1) = \sum_{k=1}^{n} (f_k b_k - f_k)$$

subject to

$$\sum_{k=1}^{n} b_k c_{kj} = \sum_{k=1}^{n} c_{kj}, \quad j = 1, 2, \ldots, m, \; 0 \leqslant b_k \leqslant 2, \; k = 1, 2, \ldots, n.$$

It is the special structure of the problem that enables a modification to the dual simplex method to perform a pivot through several bases in one iteration as was done by the algorithm of Barrodale and Roberts. However, solving the dual with the dual simplex algorithm is identical to solving the primal with the primal simplex algorithm. What we shall demonstrate in this paper is that Abdelmalek's algorithm is equivalent to the Barrodale and Roberts' algorithm. By equivalence, we mean that, given an initial basis for the dual problem and the corresponding basis for the primal problem, the two algorithms will generate corresponding bases at each iteration. The only real difference is in the formations of the simplex tableaus.

The first step will be to explicitly develop the algorithms and the manipulations that are performed on the tableaus. We shall then demonstrate the truth of the hypothesis from an algorithmic and geometrical viewpoint. An underlying assumption of the hypothesis is that the same pivot selection criteria are used and also that in case of degeneracy corresponding perturbations [6] will be employed. We shall also present a numerical example to further illuminate our presentation.

2. Development of Primal Algorithm (Barrodale and Roberts). Using matrix notation, $LP(2)$ can be rewritten as follows:

$$\text{(5)} \qquad \text{Minimize } e^T P + e^T N = e^T(P + N)$$

subject to

$$Ca + IP - IN = f, \qquad P, N \geqslant 0,$$

where e is a vector of conforming dimension whose components are 1's, $a^T=$

$(a_1, \ldots, a_m)$, $P^T = (P_1, \ldots, P_n)$, $N^T = (N_1, \ldots, N_n)$, $f^T = (f_1, \ldots, f_n)$, and C is an n by m matrix with $n \geqslant m$.

For convenience we shall assume that the matrix C has full column rank; i.e., rank $C = m$. However, we should note that rank deficiencies are easily treated within the linear programming framework (see Charnes and Cooper [7]). Because of the rank assumption, it follows directly that an optimal basis must contain all n columns of the matrix X and also $n - m$ columns from the matrices I and $-I$ associated with the $2n$ variables, P_j and N_j.

If we use the revised simplex approach to generate the tableau, then the basis matrices will be of the form (after row interchanges)

$$B_* = \begin{pmatrix} B & 0 \\ R & D \end{pmatrix},$$

where $\binom{B}{R} = C$ after row interchanges, B is an m by m matrix, R is an $(n-m)$ by m matrix, 0 is an m by $(n-m)$ zero matrix, $\binom{f_B}{f_R} = f$ after row interchanges, and D is an $(n-m)$ by $(n-m)$ matrix whose diagonal elements are either $+1$ or -1 and non-diagonal elements are zeros. Due to the format of the basis matrix, the initial tableau will also require the appropriate adjustments.

$$
\begin{array}{cc:cccccc}
 & & a & P_B & P_R & N_B & N_R & \\
\multicolumn{2}{c:}{\begin{bmatrix} B & 0 \\ R & D \end{bmatrix}^{-1}} & \begin{matrix} B \\ R \end{matrix} & \begin{matrix} I \\ 0 \end{matrix} & \begin{matrix} 0 \\ I \end{matrix} & \begin{matrix} -I \\ 0 \end{matrix} & \begin{matrix} 0 \\ -I \end{matrix} & \begin{matrix} f_B \\ f_R \end{matrix} \\
\hdashline
0 & e & 0 & e & e & e & e & 0
\end{array}
$$

$$
= \begin{array}{cc:cccccc}
\multicolumn{2}{c:}{\begin{bmatrix} B^{-1} & 0 \\ -DRB^{-1} & D \end{bmatrix}} & \begin{matrix} B \\ R \end{matrix} & \begin{matrix} I \\ 0 \end{matrix} & \begin{matrix} 0 \\ I \end{matrix} & \begin{matrix} -I \\ 0 \end{matrix} & \begin{matrix} 0 \\ -I \end{matrix} & \begin{matrix} f_B \\ f_R \end{matrix} \\
\hdashline
eDRB^{-1} & -D & 0 & e & e & e & e & 0
\end{array}
$$

$$
= \begin{array}{ccccc:c}
a & P_B & P_R & N_B & N_R & \\
I & B^{-1} & 0 & -B^{-1} & 0 & B^{-1}f_B \\
0 & -DRB^{-1} & D & DRB^{-1} & -D & D(f_R - RB^{-1}f_B) \\
\hdashline
0 & e + eDRB^{-1} & e - eD & e - eDRB^{-1} & e + eD & eD(f_R - RB^{-1}f_B)
\end{array}
$$

The marginal cost for $(n - m)$ nonbasic N_i (P_i) with the corresponding P_i (N_i) in the basis is 2. For the case where both N_i and P_i, m of each, are nonbasic, the marginal cost for P_i is $1 + eDRB_k^{-1}$, and for N_i it is $1 - eDRB_k^{-1}$, where B_k^{-1} is the kth column of B^{-1}, and $c_i = B_k = (c_{i1}, c_{i2}, \ldots, c_{in})$ is the kth row of B. At optimality, the row of the marginal cost terms will be nonnegative, or this can be replaced by the equivalent statement

$$(6) \qquad -1 \leqslant eDRB_k^{-1} \leqslant 1, \qquad k = 1, 2, \ldots, m.$$

To assist in demonstrating the equivalence of the two algorithms under consideration here, two index sets are defined as follows:

$$U = \{i \mid P_i \text{ is basic}\}, \qquad L = \{i \mid N_i \text{ is basic}\}.$$

Condition (6) can now be restated as

$$-1 \leqslant \left(\sum_{i\in U} c_i - \sum_{i\in L} c_i\right) B_k^{-1} \leqslant 1, \qquad k = 1, 2, \ldots, m,$$

or defining

$$w_k = \left(\sum_{i\in U} c_i - \sum_{i\in L} c_i\right) B_k^{-1}, \qquad -1 \leqslant w_k \leqslant 1,\ k = 1, 2, \ldots, m.$$

Once it has been ascertained that the optimality conditions are not satisfied for some i (say $i = l$), then either N_l or P_l is to enter the basis and the primal algorithm proceeds to determine the LP variable to leave the basis. The standard procedure at this stage is to take the minimum of the ratios

$$\frac{f_k - c_k B^{-1} f_B}{\mu c_k B_q^{-1}}; \qquad \mu c_k B_q^{-1} > 0,\ k \in U, \text{ and}$$

$$\frac{f_k - c_k B^{-1} f_B}{\mu c_k B_q^{-1}}; \qquad \mu c_k B_q^{-1} < 0,\ k \in L, \tag{7}$$

where q is the row of B associated with the index l $(c_l = B_q)$ and

$$\mu = \begin{cases} -1 & \text{if } P_l \text{ enters the basis,} \\ 1 & \text{if } N_l \text{ enters the basis.} \end{cases}$$

The standard primal algorithm would now perform a pivot to remove the vector chosen by the minimum ratio test from the basis and to enter either N_l or P_l into the basis. However, the special purpose algorithms may perform several interchanges of basis vectors before executing the pivot to bring either N_l or P_l into the basis. Let t be the value of k when the minimum ratio occurs in (7). Then the linear programming variable to be removed from the basis will be either N_t or P_t, whichever is in the basis. In order to clarify this process, suppose P_l is to enter the basis, and N_t is to leave the basis. If $1 + w_q + 2\,|c_t B_q^{-1}| > 0$, then the standard pivoting procedure of the simplex algorithm is carried out. If $1 + w_q + 2\,|c_t B_q^{-1}| < 0$, then P_t will replace N_t in the basis, and the row of marginal costs are updated. This calculation is achieved by adding twice the tth row of the current tableau to the row of the reduced cost factors. The algorithm now returns to the minimum ratio test and finds the ratio which is next to the smallest. The basic variable associated with this ratio is then tested, as above, for removal by the standard pivoting technique or for interchange in the basis with its corresponding dependent deviation variable. This procedure will continue until P_l is brought into the basis by the usual pivoting method, and then a new vector will be sought to enter the basis by observing the values of the marginal cost row, and the steps are repeated until optimality is reached.

3. Development of Dual Algorithm (Abdelmalek). In this section, we shall briefly develop the dual simplex method which Abdelmalek applies to LP (4). Rewriting LP (4) in matrix notation and condensing the constant term in the objective function, we obtain the following.

$$\text{Maximize } f^T b - f^T e \tag{8}$$

subject to

$$C^T b = C^T e, \qquad 0 \leqslant b_i \leqslant 2, \; i = 1, 2, \ldots, n,$$

where $b^T = (b_1, b_2, \ldots, b_n)$ is the vector of the variables for this dual problem.

We shall now present the simplex tableau formulation that is employed by Abdelmalek in solving LP (8). The notation is the same as in the preceding section.

$$\begin{bmatrix} B^T & 0 \\ \hline -f_B^T & 1 \end{bmatrix}^{-1} \quad \begin{array}{cc|c} B^T & R^T & C^T e \\ \hline -f_B^T & -f_R^T & -e^T f \end{array}$$

$$(B^{-1}f_B)^T = \begin{array}{cc} (B^{-1})^T & 0 \\ \hline (B^{-1}f_B)^T & 1 \end{array} \quad \begin{array}{cc|c} B^T & R^T & C^T e \\ \hline -f_B^T & -f_R^T & -e^T f \end{array}$$

$$= \begin{array}{cc|c} I & (RB^{-1})^T & (CB^{-1})^T e \\ \hline 0 & f_B^T(RB^{-1})^T - f_R^T & (B^{-1}f_B)^T C^T e - e^T f \end{array}$$

However, since some of the nonbasic elements associated with the columns of $(RB^{-1})^T$ may be at their upper bound, we must modify the right-hand side to account for this fact. Thus, the values of the kth basic variable $b_{B(k)}$ will be given by

$$b_{B(k)} = (B_k^{-1})^T \left[C^T e - 2 \sum_{i \in U} c_i \right] = (B_k^{-1})^T \left[Be + Re - \sum_{i \in U} c_i \right]$$
$$= 1 + (B_k^{-1})^T \left[\sum_{i \in L} c_i - \sum_{i \in U} c_i \right],$$

where U and L are the index set of the nonbasic variables which are at their upper and lower bound, respectively. (Further details on this can be found in Hadley [10].) Nonbasic variables which are at their upper bound are recognized by the fact that the associated entry in $f_B^T(RB^{-1})^T - f_R^T$ will be negative. These variables will have an x placed above their associated column in the tableau. Therefore, the tableau may have the following form:

$$\begin{array}{cc|l} I & (RB^{-1})^T & b_B \\ 0 & f_B^T(RB^{-1})^T - f_R^T & f_B^T(b_B - e) + \sum_{j \in U} f_j - \sum_{j \in L} f_j. \end{array}$$

At this point a brief description of the steps from Abdelmalek's algorithm is given.

1. Select a variable to become primal feasible (vector to leave the basis) by the rule

$$b_{B(q)} = \min\{b_1, b_2\},$$

where

$$b_1 = \min_i \{b_{B(i)}, b_{B(i)} < 0\}$$

and

$$b_2 = \min_i\{2 - b_{B(i)}, b_{B(i)} > 2\}.$$

If no minimum exists, the algorithm terminates as the optimal solution has been obtained.

Let b_l indicate the variable to become primal feasible.

Case 1. $b_{B(q)} < 0$.

2.1. Determine the index t by the following ratio test:

$$(f_t - z_t)/y_{qt} = \min_k \{(f_k - z_k)/y_{qk}, k \in U, y_{ik} > 0 \text{ and } (f_k - z_k)/y_{qk}, k \in L, y_{ik} < 0\},$$

where y_{qk} is the qth component of

$$y_k^T = c_k B^{-1} \quad \text{and} \quad z_k = f_B^T y_k.$$

2.2. If $y_{qt} < 0$, do not change b_B, go to 3.

2.3. If $y_{qt} > 0$, add $2y_t$ to b_B, remove the upper bound flag on b_t, and go to 3.

Case 2. $b_{B(q)} > 2$.

2.4. Determine the index t by the following ratio test:

$$(z_t - f_t)/y_{qt} = \min_k\{(z_k - f_k)/y_{qk}, k \in U, y_{ik} < 0 \text{ and } (z_k - f_k)/y_{qk}, \ k \in L, \ y_{ik} > 0\}.$$

2.5. If $y_{qr} > 0$, subtract $2y_l$ from b_B and flag b_l to indicate that it is at the upper bound. Go to 3.

2.6. If $y_{qt} < 0$, add $2y_t$ to and subtract $2y_l$ from b_B. Remove the upper bound flag on b_t and place the upper bound flag on b_l.

3. Determine if a standard update of $b_{B(q)}$ will produce a value in [0, 2]. If not, replace c_l by c_t and go to either Case 1 or Case 2. If $0 \leqslant b_{B(q)} \leqslant 2$, update in the standard manner and go to step 1.

4. Comparison of the Algorithms. The purpose of this section is to demonstrate that the algorithms are equivalent. We base this conclusion on the fact that the primal algorithm maintains primal feasibility and complementary slackness, while seeking to obtain dual feasibility. The dual algorithm maintains dual feasibility and complementary slackness while seeking primal feasibility. Hence, when the dual algorithm is applied to the dual problem, it is equivalent to the primal algorithm applied to the primal problem (see, for example, Charnes and Cooper [7, p. 477], Wagner [14], Simonnard [13, p. 116]).

However, the basic question here is whether the two algorithms will proceed through the same sequence of bases or extreme points since the algorithms utilize techniques which will enable them to pass through a selected sequence of extreme points before performing a simplex iteration. For the purpose of simplicity in our development, we shall assume that degeneracy is not present in our problem.

In general, we shall illustrate how the steps performed on the dual relate to the primal. A negative $z_k - f_k$ in the dual and flagging the kth column ($b_k = 2$) corresponds to having a P_k in the basis of the primal. Also, when b_k is nonbasic with $b_k = 0$ in the dual problem, N_k is in the basis of the primal. If b_k is basic in the dual, then N_k and P_k are nonbasic in the primal. The $f_k - z_k$'s in the dual are the same as the deviations in the primal. Therefore, the ranking of the minimum ratios will be in the same sequence for each of the algorithms so that corresponding vectors will enter and leave the basis during a pivot. This can be observed directly by noting that the same vector (assuming no ties exist in the ratios) must enter the basis in both algorithms, because there is a *unique* vector whose entry into basis will maintain the dual feasibility of (8) and force primal feasibility in the pivot row. Both the rules given by Abdelmalek and Barrodale and Roberts do this although they are stated in slightly different forms.

The primal algorithm attempts to place the dual variables in the interval $[-1, 1]$. The dual algorithm works towards placing the primal variables of the dual (which are the dual variables of the primal) in the interval $[0, 2]$. The difference in the intervals has been brought about by the transformation employed by Abdelmalek. Thus, there will be a difference between these variables which is created by the transformation. In addition, the differences in tableaus in terms of the signs of the elements can be accounted for by the utilization of the D matrix in the primal. This matrix is not employed by Abdelmalek in the dual, but he utilizes the x above the columns to flag the variables at their upper bounds.

Additional aspects such as the value of the objective function being the same at each extreme point of the sequence is clearly observable from the tableaus which were developed. The process of adding (or subtracting) the term $2y_t$ is the same in the algorithms.

5. Numerical Example. In this section, we shall solve a linear discrete L_1 norm problem by both the primal and dual methods following the manner in which they have been developed in the previous sections. The data is taken from a paper by Karst [11]. The primal method will utilize the revised simplex tableaus as was done in the paper by Barrodale and Roberts [5]. The asterisks will denote the sequence of pivots and the corresponding calculations that are performed on the tableaus. The addition of twice the pivot row elements to the $z_j - f_j$ row is irrelevant for the nonpivot columns, and this calculation is eliminated from the procedure.

In solving the problem by the dual method, we have employed two preliminary tables to obtain the table to begin the algorithm.

We shall find the L_1 norm estimators for the following system of equations.

$$\begin{array}{lll} a_1 - 3a_2 = -3, & a_1 - 3a_2 = 2, & a_1 + 3a_2 = 2, \\ a_1 + 2a_2 = -1, & a_1 - 2a_2 = -1, & a_1 + 4a_2 = 0, \\ a_1 - 5a_2 = 0, & a_1 + a_2 = 1, & a_1 + 4a_2 = 4. \\ & a_1 + 2a_2 = 3, & \end{array}$$

The arbitrary starting basis will consist of the first and second equations for both algorithms. This choice will clearly demonstrate the equivalence of the two algorithms as was presented in Sections 3 and 4.

TABLE I: *Primal method* (*Barrodale and Roberts*)

BASIS	RESIDUAL	N_1	N_2	RATIOS
a_1	-9/5	-2/5	3/5	-
a_2	2/5	1/5	-1/5	-
P_3	19/5	7/5	-2/5	-
P_4	25/5	5/5	0	-
P_5	8/5	4/5	1/5	8
P_6	12/5	1/5	4/5	3
P_7	20/5	0	5/5	4
P_8	13/5	-1/5	6/5**	13/6**
P_9	1/5	-2/5	7/5*	1/7*
P_{10}	21/5	-2/5	7/5	3
	119/5	-7/5	-23/5	
	162/7		-9/5	*
	39/2		3/5	**

TABLE II: *Primal method*

BASIS	RESIDUAL	N_1	P_8	RATIOS
a_1	-3/6	-3/6	3/6	-
a_2	5/6	1/6	1/6	-
P_3	28/6	8/6***	2/6	7/2***
P_4	30/6	6/6	0	5
P_5	7/6	5/6*	-1/6	7/5*
P_6	4/6	2/6**	-4/6	2**
P_7	11/6	1/6	-5/6	11
N_2	13/6	-1/6	5/6	-
N_9	17/6	1/6	7/6	17
P_{10}	7/6	-1/6	-7/6	-
	$19\frac{1}{2}$	-15/6	9/6	
	16	-5/6		*
	$15\frac{1}{2}$	-1/6		**
	$15\frac{1}{4}$	15/6		***

TABLE III: *Primal method*

BASIS	RESIDUAL	P_3	P_8	RATIO
a_1	5/4	3/8	5/8	-
a_2	1/4	-1/8	1/8	-
N_1	14/4	6/8	2/8	14
P_4	6/4	-6/8	-2/8	-
N_5	7/4	5/8	3/8	14/3
N_6	2/4	2/8	6/8*	2/3*
P_7	5/4	-1/8	-7/8	-
N_2	11/4	1/8	7/8	22/7
N_9	9/4	-1/8	9/8	2
P_{10}	7/4	1/8	-9/8	-
	15¼	1/8	-1/8	
	91/6		11/8	*

TABLE IV: *Primal method: Optimal*

BASIS	RESIDUAL	P_3	N_6
a_1	5/6	1/6	-5/6
a_2	1/6	-1/6	-1/6
N_1	20/6	4/6	-2/6
P_4	10/6	-4/6	2/6
N_5	9/6	3/6	-3/6
P_8	4/6	2/6	8/6
P_7	11/6	1/6	7/6
N_2	13/6	-1/6	-7/6
N_9	9/6	-3/6	-9/6
P_{10}	15/6	3/6	9/6
	91/6	5/6	7/6

Preliminary Tableau a: *Dual method* (*Abdelmalek*)

-3	-1	0	2	-1	1	3	2	0	4	
1	1	1	1	1	1	1	1	1	1	10
-3	2	-5	-3	-2	1	2	3	4	4	3
3	1	0	-2	1	-1	-3	-2	0	-4	-7
b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	

Preliminary Tableau b: *Dual method*

-3	-1	0	2	-1	1	3	2	0	4	
1	0	7/5	5/5	4/5	1/5	0	-1/5	-2/5	-2/5	17/5
0	1	-2/5	0	1/5	4/5	5/5	6/5	7/5	7/5	33/5
0	0	-19/5	-25/5	-8/5	-12/5	-20/5	13/5	-1/5	-21/5	119/5

Initial Tableau I: *Dual method*

		x	x	x	x	x	x	x	x				
-3	-1	0	2	-1	1	3	2	0	4	b_B	*	**	$\bar{b}$
1	0	7/5	5/5	4/5	1/5	0	-1/5	-2/5	-2/5	-7/5			
0	1	-2/5	0	1/5	4/5	5/5	6/5**	7/5*	7/5	-23/5	-9/5	3/5	½
0	0	-19/5	-25/5	-8/5	-12/5	-20/5	-13/5	-1/5	-21/5	119/5	162/7	39/2	

TABLE II: *Dual method*

		x	x	x	x	x			x						
-3	-1	0	2	-1	1	3	2	0	4		b_B	*	**	***	$\bar{b}$
1	1/6	8/6***	6/6	5/6*	2/6*	1/6	0	-1/6	-1/6	27/6	-15/6	-5/6	-1/6	15/6	15/8
0	5/6	-2/6	0	1/6	4/6	5/6	1	7/6	7/6	33/6	-9/6				
0	13/6	-28/6	-30/6	-7/6	-4/6	-11/6	0	17/6	-7/6	39/2	39/2	16	15½	15¼	

TABLE III: *Dual method*

			x			x			x				
-3	-1	0	2	-1	1	3	2	0	4		b_B	*	$\bar{b}$
6/8	1/8	1	6/8	5/8	2/8	1/8	0	-1/8	-1/8	27/8	15/8		
2/8	7/8	0	2/8	3/8	6/8*	7/8	1	9/8	9/8	53/8	17/8	5/8	5/6
7/2	11/4	0	-3/2	7/4	1/2	-5/4	0	9/4	-7/4	15¼	15¼	91/6	

TABLE IV: *Dual method* (*Optimal*)

			x			x	x		x		
-3	-1	0	2	-1	1	3	2	0	4		b_B
4/6	-1/6	1	4/6	3/6	0	-1/6	-2/6	-3/6	-3/6	7/6	11/6
2/6	7/6	0	2/6	3/6	1	7/6	8/6	9/6	9/6	53/6	1/6
20/6	13/6	0	-10/6	9/6	0	-11/6	-4/6	9/6	-15/6	91/6	91/6

6. Conclusions. Because Abdelmalek's algorithm is essentially equivalent to the Barrodale and Roberts' algorithm, Abdelmalek's comment that the methods are comparable in terms of both speed and number of iterations is not surprising. A set of comprehensive tests performed by the National Bureau of Standards [9] on four L_1 norm codes clearly demonstrates this fact. The only differences between the two algorithms involve the choice of an initial basis and the heuristic rules for breaking ties. Of course, there are an infinite number of ways any linear programming "algorithm" can be implemented – e.g., full tableau, revised simplex with product form or explicit inverse, multiple pricing, etc. The unique thing about the Barrodale and Roberts' algorithm is the ability to combine several standard iterations into one.

It does appear that the simplex algorithm of linear programming is the most efficient method to solve the linear discrete L_1 norm problem. A superior nonsimplex based algorithm would amount to finding a better way to solve a linear programming problem. This topic has been studied for many years, and the iterative technique of passing from extreme point to extreme point common to virtually all linear programming codes remains the most efficient.

Current research on solving the linear discrete L_1 norm problem seems to involve extensions of the Barrodale and Roberts' algorithm. McCormick and Sposito [12] demonstrate how the L_2 norm estimate can be utilized to accelerate the convergence of the Barrodale and Roberts' algorithm. Armstrong and Hultz [4] extend the algorithm to handle additional linear constraints on the parameters. Finally, Armstrong and Frome [3] demonstrate how a network structure can be utilized when obtaining L_1 norm estimates for two-way tables.

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