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On Maximal Finite Irreducible Subgroups of $GL(n, \mathbf{Z})$ IV. Remarks on Even Dimensions with Applications to $n = 8$

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Abstract. The general methods for the determination of maximal finite absolutely irreducible subgroups of $GL(n, \mathbf{Z})$ developed in Part I of this series of papers [6] are refined for even n. Applications are made to $n = 8$ in view of Part V [7], where a complete classification is obtained.

1. Introduction. The general procedure for the determination of the $\mathbf{Z}$-classes of maximal finite irreducible (i.e. $\mathbf{C}$-irreducible) subgroups of $GL(n, \mathbf{Z})$ suggested in Part I [6] consists of three steps: finding representatives of the $\mathbf{Q}$-classes of the minimal irreducible finite subgroups of $GL(n, \mathbf{Z})$, calculating the $\mathbf{Z}$-classes of these groups by the centering algorithm, and computing the $\mathbf{Z}$-automorphism groups of the quadratic forms fixed by the minimal irreducible finite subgroups of $GL(n, \mathbf{Z})$. These methods turned out to be very effective for odd dimensions such as $n = 5, 7, 9$, where we had to consider only two, respectively three, $\mathbf{Q}$-classes of minimal irreducible subgroups of $GL(n, \mathbf{Z})$. On the other hand, for $n = 6$ this number is already 33 and a cautious estimate yields more than a hundred $\mathbf{Q}$-classes for $n = 8$. The main reason for these big numbers is that there exist many possibilities for the decomposition scheme of normal abelian subgroups of irreducible matrix groups in $GL(n, \mathbf{Q})$, if n has many even divisors. Moreover, if n is a power of two, a lot of 2-groups occur. Therefore, it is desirable to have a method which allows us to avoid the determination of all minimal irreducible finite subgroups of $GL(n, \mathbf{Z})$ in case $n = 2r$, $r \in \mathbf{N}$. In Section 2 we describe a method which provides all quadratic forms fixed by a finite irreducible subgroup G of $GL(2r, \mathbf{Z})$, where G has a $\mathbf{Q}$-reducible subgroup of index 2. Note that these groups include all 2-groups in case $2r$ is a power of 2. The method requires information about the finite irreducible subgroups of $GL(r, \mathbf{Z})$.

In Section 3 we carry out the computations for $2r = 8$ and obtain 17 of 26 primitive positive definite integral quadratic forms the automorphism groups of which are the maximal finite irreducible subgroups of $GL(8, \mathbf{Z})$. The remaining discussions for $n = 8$ and a complete description of the results for dimensions less than 10 appear in Part V [7].

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2. Irreducible Subgroups of $GL(2r, \mathbf{Z})$ Derived from Subgroups of $GL(r, \mathbf{Z})$ and Associated Forms. As we already mentioned in the introduction it is desirable to avoid the computation of all $\mathbf{Q}$-classes of minimal irreducible finite subgroups of $GL(n, \mathbf{Z})$, $n = 2r$. Therefore, we discuss the following two types of these groups G separately.

Type (α): G has a $\mathbf{Q}$-reducible subgroup of index two.

Type (β): G has no $\mathbf{Q}$-reducible subgroup of index two.

For $n = 8$ the majority of the groups belongs to Type (α), for instance all those groups G the order of which is a power of two. In the cases already treated we computed all centerings of G and the corresponding quadratic forms. Since only the forms are used to determine the maximal finite subgroups of $GL(n, \mathbf{Z})$, we shall develop a method for groups of Type (α) to find the forms without computing all centerings. Moreover, the method allows us to treat many groups simultaneously.

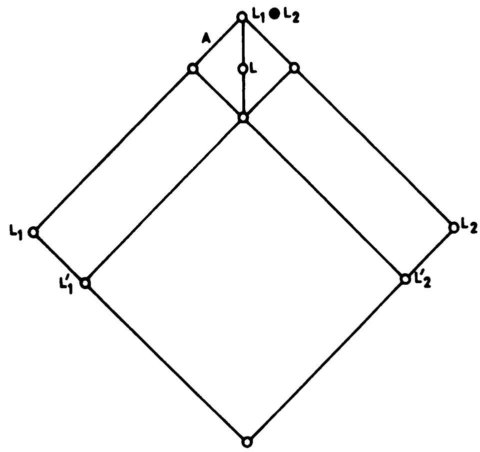
Let G be an irreducible finite subgroup of $GL(2r, \mathbf{Z})$ ($r \in \mathbf{N}$) with a $\mathbf{Q}$-reducible subgroup N of index two. Since N is normal in G, the restriction $\Delta|_N$ of the natural representation Δ of G to N can be assumed to be of the form $\Delta|_N = \Gamma_1 \dot{+} \Gamma_2$, where Γ_1, Γ_2 are rationally inequivalent irreducible integral representations of N with $\Gamma_1(N) = \Gamma_2(N)$ (Corollary (6.19) in [3] and Theorem (3.1) in [6]).

A short calculation shows that G can be chosen as $G = N \,\dot{\cup}\, \left(\begin{smallmatrix} 0 & h \\ I_r & 0 \end{smallmatrix}\right) N$, where N consists of block diagonal matrices and $h \in \Gamma_1(N) = \Gamma_2(N)$. If X denotes the integral positive definite matrix representing a form fixed by $\Gamma_1(N)$, then $I_2 \otimes X$ represents the corresponding form of G. We are interested in the integral forms which are induced on the centerings of G. Let $M = \mathbf{Z}^{2r \times 1}$ be the natural representation module of G and L a centering of M. We use a description of M as $\mathbf{Z}N$-module given in [5]. M splits into a direct sum, say $M = M_1 \oplus M_2$ with associated projections π_1, π_2

$$\left(M_1 := \left\{ \begin{pmatrix} l_1 \\ 0 \end{pmatrix} \in M | l_1 \in \mathbf{Z}^{r \times 1} \right\}, M_2 := \left\{ \begin{pmatrix} 0 \\ l_2 \end{pmatrix} \in M | l_2 \in \mathbf{Z}^{r \times 1} \right\} \right).$$

We define N-centerings of M_i: $L_i := \pi_i(L)$, $L_i' := M_i \cap L_i$ ($i = 1, 2$) and the finite $\mathbf{Z}N$-module $A := (L_1 \oplus L_2)/L$. Then $A \cong L_i/L_i'$ ($i = 1, 2$) holds, and there exist $\mathbf{Z}N$-epimorphisms μ_i: $L_i \longrightarrow A$ such that the kernel of $\mu_1 \oplus \mu_2$: $L_1 \oplus L_2 \longrightarrow A$: $(l_1, l_2) \longrightarrow \mu_1(l_1) + \mu_2(l_2)$ is equal to L. Furthermore, L is not only an N-centering but also a G-centering. Therefore, $\left(\begin{smallmatrix} 0 & h \\ I_r & 0 \end{smallmatrix}\right) L = L$ and $\left(\begin{smallmatrix} 0 & h \\ I_r & 0 \end{smallmatrix}\right) L_1 = L_2$ holds. Hence, μ_1, μ_2 can be chosen in such a way that $\mu_1 \oplus \mu_2$ is a $\mathbf{Z}G$-epimorphism. Now, let us assume $L_1 = M_1$.* Then $L_2 = M_2$ follows. Clearly, A can be generated by $m \leqslant r$ elements. Therefore, homomorphisms of L_i ($i = 1, 2$) into A, respectively endomorphisms of A, can be described by $m \times r$, respectively $m \times m$, matrices over $\mathbf{Z}/a\mathbf{Z}$ with $a := \exp(A)$. The matrix of such a mapping β is denoted by $\bar{\beta}$. Since A is a $\mathbf{Z}G$-module there is a homomorphism α: $G \longrightarrow \mathrm{Aut}_{\mathbf{Z}}(A)$ with $(\mu_1 \oplus \mu_2)g = \alpha(g)(\mu_1 \oplus \mu_2)$ for all $g \in G$. In particular, we obtain $(\mu_1 \oplus \mu_2)\left(\begin{smallmatrix} 0 & h \\ I_r & 0 \end{smallmatrix}\right) = \chi(\mu_1 \oplus \mu_2)$ with $\chi = \alpha\left(\left(\begin{smallmatrix} 0 & h \\ I_r & 0 \end{smallmatrix}\right)\right)$. Hence, $\bar{\mu}_2 = \bar{\chi}\bar{\mu}_1$ and $\bar{\mu}_1 h = \bar{\chi}\bar{\mu}_2$. Note that $\chi^2 \in \alpha(N)$ because of $\left(\begin{smallmatrix} 0 & h \\ I_r & 0 \end{smallmatrix}\right)^2 \in N$.

*Compare discussion in the second paragraph following (2.1).



In our later computations we want to deal with all groups G just discussed which simultaneously fix the same quadratic form. This can be done because of the considerations of this paragraph and especially because of the following lemma.

(2.1) LEMMA. *Let G, μ_1, μ_2, X as above and $k_1, k_2 \in \mathrm{Aut}_{\mathbf{Z}}(X)$. Then the quadratic form belonging to the centering $L = \ker(\mu_1 \oplus \mu_2)$ is equal to the one induced by X on $\ker(\mu_1 k_1 \oplus \mu_2 k_2)$.*

Proof. Let S be the matrix of the basis transformation from M to L. The form belonging to M is $S^t \mathrm{diag}(X, X)S$. Then $\mathrm{diag}(k_1^{-1}, k_2^{-2})S$ is the matrix of a basis transformation from M to $\ker(\mu_1 k_1 \oplus \mu_2 k_2)$. Therefore, the form induced by $\mathrm{diag}(X, X)$ on $\ker(\mu_1 k_1 \oplus \mu_2 k_2)$ is given by

$$(\mathrm{diag}(k_1^{-1}, k_2^{-1})S)^t \ \mathrm{diag}(X, X)(\mathrm{diag}(k_1^{-1}, k_2^{-1})S)$$

which is equal to $S^t \mathrm{diag}(X, X)S$, since k_1^{-1}, k_2^{-1} are elements of $\mathrm{Aut}_{\mathbf{Z}}(X)$. Q.E.D.

We explicitly describe the form induced by $I_2 \otimes X$ on $L = \ker(\mu_1 \oplus \mu_2)$. Let $B \in \mathbf{Z}^{r \times r}$ be the matrix of the basis transformation from L_1 to L_1'. Since μ_1 is an epimorphism there is a matrix $C \in \mathbf{Z}^{r \times r}$ which satisfies $\bar{\mu}_1 C + \bar{\chi}\bar{\mu}_1 = 0$. Then $S := \left(\begin{smallmatrix} B & C \\ 0 & I_r \end{smallmatrix}\right)$ is the matrix of the basis transformation from M to L. The form induced on M is $S^t \mathrm{diag}(X, X)S$.

For our discussion we assumed $L_1 = M_1$ so far. If L_1 is a proper N-centering of M_1, we transform G by $\mathrm{diag}(D, D)$, where D is the matrix of a basis transformation from M_1 to L_1. Note that the new group has the same "block pattern" as G. However, $\Gamma_1(N)$ is replaced by a rationally equivalent integral group. This leads us to associate the quadratic forms of the centerings of G with the N-centerings of M_1. Therefore, we proceed in our computation as follows.

Let $X \in \mathbf{Z}^{r \times r}$ be the matrix of a positive definite quadratic form with irreducible $\mathrm{Aut}_{\mathbf{Z}}(X)$, and let $K_1, \ldots, K_l$ be all minimal irreducible subgroups of $\mathrm{Aut}_{\mathbf{Z}}(X)$ up to $\mathbf{Z}$-equivalence. By $M = \mathbf{Z}^{r \times 1}$ we denote the natural representation module of $\mathrm{Aut}_{\mathbf{Z}}(X)$. For each submodule M' of M which is a centering with respect to one of the K_i, $i \in \{1, \ldots, l\}$, we define $K(M')$ to be the biggest subgroup of $\mathrm{Aut}_{\mathbf{Z}}(X)$ leaving M' invariant and α' to be the homomorphism of $K(M')$ into $\mathrm{Aut}_{\mathbf{Z}}(M/M')$ which describes the action of $K(M')$ on M/M'.

There are two possibilities for M' to yield a group G as considered above for which Γ_1, Γ_2 are inequivalent. Either $K(M')$ does not act faithfully on M/M' or α' is injective and there exists an irreducible subgroup H of $K(M')$ with an outer automorphism c subject to the following three properties: c^2 is an inner automorphism, c is induced by the normalizer of $\alpha'(H)$ in $\mathrm{Aut}_{\mathbf{Z}}(M/M')$ corresponding to H, and c is not induced by the normalizer of H in $GL(r, \mathbf{Q})$.

Let $N(M')$ be the set of all $\chi \in \mathrm{Aut}_{\mathbf{Z}}(M/M')$ for which an irreducible subgroup H of $K(M')$ exists which is normalized by χ and for which $\chi^2 \in \alpha'(H)$. In case α' is injective χ must correspond to an automorphism c described above. The forms we are interested in are induced by $I_2 \otimes X$ on $\ker(\mu_1 \oplus \mu_2)$ with $\bar{\mu}_2 = \bar{\chi}\bar{\mu}_1$ for $\chi \in N(M')$. However, we know from Lemma (2.1) that the χ lying in the same coset of $\mathrm{Aut}_{\mathbf{Z}}(M/M')$ modulo $\alpha'(K(M'))$ provide the same form. Hence, it suffices to pick one χ out of each coset. If M' runs through a set of representatives of the $\mathrm{Aut}_{\mathbf{Z}}(X)$-orbits of the centerings of M as discussed above we get all forms derived from $I_2 \otimes X$. Moreover, if X runs through a set of representatives of integral positive definite primitive r-ary forms with an irreducible automorphism group, we obtain all integral $2r$-ary forms an automorphism group of which is irreducible and has a $\mathbf{Q}$-reducible subgroup of index 2. This procedure is performed in the next paragraph for $r = 4$.

There can be made further simplifications the underlying ideas of which are demonstrated by the following example.

(2.2) LEMMA. *If $N(2M)$ is contained in the subgroup induced by $\mathrm{Aut}_{\mathbf{Z}}(X)$ in $\mathrm{Aut}_{\mathbf{Z}}(M/2M)$, each centering $M' \subseteq 2M$ provides only multiples of quadratic forms which can already be obtained by centerings M'' which are not contained in $2M$.*

Proof. For $M' = 2M$ we have $\bar{\mu}_1 = \bar{\mu}_2 = I_n \in \mathbf{Z}_2^{r \times r}$. A matrix for the basis transformation is $S = \begin{pmatrix} I_r & I_r \\ I_r & -I_r \end{pmatrix}$. Because of $S^t(I_2 \otimes X)S = 2(I_2 \otimes X)$ the result follows. Q.E.D.

3. Irreducible Subgroups of $GL(4, \mathbf{Z})$ and Derived Octonary Forms. There are—up to $\mathbf{Z}$-equivalence—six quaternary integral primitive quadratic forms admitting an irreducible automorphism group [1], [2]. They are represented by the following matrices:

$$Q_1 = I_4, \quad Q_2 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}, \quad Q_3 = I_2 \otimes \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

$$Q_4 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \otimes \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad Q_5 = I_4 + J_4, \quad Q_6 = 5I_4 - J_4,$$

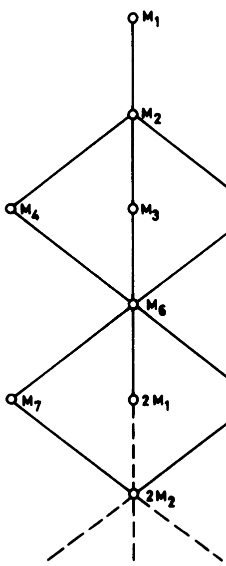
where all entries of J_4 are 1.

For each of these forms we have to proceed as described in the last paragraph in order to obtain the forms belonging to the centerings of the irreducible subgroups of $\mathrm{Aut}_{\mathbf{Z}}(I_2 \otimes Q_i)$ $(i = 1, \ldots, 6)$. For this we can make use of the list of the finite subgroups of $GL(4, \mathbf{Z})$ in [1].

Ad Q_1. The automorphism group of Q_1 is the full monomial group H_4 of order $2^4 4!$. From Theorem (3.2) in [6] one sees immediately that the minimal irreducible subgroups of H_4 are 2-groups. Hence we only have to consider 2-centerings of the natural representation module $L = \mathbf{Z}^{4 \times 1}$. For instance, the extraspecial 2-group

$$\left\{ g \otimes h \mid g, h \in \left\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \right\}$$

of order 32 has the following lattice of centerings which contains the centerings of all other irreducible subgroups of H_4.



The orbits under the action of H_4 are $\{M_1\}$, $\{M_2\}$, $\{M_3, M_4, M_5\}$, $\{M_6\}$, $\{M_7, M_8\}$, $\{2M_1\}$, ... The corresponding bases are expressed in the basis of M_1 via the transformation matrices:

$$\text{(i)}\ B(M_1) = I_4, \quad \text{(ii)}\ B(M_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad \text{(iii)}\ B(M_3) = I_2 \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$\text{(iv)}\ B(M_6) = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad \text{(v)}\ B(2M_1) = 2I_4,$$

(vi) $B(M_7) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

(3.1) LEMMA. *Only the forms induced by* $I_2 \otimes Q_1 = I_8$ *on* $\ker(\mu_1 \oplus \mu_2)$ *need to be considered*:

(i) $\bar{\mu}_1 = \bar{\mu}_2 = (0\ 0\ 0\ 0) \in \mathbf{Z}_2^{1\times 4}$,

(ii) $\bar{\mu}_1 = \bar{\mu}_2 = (1\ 1\ 1\ 1) \in \mathbf{Z}_2^{1\times 4}$,

(iii)
$$\bar{\mu}_1 = \bar{\mu}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \in \mathbf{Z}_2^{2\times 4},$$

(iv)
$$\bar{\mu}_1 = \bar{\mu}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \in \mathbf{Z}_2^{3\times 4},$$

(v)
$$\bar{\mu}_1, \bar{\mu}_2 \in \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \right\} \subseteq \mathbf{Z}_2^{4\times 4}.$$

(3.2) PROPOSITION. *The forms obtained from Lemma* (3.1) *are*:

(i) $F_1 = I_8$, $\det F_1 = 1^8$;

(ii)
$$F_2 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \end{pmatrix}, \det F_2 = 1^6 2^2;$$

(iii)
$$F_3 = I_2 \otimes \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}, \det F_3 = 1^4 2^4;$$

(iv)
$$F_4 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 \end{pmatrix}, \det F_4 = 1^2 2^6;$$

(v) $2F_1$; $2F_5$ with $F_5 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$, $\det F_5 = 1^8$; $2F_1$.

Proof of (3.1). An easy calculation yields that $K(M')$ acts faithfully on M_1/M' except for those centerings M' containing $2M_1$. We consider these exceptions first. In the cases (i) and (ii) we have $\bar{\mu}_1 = \bar{\mu}_2$ since the automorphism group of M_1/M_2 is trivial. In case (iii) $K(M_3)$ is a subgroup of index 3 of H_4 and its image in $\mathrm{Aut}_{\mathbf{Z}}(M_1/M_3)$ is isomorphic to C_2. Since the image of $K(M_3)$ in $\mathrm{Aut}_{\mathbf{Z}}(M_1/M_3)$ is self normal, we again obtain $\bar{\mu}_1 = \bar{\mu}_2$. In case (iv) we see that $K(M_6)$ is equal to H_4 and the subgroup induced by $K(M_6)$ in $\mathrm{Aut}_{\mathbf{Z}}(M_1/M_6)$ is isomorphic to the symmetric group S_4 which is the biggest subgroup of $\mathrm{Aut}_{\mathbf{Z}}(M_1/M_6)$ transforming M_2/M_6 into itself. Since M_2 is the only centering of M_1 of index 2 with respect to an irreducible subgroup of H_4 any element of $\mathrm{Aut}_{\mathbf{Z}}(M_1/M_6)$ normalizing the image of an irreducible subgroup of H_4 leaves M_2/M_6 invariant and, hence, lies in the image of H_4. Therefore, $\bar{\mu}_1 = \bar{\mu}_2$ holds. Also, in case (v) $K(2M_1)$ is equal to H_4 and the subgroup induced in $\mathrm{Aut}_{\mathbf{Z}}(M_1/M_2)$ consists of all permutation matrices of degree 4 with respect to the standard basis. Clearly the normalizer of this group is the direct product of the group itself and its centralizer which is easily seen to be generated by $J_4 + I_4$ (over $\mathbf{Z}_2$). The images of the irreducible subgroups of H_4 are the transitive groups of 4×4-permutation matrices. Some standard arguments show that their normalizers are contained in the normalizer of the group of all permutation matrices. Therefore, we end up with two possibilities for $\bar{\mu}_2$.

It remains to discuss the cases in which $K(M')$ acts faithfully on M_1/M'. For $M' = M_7$ (or $M' = M_8$) $K(M')$ is a subgroup of H_4 of index 2 and its image in $\mathrm{Aut}_{\mathbf{Z}}(M_1/M_7)$ is the biggest automorphism group leaving M_2/M_7 invariant. (Note M_6/M_7 is the Frattini-subgroup of M_1/M_7.) The same argument as in case (iv) shows that no μ_2 can exist.

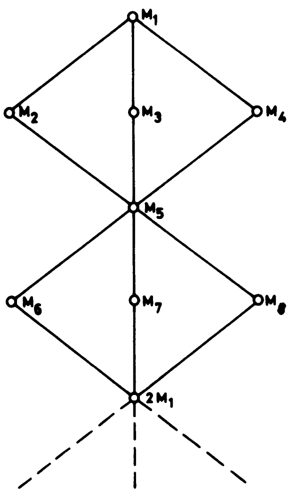
For $M' = 2M_2$ the module $\ker(\mu_1 \oplus \mu_2)$ is contained in

$$L := \langle 2(M_1 \oplus M_1), \ker(\mu_1 \oplus \mu_2)\rangle$$

of index 2. Clearly, there are two possible forms which can be induced on L, namely $2F_1$ or $2F_5$; compare (3.2). In the first case we can only obtain a form which already occurred in (ii). In the second we observe that the automorphism group of F_5 is the Weyl-group $W(E_8)$ of the root system E_8. The $2^8 - 1$ subgroups of index 2 of the lattice on which $W(E_8)$ acts fall into two orbits of length 135 and 120. The stabilizer of a lattice in the first orbit is irreducible of order $2^7 8!$ and the corresponding form is F_2. The stabilizers of lattices in the second orbit are reducible since they permute all roots which are orthogonal to a given root. Hence, we can only obtain the form F_2.

If M' is a centering properly contained in $2M_2$, then there must exist two centerings L_1, L_2 between $M_1 \oplus M_1$ and $\ker(\mu_1 \oplus \mu_2)$ which correspond to M_6 and $2M_2$ and are of index 2^3, respectively 2^5, in $M_1 \oplus M_1$. Clearly, we have $L_1 \supset L_2$; and there must be at least one more centering between L_1 and L_2. The forms induced on the subgroups S with $L_1 \supset S \supset L_2$ are easily seen to be equivalent to $2F_1$ in one case and $2F_5$ in two cases. Therefore, the subgroup with the form $2F_1$ must be a centering and a multiple of the form induced on $\ker(\mu_1 \oplus \mu_2)$ has already been obtained. Q.E.D.

Ad Q_2. The automorphism group of Q_2 is the Weyl group of the root system F_4. It is of order $2^7 3^2 = 1152$ and has a subgroup $\widetilde{H}_4$ of index 3 which is rationally equivalent to H_4. First, we consider the minimal irreducible subgroups which are rationally equivalent to a subgroup of H_4. They yield the following centerings:



The orbits under the action of $\mathrm{Aut}_Z(Q_2)$ are $\{M_1\}$, $\{M_2, M_3, M_4\}$, $\{M_5\}$, $\{M_6, M_7, M_8\}$, $\{2M_1\}, \ldots$. Transformation matrices of the corresponding bases:

$$\text{(i)}\quad B(M_1) = I_4,$$

$$\text{(ii)}\quad B(M_2) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad \text{(iii)}\quad B(M_5) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & -1 & 1 & 1 \\ 0 & -1 & 1 & -1 \end{pmatrix},$$

$$\text{(iv)}\quad B(M_6) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix}, \quad \text{(v)}\quad B(2M_1) = 2I_4.$$

(3.3) LEMMA. *Only the forms induced by* $I_2 \otimes Q_2 = F_3$ *on* $\ker(\mu_1 \oplus \mu_2)$ *need to be considered*:

(i) $\bar{\mu}_1 = \bar{\mu}_2 = (0\ 0\ 0\ 0) \in \mathbf{Z}_2^{1\times 4}$,

(ii) $\bar{\mu}_1 = \bar{\mu}_2 = (0\ 1\ 0\ 0) \in \mathbf{Z}_2^{1\times 4}$,

(iii)
$$\bar{\mu}_1 = \bar{\mu}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \in \mathbf{Z}_2^{2\times 4},$$

(iv)
$$\bar{\mu}_1 = \bar{\mu}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \in \mathbf{Z}_2^{3\times 4}.$$

(3.4) PROPOSITION. *The forms obtained from Lemma* (6.3) *are*:

(i) F_3; (ii) F_4; (iii) $2F_5$; (iv) $2F_2$.

They occurred already in (3.2).

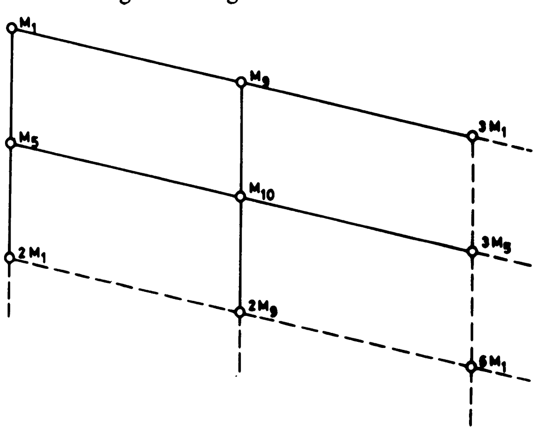
Proof of (3.3). The centerings M' for which $K(M')$ acts faithfully on M_1/M' are properly contained in $2M_1$. In the cases (i), (ii) and (iii) $K(M')$ induces the full automorphism group of M_1/M'. Therefore, $\bar{\mu}_1 = \bar{\mu}_2$ holds. The argument in the cases (iv) and (v) is exactly the same as in (3.1), case (iv). The rest follows from Lemma (2.2). Q.E.D.

Next, we discuss the minimal irreducible subgroups of $\mathrm{Aut}_{\mathbf{Z}}(Q_2)$ which are not rationally equivalent to a subgroup of H_4.

(3.5) LEMMA. *If U is an absolutely irreducible subgroup of* $\mathrm{Aut}_{\mathbf{Z}}(Q_2)$ *which has more 2-centerings than* $\mathrm{Aut}_{\mathbf{Z}}(Q_2)$ *itself, then U is rationally equivalent to a subgroup of H_4.*

Proof. The 2-centerings of $\mathrm{Aut}_{\mathbf{Z}}(Q_2)$ are $M_1, M_5, 2M_1, \ldots$. If M_1/M_5 or $M_5/2M_1$ become reducible as $\mathbf{Z}_2 U$-modules, the statement is obvious, since the forms induced on $M_2, M_3, M_4, M_6, M_7, M_8$ are multiples of Q_1. If M_1/M_5 and $M_5/2M_1$ stay irreducible, then the lattice of 2-centerings with respect to U is not linearly ordered and the order of U divides 72. Since 2^2 does not divide 72/4 this yields a contradiction to Corollary 3.6 in [4]. Q.E.D.

We obtain the following centerings and transformation matrices:



(vi) $$B(M_9) = \begin{pmatrix} 2 & 0 & 0 & -1 \\ -1 & 2 & 1 & 3 \\ 1 & 0 & 0 & -2 \\ 1 & -1 & -2 & -1 \end{pmatrix},$$

(vii) $$B(M_{10}) = \begin{pmatrix} 6 & 0 & -1 & 3 \\ 0 & 6 & -2 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

(viii) $$B(3M_1) = 3I_4.$$

(3.6) LEMMA. *We have only to consider forms which are induced by* $I_2 \otimes Q_2$ *on* $\ker(\mu_1 \oplus \mu_2)$:

(vi) $$\bar{\mu}_1 = \bar{\mu}_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \end{pmatrix} \in \mathbf{Z}_3^{2\times 4},$$

(vii) $$\bar{\mu}_1 = \begin{pmatrix} 2 & 3 & 2 & 0 \\ 3 & 2 & 1 & 1 \end{pmatrix}, \quad \bar{\mu}_2 = \begin{pmatrix} -1 & 0 & -1 & 3 \\ 0 & -1 & -2 & -2 \end{pmatrix} \in \mathbf{Z}_6^{2\times 4}.$$

(3.7) PROPOSITION. *From* (3.6) *we obtain the forms*

(vi) $$F_6 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \otimes \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}, \quad \det F_6 = 1^4 6^4;$$

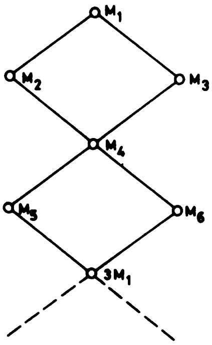
(vii) $$F_7 = I_4 \otimes \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \det F_7 = 1^4 3^4.$$

Proof of (3.6). M_9 is the only centering M' for which $K(M')$ acts faithfully on M_1/M' which we have not yet discussed. $K(M_9)$ has order 144 and induces the full automorphism group of M_1/M_9. Hence, in case (iv) we have $\bar{\mu}_1 = \bar{\mu}_2$. As for case (vii) we have $K(M_{10}) = K(M_9)$ and $K(M_{10})$ acts faithfully on M_1/M_{10}. The automorphism group $\overline{K(M_{10})}$ of M_1/M_{10} induced by $K(M_{10})$ is of index two in $\mathrm{Aut}_{\mathbf{Z}}(M_1/M_{10})$. Hence $\overline{\mu_2} = \epsilon\overline{\mu_1}$, where ϵ lies in $\mathrm{Aut}_{\mathbf{Z}}(M_1/M_{10})\backslash\overline{K(M_{10})}$. The centerings contained in $2M_1$ need not be considered because of Lemma (2.2). The centerings contained in $3M_1$ need not be considered, since $\mathrm{Aut}_{\mathbf{Z}}(Q_2)$ acts faithfully on $M_1/3M_1$ and different representations Δ of the irreducible subgroups H of $\mathrm{Aut}_{\mathbf{Z}}(Q_2)$ with $\Delta(H) = H$ can be distinguished by the signs of their character values which can already be determined from the action on $M_1/3M_1$. (All character tables are listed in [1].) Q.E.D.

Ad Q_3. The automorphism group of Q_3 is the wreath product $\mathrm{Aut}_{\mathbf{Z}}((\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix})) \sim C_2 \cong D_{12} \sim C_2$ of order $12^2 2$. As in the case of the form Q_2, one recognizes that the subgroups of $\mathrm{Aut}_{\mathbf{Z}}(Q_3)$ are of two kinds. Either their nontrivial centerings are 3-centerings, or they have just two $\prec$-maximal 3-centerings and nontrivial 2-centerings. If G is a subgroup of the first type, the centerings of G are also centerings of

$$\left\langle \mathrm{diag}\left(\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), \mathrm{diag}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \begin{pmatrix} \mathbf{0} & I_2 \\ I_2 & \mathbf{0} \end{pmatrix} \right\rangle.$$

These are given by:



The orbits under the action of $\mathrm{Aut}_{\mathbf{Z}}(Q_3)$ are $\{M_1\}$, $\{M_2, M_3\}$, $\{M_4\}$, $\{M_5, M_6\}$, $\{3M_1\}, \ldots$.

The corresponding bases are given by the transformation matrices:

$$\text{(i) } B(M_1) = I_4, \quad \text{(ii) } B(M_2) = \begin{pmatrix} 1 & 1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{(iii) } B(M_4) = I_2 \otimes \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix},$$

$$\text{(iv) } B(M_5) = \begin{pmatrix} 3 & 0 & -1 & -1 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad \text{(v) } B(3M_1) = 3I_4.$$

(3.8) LEMMA. *Only those forms have to be considered which are induced by* $I_2 \otimes Q_3$ *on* $\ker(\mu_1 + \mu_2)$:

(i) $\bar{\mu}_1 = \bar{\mu}_2 = (0\ 0\ 0\ 0) \in \mathbf{Z}_3^{1 \times 4}$,

(ii) $\bar{\mu}_1 = \bar{\mu}_2 = (1\ 1\ 1\ 1) \in \mathbf{Z}_3^{1 \times 4}$,

(iii) $$\bar{\mu}_1, \bar{\mu}_2 \in \left\{ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \right\} \subset \mathbf{Z}_3^{2 \times 4},$$

(iv) $$\bar{\mu}_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad \bar{\mu}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \in \mathbf{Z}_3^{3 \times 4}.$$

(3.9) PROPOSITION. *From* (3.8) *we obtain the following forms*

(i) $F_7 = I_4 \otimes \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ (*occurred already in* (3.7));

(ii) $$F_8 = \begin{pmatrix} 4 & -2 & 0 & 0 & -2 & 1 & 1 & 1 \\ -2 & 4 & 0 & 0 & 1 & -2 & -2 & 1 \\ 0 & 0 & 4 & -2 & -1 & 2 & -1 & -1 \\ 0 & 0 & -2 & 4 & -1 & -1 & 2 & -1 \\ -2 & 1 & -1 & -1 & 4 & -2 & -2 & 1 \\ 1 & -2 & 2 & -1 & -2 & 4 & 1 & -2 \\ 1 & -2 & -1 & 2 & -2 & 1 & 4 & -2 \\ 1 & 1 & -1 & -1 & 1 & -2 & -2 & 4 \end{pmatrix}, \quad \det F_8 = 1^3 3^4 9;$$

(iii) $F_9 = I_2 \otimes \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$, $\det F_9 = 1^2 3^4 9^2$; $3F_5$; F_9;
(iv) $3F_{10}$ *with* $F_{10} = I_8 + J_8$, $\det F_{10} = 1^7 9$.

Proof of (3.8). $K(M')$ acts faithfully on M_1/M' exactly for those centerings which are properly contained in M_4. In the cases (i) and (ii) $K(M')$ clearly induces the full automorphism group of M_1/M', hence $\bar{\mu}_1 = \bar{\mu}_2$.

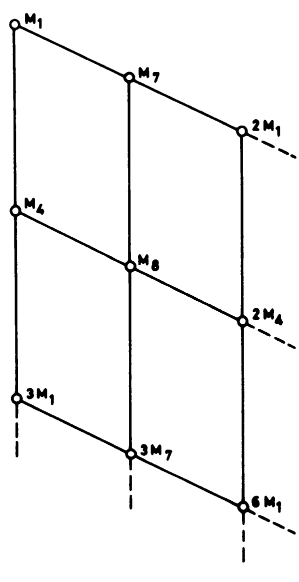
In case (iii) we obtain $K(M_4) = \mathrm{Aut}_{\mathbf{Z}}(Q_3)$ and the induced subgroup of $\mathrm{Aut}_{\mathbf{Z}}(M_1/M_4)$ is isomorphic to the dihedral group of order 8. Because of $|\mathrm{Aut}_{\mathbf{Z}}(M_1/M_4)| = 48$ there are two possibilities for $\bar{\mu}_2$.

In case (iv) $K(M_5)$ acts faithfully on M_1/M_5 and is of order 144. If $\overline{K(M_5)}$ denotes the induced subgroup of $\mathrm{Aut}_{\mathbf{Z}}(M_1/M_5)$, then M_4/M_5 has to be invariant under the normalizer of $\overline{K(M_5)}$. Hence, the normalizer is contained in a subgroup of order $3^2 2 \cdot 48$. The index is 6, and there is at most one possibility for a relevant outer automorphism. We end up with $\bar{\mu}_1, \bar{\mu}_2$ as given. Finally, if M' is contained in $3M_1$, the same argument as at the end of the proof of (3.6) applies. Q.E.D.

If G is a subgroup of $\mathrm{Aut}(Q_3)$ with nontrivial 2-centerings, those centerings are also centerings of the group

$$\left\langle \mathrm{diag}\left(\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\right), \mathrm{diag}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \mathrm{diag}(-I_2, I_2), \begin{pmatrix} \mathbf{0} & I_2 \\ I_2 & \mathbf{0} \end{pmatrix} \right\rangle.$$

They are given by:



The bases are described by the following transformation matrices:

$$\text{(vi)}\ B(M_7) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad \text{(vii)}\ B(M_8) = \begin{pmatrix} 6 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 6 & 3 & 2 \end{pmatrix},$$

$$\text{(viii)}\ B(2M_1) = 2I_4, \quad \text{(ix)}\ B(2M_4) = 2I_2 \otimes \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}.$$

(3.10) LEMMA. *We have to consider only forms which are induced by $I_2 \otimes Q_3$ on* $\ker(\mu_1 \oplus \mu_2)$:

$$\text{(vi)} \qquad \bar{\mu}_1 = \bar{\mu}_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \in \mathbf{Z}_2^{2\times 4},$$

$$\text{(vii)} \qquad \bar{\mu}_1 = \begin{pmatrix} 1 & -2 & 3 & 0 \\ 0 & 3 & -2 & 1 \end{pmatrix}, \quad \bar{\mu}_2 = \begin{pmatrix} 1 & -2 & -1 & 2 \\ -2 & 1 & -2 & 1 \end{pmatrix} \in \mathbf{Z}_6^{2\times 4}.$$

(3.11) PROPOSITION. *Lemma* (3.10) *provides the forms*

(vi) F_6 (*occurred already in* (3.7)),

(vii) $3F_3$ (F_3 *occurred already in* (3.2)).

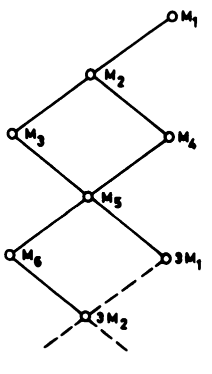
Proof of (3.10). $K(M')$ acts faithfully on M_1/M' except for $M' \in \{M_4, M_7, 2M_1\}$. The case $M' = M_4$ was already treated in (3.8). In case (v) the group $K(M_7)$ has order 48 and induces the full automorphism group of M_1/M_7. Hence, $\bar{\mu}_1 = \bar{\mu}_2$ holds. In case (vi) we have $K(M_8) = K(M_7)$, and $K(M_8)$ induces a subgroup of index 6 in $\mathrm{Aut}_{\mathbf{Z}}(M_1/M_7)$. Therefore, we have at most one possibility for a relevant outer automorphism yielding $\overline{\mu_2}$.

For $M' \subseteq 2M_1$ we can apply Lemma (2.2), since $\mathrm{Aut}_{\mathbf{Z}}(Q_3)$ induces a maximal imprimitive and, hence, self-normal subgroup of $\mathrm{Aut}_{\mathbf{Z}}(M_1/2M_1)$. The cases $M' \subseteq 3M_1$ were already discussed in (3.8). Q.E.D.

Ad Q_4. The automorphism group of Q_4 is rationally equivalent to a subgroup of order 144 of $\mathrm{Aut}_{\mathbf{Z}}(Q_3)$. There is no irreducible subgroup of $\mathrm{Aut}_{\mathbf{Z}}(Q_4)$ which admits nontrivial 2-centerings. All possible 3-centerings already occur as centerings of

$$\left\langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\rangle.$$

We obtain the following lattice of centerings:



The orbits under the action of $\mathrm{Aut}_{\mathbf{Z}}(Q_4)$ are $\{M_1\}, \{M_2\}, \{M_3, M_4\}, \{M_5\}, \{M_6\}, \{3M_1\}, \ldots$.

Corresponding bases:

$$\text{(i)}\ B(M_1) = I_4, \quad \text{(ii)}\ B(M_2) = \begin{pmatrix} -1 & -1 & -1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

$$\text{(iii)}\ B(M_3) = I_2 \otimes \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}, \quad \text{(iv)}\ B(M_5) = \begin{pmatrix} 1 & 0 & 0 & 3 \\ -1 & 3 & 0 & 0 \\ -1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\text{(v)}\ B(M_6) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \otimes \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

(3.12) LEMMA. *Only forms induced by* $I_2 \otimes Q_4$ *on* $\ker(\mu_1 \oplus \mu_2)$ *need to be considered*:

(i) $\bar{\mu}_1 = \bar{\mu}_2 = (0\ 0\ 0\ 0) \in \mathbf{Z}_3^{1\times 4}$,

(ii) $\bar{\mu}_1 = \bar{\mu}_2 = (1\ 1\ 1\ 1) \in \mathbf{Z}_3^{1\times 4}$,

(iii) $$\bar{\mu}_1 = \bar{\mu}_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \in \mathbf{Z}_3^{2\times 4},$$

(iv) $$\bar{\mu}_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad \bar{\mu}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \in \mathbf{Z}_3^{3\times 4}.$$

(3.13) PROPOSITION. *From Lemma* (3.12) *we obtain the forms*:

(i) F_9 (*occurred already in* (3.9));

(ii) $$F_{11} = \begin{pmatrix} 6 & 0 & 0 & 0 & 0 & 3 & 3 & 0 \\ 0 & 6 & 0 & 3 & 0 & 0 & -3 & 0 \\ 0 & 0 & 6 & 0 & -3 & 0 & 0 & 3 \\ 0 & 3 & 0 & 6 & 0 & 0 & -3 & -3 \\ 0 & 0 & -3 & 0 & 6 & 0 & 3 & 0 \\ 3 & 0 & 0 & 0 & 0 & 6 & 3 & 3 \\ 3 & -3 & 0 & -3 & 3 & 3 & 8 & 4 \\ 0 & 0 & 3 & -3 & 0 & 3 & 4 & 8 \end{pmatrix}, \quad \det F_{11} = 1 \cdot 3^4 \cdot 9^3;$$

(iii) $$F_{12} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \otimes \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \otimes \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \det F_{12} = 1 \cdot 3^3 \cdot 9^3 27;$$

(iv) $$F_{13} = 9I_8 - J_8, \quad \det F_{13} = 1 \cdot 9^7.$$

Proof of (3.12). $K(M')$ acts faithfully on M_1/M' except for $M' \in \{M_1, M_2, M_3, M_4, 2M_1\}$. In cases (i), (ii) we clearly have $\bar{\mu}_1 = \bar{\mu}_2$. In case (iii)

$K(M_3)$ has order 72 and induces a subgroup isomorphic to $C_2 \times S_3$ in $\mathrm{Aut}_{\mathbf{Z}}(M_1/M_3)$ which is already the biggest subgroup leaving M_2/M_3 invariant. Hence this subgroup is self-normalizing (M_2 is an $\mathrm{Aut}_{\mathbf{Z}}(Q_4)$-centering!), and we obtain $\bar{\mu}_1 = \bar{\mu}_2$. In case (iv) $K(M_5)$ acts faithfully on M_1/M_5 and is equal to $\mathrm{Aut}_{\mathbf{Z}}(Q_4)$. Since the index of the induced subgroup in $\mathrm{Aut}_{\mathbf{Z}}(M_1/M_5)$ in the biggest subgroup leaving M_2/M_5 invariant is 6, there is at most one possibility for $\bar{\mu}_2$ (compare (3.10)).

A similar argument shows that M_6 can yield at most one form. This form would necessarily be $9F_5$, since the automorphism group of the unique form obtained from M_5, namely $9I_8 - J_8$, has a unique centering of index 3. The form provided by this centering is $9F_5$ (for a similar argument compare the proof of (3.1)).

The cases $M' \subseteq 3M'$ and $M' \subseteq 2M'$ are treated as in (3.10). Q.E.D.

Ad Q_5. The automorphism group of Q_5 is isomorphic to $C_2 \times S_5$ of order 240. The orders of the irreducible subgroups are all divisible by 5, hence we have only nontrivial 5-centerings. They already occur as centerings of

$$\left\langle \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\rangle.$$

The lattice of 5-centerings is linearly ordered: $M_1 \supset M_2 \supset M_3 \supset M_4 \supset 5M_1 \supset \cdots$. Bases for M_i are:

$$B(M_i) = \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix} - I_4 \right)^{i-1} \qquad (i = 1, 2, 3, 4).$$

(3.13) Lemma. *We have to consider only forms which are induced by* $I_2 \otimes Q_5$ *on* $\ker(\mu_1 \oplus \mu_2)$:

(i) $\bar{\mu}_1 = \bar{\mu}_2 = (0\ 0\ 0\ 0) \in \mathbf{Z}_5^{1 \times 4}$,

(ii) $\bar{\mu}_1 = \bar{\mu}_2 = (1\ 2\ 3\ 4) \in \mathbf{Z}_5^{1 \times 4}$.

(3.14) Proposition. (3.13) *provides the forms*:

(i) $F_{14} = I_2 \otimes (I_4 + J_4)$, $\det F_{14} = 1^6 5^2$;

(ii) $$F_{15} = \begin{pmatrix} 4 & -2 & 1 & 1 & -1 & -2 & 0 & 2 \\ -2 & 4 & -1 & 0 & -1 & 1 & 1 & -1 \\ 1 & -1 & 4 & -1 & 1 & -2 & -2 & 0 \\ 1 & 0 & -1 & 4 & 1 & 1 & 2 & 1 \\ -1 & -1 & 1 & 1 & 4 & 1 & -1 & -2 \\ -2 & 1 & -2 & 1 & 1 & 4 & 2 & 0 \\ 0 & 1 & -2 & 2 & -1 & 2 & 4 & 1 \\ 2 & -1 & 0 & 1 & -2 & 0 & 1 & 4 \end{pmatrix}, \quad \det F_{15} = 1^4 5^4.$$

Proof of (3.13). $K(M')$ acts faithfully on M_1/M' except for $M' \in \{M_1, M_2, 2M_1\}$. The order of $K(M_2)$ is 40 and $K(M_2)$ already induces the full automorphism group of M_1/M_2. Hence $\bar{\mu}_1 = \bar{\mu}_2$ holds in case (ii). Because of $K(M_2) = K(M_3)$ one easily recognizes that there is just one representation Δ of $K(M_3)$ with $\Delta(K(M_3)) = K(M_3)$. Hence M_3 need not be considered, since $K(M_3)$ acts faithfully on M_1/M_3. Similar arguments work for all M' contained in M_3. If M' is contained in $2M_1$, Lemma (2.2) can be applied because the outer automorphism group of S_5 is trivial. Q.E.D.

Ad Q_6. The automorphism group of Q_6 is rationally equivalent to the automorphism group of Q_5. Hence, we are only concerned with 5-centerings which already occur as centerings of

$$\left\langle \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\rangle.$$

Again the lattice of 5-centerings is linearly ordered: $M_1 \supset M_2 \supset M_3 \supset M_4 \supset 5M_1 \supset \cdots$. Bases for M_i are:

$$B(M_i) = \left(\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} - I_4 \right)^{i-1} \qquad (i = 1, 2, 3, 4).$$

(3.15) Lemma. *Only the forms induced by* $I_2 \otimes Q_6$ *on* $\ker(\mu_1 \oplus \mu_2)$ *need to be considered*:

(i) $\bar{\mu}_1 = \bar{\mu}_2 = (0\ 0\ 0\ 0) \in \mathbf{Z}_5^{1\times 4}$,

(ii) $\bar{\mu}_1, \bar{\mu}_2 \in \{(1\ 1\ 1\ 1), (2\ 2\ 2\ 2)\} \subseteq \mathbf{Z}_5^{1\times 4}$.

(3.16) Proposition. (3.15) *yields the forms*:

(i) $F_{16} = I_2 \otimes (5I_4 - J_4)$, $\det F_{16} = 1^2 5^6$;

(ii) $$F_{17} = \begin{pmatrix} 8 & 3 & 3 & 3 & -3 & -3 & -2 & -2 \\ 3 & 8 & 3 & 3 & 2 & 2 & -2 & -2 \\ 3 & 3 & 8 & 3 & 2 & 2 & -2 & -2 \\ 3 & 3 & 3 & 8 & 2 & 2 & -2 & 3 \\ -3 & 2 & 2 & 2 & 8 & 3 & 2 & 2 \\ -3 & 2 & 2 & 2 & 3 & 8 & 2 & 2 \\ -2 & -2 & -2 & -2 & 2 & 2 & 8 & 3 \\ -2 & -2 & -2 & 3 & 2 & 2 & 3 & 8 \end{pmatrix}, \quad \det F_{17} = 1 \cdot 5^6 25;\ 5F_5.$$

Proof of (3.15). The proof is analogous to the one of (3.13) the main difference being $K(M_2) = \mathrm{Aut}_{\mathbf{Z}}(Q_6)$. Q.E.D.

The automorphism groups of the forms $F_1, \ldots, F_{17}$ obtained in this paragraph are discussed in Part V [7]. It turns out that all these groups are irreducible which is not completely clear from Section 2. In Part V we also determine the remaining forms of degree 8, i.e. those forms F for which there is no subgroup H in $\mathrm{Aut}_{\mathbf{Z}}(F)$ having a $\mathbf{Q}$-reducible subgroup of index 2. We finally obtain 26 $\mathbf{Z}$-classes of maximal finite irreducible subgroups of $GL(8, \mathbf{Z})$.

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