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NUMERICAL ANALYSIS

edited by<br>Gene H. Golub and Joseph Oliger

## Lecture Notes from the Short Course sponsored by the AMS, Atlanta, January 3-4, 1978


#### Abstract

This is the collection of texts prepared by the lecturers of the Numerical Analysis Short Course given at the A.M.S. meeting in Atlanta, Georgia in January 1978. Computational linear algebra, optimization and the solution of nonlinear equations, the approximation of functions and functionals, and approximations for initial and boundary value problems for ordinary and partial differential equations are discussed. Methods such as the QR factorization, singular value decomposition, quasiNewton and secant methods, finite difference, finite element and collocation methods are included in these discussions.

The subject matter was chosen to emphasize prom inent research areas and attitudes in numerical analysis. These are introductory lectures on the subject matter for presentation to an audience of scientists from other areas or disciplines. Typically there is an introduction to a given problem area and to techniques used, an application to applied problems, and a discussion of current research questions or directions.

Several trends in modern numerical analysis are discussed in these lectures. There has always been the quest to find the best way to do things. More realistic notions of "best" are evolving which in-


corporate the classical notions and realistic costs of producing the desired result. The discussion of good vs. best approximation is an example. More attention is being given to providing not only an answer, but a computed guarantee that it is a good answer-or a poor one. Easily computed and sharp a posteriori estimates are needed. The discussion of estimates of condition numbers is an example. There is progress being made in algorithm design based on operator splittings which allow one to take advantage of being able to solve simpler subproblems very efficiently. Updating strategies for optimization and splitting methods for differential equations are examples.
These texts should be useful to the practicing users of numerical methods, programmers, scientists, and engineers who would like to know what prog ress is being made on the theoretical and developmental side of the subject. They should be useful to numerical analysts to review progress in areas other than their own, and to mathematicians in general who would like to understand what the concerns of numerical analysts are. The texts should be useful for the development of seminars and reading courses in the academic environment. Many will probably find the bibliographies of current work most useful.

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# REVIEWS IN GRAPH THEORY 

Compiled and Edited by William G. Brown<br>McGILL UNIVERSITY DEPARTMENT OF MATHEMATICS

This publication is a four-volume compendium of about 9,600 reviews in graph theory published by Mathematical Reviews in Volumes 1 through 56, i.e. between 1940 and 1978 inclusive. Reviews were selected from the several sections of Mathematical Reviews which were the usual repositories of such items; from the subject lists in Mathematical Reviews indexes, where available; and through a systematic perusal of about half of all reviews published by Mathematical Reviews during the 39 years under investigation. Every review cited in a selected review was also read, and the process iterated until stable.

A classification scheme containing over 500 categories was developed for the purpose. Every review has been assigned one primary classification and, on the average, one secondary classification. Reviews are reprinted in strict chronological order of Mathematical Review numbers in their primary subject area, with a brief citation at each secondary location.

The final volume provides a detailed author index, which can serve as an effective bibliography of the subject.

These volumes are a research tool. They are directed to anyone who has occasion to consult the literature of graph theory: mathematicians, computer scientists, engineers, and management scientists, as well as students, teachers, and practicing researchers.

The potential reader requires no more background than would be required to read papers which are reviewed in the compendium. These vary from highly erudite papers in other areas of mathematics where graph theory is used as a tool to solve specific problems, to elementary descriptive papers which would be understandable to high school students.

A few of the reviews are themselves gems of the mathematical literature. But, for the most part, the reader will use this book as a research tool-to determine what has been done in a particular area of the subject, or to locate known papers when the values of not all parameters are available.

There has been nothing of this scope or magnitude in the subject before. This is the first major bibliography in graph theory which incorporates reviews.

The editor's previous work includes research papers in graph theory and related fields, and many reviews.

It is estimated that Reviews in Graph Theory will be 2,156 pages. Publication is scheduled for December 1980. For prices call or write the Sales Department of the Society after December 1.

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# Mathematical Surveys <br> Recent volumes in this distinguished series <br> APPROXIMATION BY POLYNOMIALS WITH INTEGRAL COEFFICIENTS 

Le Baron O. Ferguson

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Aside from the intrinsic interest to the pure mathematician, there is the likelihood of im-
portant applications to other arear $v$ mathematics; for example, in the simulation of transcendental functions on computers. In most computers, fixed point arithmetic is faster than floating point arithmetic and it may be possible to take advantage of this fact in the evaluation of integral polynomials to create more efficient simulations. Another promising area for applications of this research is in the design of digital filters. A central step in the design procedure is the approximation of a desired system function by a polynomial or rational function. Since only finitely many binary digits of accuracy actually can be realized for the coefficients of these functions in any real filter, the problem amounts (to within a scale factor) to approximation by polynomials or rational functions with integral coefficients.

This book should make the task of finding out what is known in this field significantly easier as it presents an introduction to most of the known results in the area of approximation by polynomials with integral coefficients and pointers to the literature for the rest. It is accessible to students at the graduate level and above.

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