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Some Large Primes and Amicable Numbers

By W. Borho

Abstract. Some new large primes of the form $3 \cdot 2^n - 1$ and $9 \cdot 2^n - 1$, related to amicable numbers, are given. Two new large amicable number pairs are found by the method of so-called "Thabit rules".

1. The numbers $3 \cdot 2^{12676} - 1$ and $9 \cdot 2^{11547} - 1$ *are primes*. The only larger known primes seem to be the four recent Mersenne primes $2^n - 1$ for $n = 19937$, 21701, 23209 and 44497, see [3]. *The only primes of the form* $3 \cdot 2^n - 1$ *resp.* $9 \cdot 2^n - 1$ *with* $n \leqslant 10^4$ *are* obtained for

$$n = 1274, 3276, 4204, 5134, 7559 \quad \text{resp.}$$
$$n = 1551, 3115, 3349, 5589, 5815, 5893, 7939, 8007,$$

and those $n \leqslant 10^3$ given by H. Riesel [2]. In conclusion, the primes $h \cdot 2^n - 1$ seem to be distributed very similarly for $h = 1, 3$ resp. 9.

2. As is well known, Mersenne primes are related to perfect numbers. Similarly, primes $p_n = 3 \cdot 2^n - 1$ and $q_n = 9 \cdot 2^n - 1$ are related to amicable numbers: *If* p_n, p_{n-1}, *and* q_{2n-1} *are simultaneously prime, then the two numbers* $2^n p_n p_{n-1}$, *and* $2^n q_{2n-1}$ *are amicable* (i.e. each is the sum of aliquot divisors of the other). This rule of Thabit ibn Kurrah (9th century) produces amicable numbers for $n = 2, 4, 7$, but *for no other* $n \leqslant 20{,}000$. Let us mention that the example $n = 4$, hitherto attributed to Fermat, was already discovered by the Arab Ibn Al-Bannā' (1256–1321); see [4].

3. We found, however, two new large amicable number pairs by means of "Thabit rules" of the following different type: *If for some* $n > 1$ *the two numbers* $q_1 = (u + 1)p^n - 1$, *and* $q_2 = (p - u)(u + 1)p^n - 1$ *are simultaneously prime, then* $aup^n q_1$ *and* $ap^n q_2$ *are amicable*. The constants a, u, p have to be specifically chosen, for instance $a = 2^3 \cdot 37 \cdot 59$, $u = 23 \cdot 52391$, $p = 2462401$ resp. $a = 3^4 \cdot 7^2 \cdot 11 \cdot 19$, $u = 233 \cdot 3821$, $p = 1784641$, give rules. We generated 37 new such rules by the procedure described in [1, Theorem 4]. *The two rules mentioned above actually give amicable numbers, both for* $n = 2$. The numbers have 42 (resp. 43) digits. We also confirmed and extended te Riele's computations [5], who found three large amicable pairs by means of Thabit rules derived from known amicable pairs applying Theorem 3 of [1].

4. For all computations on primes mentioned, we used primality tests of Lucasian type, with certain new simplifications, adopting them to the specific type of numbers considered in Section 1 resp. 3. These slight improvements in theory are to be published in some detail elsewhere, along with more material on the

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computations, and a list of 25 new amicable pairs (*Mitt. Math. Gesellsch. Hamburg*). But let us state here *as an example* the *version of the Lucas Test* that was used *for the numbers* $q_n = 9 \cdot 2^n - 1$: They can be prime only if $n = 12t + r$ with $r = 1, 3, 7$, or 9. Put $a = 2^{t+1}$ if $r = 3$ or 7, and $a = 1$ if $r = 1$ or 9. Consider the Fibonacci sequence $v_{i+1} = av_i + v_{i-1}$ starting with $v_0 = 2$, $v_1 = a$, and the Lucas sequence $r_{i+1} = r_i^2 - 2$ with start-value $r_1 = v_{18}$. *Then* q_n *is prime if and only if* $r_{n-1} \equiv 0 \bmod q_n$. In comparison, the method used in [2] would involve trial and error for computing an appropriate starting value r_1, without guarantee of success within reasonable bounds.

5. The computations were carefully done and checked by J. Buhl (Section 1) J. Buhl and S. Mertens (Section 2) resp. H. Hoffmann, E. Nebgen, and R. Reckow (Section 3) on the CDC Cyber 76 at the computer centres of Wuppertal and Köln.

FB Mathematik
Universität-Gesamthochschule Wuppertal
Gaussstrasse 20
D-5600 Wuppertal 1, West Germany

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