

## Solutions of the Diophantine Equation

$$A^4 + B^4 = C^4 + D^4$$

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**Abstract.** A survey is presented of the more important solution methods of the equation of the title. When space permits, a brief description of the methods and numerical examples are also given. The paper concludes with an incomplete list of 218 primitive nontrivial solutions in rational integers not exceeding  $10^6$ .

### 1. Introduction. The Diophantine equation

$$(1) \quad A^4 + B^4 = C^4 + D^4$$

was first proposed by Euler [1] in 1772 and has since aroused the interest of numerous mathematicians. Among quartic Diophantine equations it has a distinct feature for its simple structure, the almost perfect symmetry between the variables and the close relationship with the theory of elliptic functions. The latter is demonstrated by the fact that Eq. (1) is satisfied by the four elliptic theta functions of Jacobi,  $\vartheta_1$ ,  $\vartheta_3$ ,  $\vartheta_2$  and  $\vartheta_4$ , in that order [6].

One of the intriguing aspects of the equation is that numerical solutions are not easy to come by. Naturally, we are interested only in primitive and nontrivial solutions in real (and, occasionally, in Gaussian complex\*\*) integers. The first known examples of solutions, and among these the solution in "least integers", i.e.  $(A, B, C, D) = (134, 133, 158, 59)$ , were computed already by Euler [1], [2], [3]. Some others were found by later researchers (see [4, pp. 644-647]), but it was not until the advent of computers that systematic searches could be conducted. The most extensive lists published to date are due to Lander and Parkin [16] and Lander, Parkin and Selfridge [17]. These lists, to be called LPS lists, contain 31 and 15 solutions, respectively, and are complete in their respective ranges.

In this paper we discuss the more important solution methods and in conclusion present a list of 218 numerical solutions. This contains all presently known primitive and nontrivial solutions in the range  $\max(A, B, C, D) < 10^6$ . Motivation to produce the list has come from the need for empirical material to study Eq. (1). To produce a largest possible selection of varied numerical examples, we have used all available methods at our disposal, while making no effort for completeness.

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\*\* Results about solutions in Gaussian complex integers have been found by Ogilvy [15], Lander [18] and the present author, but are not included in this report.

**2. Some Preliminary Remarks.** Due to its special form, Eq. (1) is invariant under the transformations,

$$(2) \quad S_A: A' = -A; \quad S_B: B' = -B; \quad S_C: C' = -C; \quad S_D: D' = -D,$$

$$(3) \quad P_{AB}: A' = B, B' = A; \quad P_{CD}: C' = D, D' = C,$$

$$(4) \quad P_{AC} \cdot P_{BD}: A' = C, B' = D, C' = A, D' = B,$$

and their products. We shall call these the *elementary transformations* of Eq. (1). Solutions obtained by elementary transformations from a given solution will not be considered different, but different forms of the same solution. Of the  $2^7 = 128$  forms of any nontrivial solution we shall choose one as the *normal form* and define this by the following criteria:

- (i) All four numbers  $A, B, C, D$  are positive.
- (ii)  $B$  and  $D$  are odd.
- (iii) The peak, i.e.  $\max(A, B, C, D)$ , is equal to  $B$  when it is odd and to  $C$  when it is even.

When there is no reason to do otherwise, the numerical solutions are quoted in their normal forms.

Following Euler, we shall use the notations  $p = (C + A)/2$ ,  $q = (C - A)/2$ ,  $r = (B + D)/2$ ,  $s = (B - D)/2$ . With these substitutions we have

$$(5) \quad pq(p^2 + q^2) = rs(r^2 + s^2),$$

an equation equivalent to (1). When computed from normal forms, all four numbers  $p, q, r, s$  are integers and positive.

**3. Solution Methods.** In contrast to the analogous cubic equation, no formula exists for the complete solution of Eq. (1). In its absence we have a large variety of methods at our disposal, each of which supplies a different set of solutions. The methods can be classified as (i) arithmetic methods, (ii) computer methods and (iii) mixed methods.

In the case of arithmetic methods we make special assumptions and use existing solutions to derive new ones. Since both the initial and derived solutions necessarily satisfy the same special conditions, no arithmetic method can yield all the solutions. However, it is possible to produce, at least in principle, a complete list of solutions in any given range by the application of computer methods.\*\*\* Naturally, in practice, the range of search is limited by the processing capacities of the computer used.

A pure computer method was used to produce the LPS lists. This is described in [16] and hence will not be discussed here.

In the case of mixed methods the computer search is coupled with an arithmetic preparation and subsequent algebraic calculations. In all known mixed methods the computer is used to check if a given algebraic expression takes the value of a perfect square. When this occurs, a solution of (1) is obtained by a further simple calculation.

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\*\*\* As a matter of fact, at least for the present, this is possible only by computer methods. The fact that the solution  $(A, B, C, D) = (134, 133, 158, 59)$  is the "smallest", was established also by the use of computer [14], and no other proof is known.

Of all the methods the simplified "Pythagorean triplets" method, a mixed method, has proved in practice the most efficient. The majority of solutions in the list was obtained by this method. We shall discuss it within the next section.

**4. The Method of Pythagorean Triplets (PT).** This method, in its original form, can be summed up as follows. Let  $(a_1, b_1, c_1)$  and  $(a_2, b_2, c_2)$  be Pythagorean triplets, i.e. numbers representable in the forms:

$$a_1 = 2u_1v_1, \quad b_1 = u_1^2 - v_1^2, \quad c_1 = u_1^2 + v_1^2$$

and

$$a_2 = 2u_2v_2, \quad b_2 = u_2^2 - v_2^2, \quad c_2 = u_2^2 + v_2^2$$

for some integers  $u_1, v_1, u_2, v_2$ . If the triplets are such that

$$(6) \quad (a_1c_1 + a_2c_2)^2 + (b_1c_1 + b_2c_2)^2 = \text{perfect square,}$$

then a solution of Eq. (1) is readily at hand. To obtain it, we first remove the common factor  $\rho$  of  $\frac{1}{2} \cdot (a_1c_1 + a_2c_2)$  and  $b_1c_1 + b_2c_2$  and then solve for  $U$  and  $V$  the system

$$(7) \quad 2UV = \frac{1}{\rho} \cdot (a_1c_1 + a_2c_2), \quad U^2 - V^2 = \frac{1}{\rho} \cdot (b_1c_1 + b_2c_2).$$

Then with

$$(8) \quad p = Uu_1 + Vv_1, \quad q = Uv_1 - Vu_1, \quad r = Uu_2 + Vv_2, \quad s = -Uv_2 + Vu_2$$

we have  $A = p - q$ ,  $B = r + s$ ,  $C = p + q$ ,  $D = r - s$  as solution of Eq. (1). Finally, we simplify by possible common factors of  $A, B, C, D$  and set them in normal form. The solution is nontrivial if the greatest common factor of  $u_1^2 + v_1^2$ ,  $u_2^2 + v_2^2$  and  $u_1u_2 + v_1v_2$  equals 1. The formulas used in the method can easily be verified by applying (5).

When using this method, every solution will be obtained sooner or later. Moreover, it can be shown that every nontrivial solution can be computed from four different sets of parameters  $u_1, v_1, u_2, v_2$ , if their selection is subject to the restrictions (i)  $u_1, v_1, u_2 > 0$ , (ii)  $u_1 > v_1, u_2, |v_2|$ , (iii) the greatest common factor of  $u_1^2 + v_1^2, u_2^2 + v_2^2$  and  $u_1u_2 + v_1v_2$  is equal to 1. E.g. for the solution (134, 133, 158, 59) these sets are (26, 8, 14, 13), (45, 22, 6, -35), (55, 16, 40, -7) and (56, 34, 34, -31).

The disadvantage of the method lies in the difficulty of computing with high enough precision square roots of functional values of 8th degree polynomials. Nevertheless, when the method was first tried at the University of Zambia in 1972, 47 nontrivial solutions of (6) were obtained during one weekend night. Of these, 17 correspond to solutions not present in the LPS lists. The computer search was conducted by my former colleague, Jorma Pihlatie, using a relatively simple FORTRAN program and an IBM 1130 computer.

In a significant number of cases we have  $v_1 = u_2$  or  $u_1 = v_2$ , and this observation has led to a modification of the method. For, if  $v_1 = u_2$ , then

$$a_1c_1 + a_2c_2 = 2x(u_1 + v_2)(u_1^2 - u_1v_2 + v_2^2 + x^2)$$

and

$$b_1c_1 + b_2c_2 = u_1^4 - v_2^4 = (u_1 + v_2)(u_1 - v_2)(u_1^2 + v_2^2),$$

where  $x$  denotes the common value  $v_1 = u_2$ . Hence  $(u_1 + v_2)^2$  can be removed from the left-hand side of (6), which then reduces to

$$(9) \quad 4x^2(u_1^2 - u_1v_2 + v_2^2 + x^2)^2 + (u_1 - v_2)^2(u_1^2 + v_2^2)^2 = \square.$$

This equation, or its simplified form,

$$(10) \quad x^2(x^2 + 3y^2 + z^2)^2 + 4y^2(y^2 + z^2)^2 = \square,$$

(where  $y = (u_1 - v_2)/2$  and  $z = (u_1 + v_2)/2$ ), contains only 6th degree polynomials in three variables. Both the numerical work of polynomial evaluations and the dimension of search are hence reduced.

The majority of solutions marked by "PT" in the list was obtained by this simplified method. A search on the PDP-10 computer of the State University of Campinas has produced 222 nontrivial primitive solutions of (9) in the range  $u_1, v_1 \leq 1061$ ,  $u_1 > |v_2|$ . However, not all corresponding solutions of Eq. (1) are contained in the list. Excluded are 26 solutions whose peaks exceed  $10^6$ . Further, there are many instances of coincidence, i.e. different solutions of (9) leading to the same solution of (1). (Every nontrivial solution of Eq. (1) can be obtained from 8 different primitive sets of parameters.) Hence the number of solutions marked by "PT" falls well below 222.

**5. Semisolution Methods.** By a *semisolution* of Eq. (1) we mean a parametric solution  $A = A(u, v, t)$ ,  $B = B(u, v, t)$ ,  $C = C(u, v, t)$ ,  $D = D(u, v, t)$ , where the parameters  $u, v$  and  $t$  have to satisfy a further Diophantine equation  $Q(u, v) = t^2$ . Here  $Q(u, v)$  denotes a homogeneous quartic polynomial. Through the semisolutions the problem of solving Eq. (1) is thus reduced to the problem of making a quartic a perfect square.

Quite frequently, in lieu of  $A, B, C, D$ , the numbers  $p, q, r, s$  are given as functions of  $u, v, t$ , as e.g. in the semisolution

$$(11) \quad p = ft, \quad q = gu(f^2u - g^2v), \quad r = gt, \quad s = fv(f^2u - g^2v),$$

which goes back to Euler [2]. Here  $f$  and  $g$  are integral constants (free parameters) and

$$(12) \quad t^2 = (f^2u - g^2v)(f^2v^3 - g^2u^3).$$

The equation  $Q(u, v) = t^2$  will be referred to as the *quartic equation* of the semisolution in question. We shall agree that a solution of a quartic equation is termed nontrivial, if it leads to a nontrivial solution of Eq. (1).

The importance of solving quartic equations was recognized already by Euler who himself gave three different methods to make a quartic a perfect square. Following Euler several other methods have become known, but the basic problem, namely to find the *complete* solution of *any* given quartic equation has still remained unsolved. A method most frequently used is the following. Suppose we have found a representation of the quartic  $Q(u, v)$  in discriminant form, i.e.

$$(13) \quad Q(u, v) = \beta^2(u, v) - \alpha(u, v) \cdot \gamma(u, v),$$

where  $\alpha(u, v)$ ,  $\beta(u, v)$  and  $\gamma(u, v)$  denote quadratic homogeneous polynomials in  $u$  and  $v$ . Suppose further that a solution  $u_0, v_0, t_0$  of the quartic equation is already known. Then the roots of the quadratic equation

$$\alpha(u_0, v_0) \cdot x^2 - 2\beta(u_0, v_0) \cdot xy + \gamma(u_0, v_0) \cdot y^2 = 0$$

are rational numbers, namely

$$\frac{x_0}{y_0} = \frac{\beta(u_0, v_0) - t_0}{\alpha(u_0, v_0)} \quad \text{and} \quad \frac{x_1}{y_1} = \frac{\beta(u_0, v_0) + t_0}{\alpha(u_0, v_0)}.$$

Now the equation

$$(14) \quad \alpha(u, v) \cdot x^2 - 2\beta(u, v) \cdot xy + \gamma(u, v) \cdot y^2 = 0$$

is quadratic and homogeneous in  $u$  and  $v$ . Moreover, when we put  $x/y = x_n/y_n$ ,  $n = 0, 1$  it has one rational solution for  $u/v$ , namely  $u_0/v_0$ . It follows that the other solution must also be rational. In this way we obtain solutions  $u_{-1}/v_{-1}$  (when  $x/y = x_0/y_0$ ) and  $u_1/v_1$  (when  $x/y = x_1/y_1$ ). Repeating this argument with the new values  $u_{-1}, v_{-1}$  and  $u_1, v_1$ , we obtain further solutions, etc. The ratios  $u_n/v_n$  and  $x_n/y_n$  form, in general, a doubly infinite chain,<sup>†</sup>

$$(15) \quad \dots, \frac{u_{-1}}{v_{-1}}, \frac{x_0}{y_0}, \frac{u_0}{v_0}, \frac{x_1}{y_1}, \frac{u_1}{v_1}, \frac{x_2}{y_2}, \dots$$

determined by the equation (14) and an initial ratio  $u_0/v_0$ . Accordingly, we shall call Eq. (14) a *chain-generating equation*.

For a detailed account of the various arithmetic methods see Dickson [4, pp. 639–644]. All these are equivalent, in one way or another, to the chord and tangent process of finding rational points on an elliptic curve (see [8, Chapter 16]). It is a well-known fact that, by applying this process, all rational points can be generated from a finite set of them. Consequently, all solutions of a quartic Diophantine equation,  $Q(u, v) = t^2$ , can be found from a finite set of solutions by arithmetic methods. The main difficulty is that no known method exists to determine this finite set in the general case. Otherwise it would be possible to determine it e.g. for the quartic equation

$$u^4 - Mv^4 = t^2$$

with general integral  $M$ . For this equation it is known that when  $M$  is representable in the form  $a^4 + b^4$ , it has the independent solutions

$$u = a^2 + ab + b^2, \quad v = a + b, \quad t = ab \cdot (2a^2 + 3ab + 2b^2)$$

and

$$u = a^2 - ab + b^2, \quad v = a - b, \quad t = ab \cdot (2a^2 - 3ab + 2b^2)$$

When  $M$  is representable as a sum of two biquadrates in two different ways, we have two more solutions of this kind. The number of solutions in the finite set hence depends, among others, on the number of ways of representing  $M$  in the form  $a^4 + b^4$ . Thus the problem goes back to solving Eq. (1).

<sup>†</sup> The other possibility is a cycle.

The possible failure of arithmetic methods notwithstanding, computers can always be used within the limits of their capacities to solve quartic equations. Then the result is a complete list of solutions in the range of search. When the quartic equation of a semisolution is solved this way, we have another instance of solution methods of mixed type.

**6. Some Examples of Semisolutions.** The second example of semisolutions that appeared in the literature, was also given by Euler [3]. In simplified formulation we quote it as follows:

$$\begin{aligned} A &= 2Pu^2 + Qv^2, \\ SB &= t, \\ C &= 2Pu^2 + 2(Q - P)uv - Qv^2, \\ SD &= -2P(Q - P)u^2 + 4PQuv + Q(Q - P)v^2, \end{aligned}$$

with the quartic equation,

$$\begin{aligned} t^2 &= 4P^2(Q - P)^2u^4 + 8P(Q - P)(Q^2 + P^2)u^3v \\ &\quad + 4(Q^4 - 3Q^3P - 3QP^3 + P^4)u^2v^2 \\ &\quad - 4Q(Q - P)(Q^2 + P^2)uv^3 + Q^2(Q - P)^2v^4. \end{aligned}$$

The parameters  $P, S, Q$  form a Pythagorean triplet, i.e.  $P^2 + S^2 = Q^2$ , but are otherwise unspecified. We have nontrivial solutions when  $Q^2 + QP + P^2$  is a perfect square, as in the case  $P = 3, Q = 5$ , observed by Euler and leading to Solution 5, (see also [9]). However, these are not the only nontrivial solutions. An example when  $Q^2 + QP + P^2$  is not a perfect square is the following:  $P = 400, Q = 689, S = 561$ . Then we have the solution  $u = 51, v = 20, t = 761210360$ , leading to Solution 43 of the list.

Strictly speaking, the simplified PT method is also a semisolution method, since Eq. (10) turns into a quartic equation by substituting  $u/v$  for  $z$  and multiplying every term by  $v^4$ . Another semisolution can be derived from the original PT method by assuming that  $v_2 = 0$ . Then we have  $a_2 = 0, b_2 = c_2 = u_2^2$ , and hence by (7)

$$2UV = \frac{1}{\rho} \cdot 2u_1v_1(u_1^2 + v_1^2) \quad \text{and} \quad U^2 - V^2 = \frac{1}{\rho} \cdot (u_1^4 - v_1^4 + u_2^4).$$

Without loss of generality we may set  $U = \kappa(u_1^2 + v_1^2), V = u_1v_1/(\kappa\rho)$ , with  $\kappa$  denoting an appropriate constant. Substituting these in the second equation, we have

$$\kappa^2(u_1^2 + v_1^2)^2 - \frac{1}{\kappa^2\rho^2} \cdot u_1^2v_1^2 = \frac{1}{\rho} \cdot (u_1^4 - v_1^4 + u_2^4).$$

In the simplest case, i.e. when  $\kappa^2\rho = 1$ , this last equation reduces to

$$u_1^2v_1^2 = u_2^4 - 2v_1^4.$$

Now the equation

$$(16) \quad t^2 = u^4 - 2v^4$$

is known to have infinitely many nontrivial solutions (see [8, pp. 72–74]). Using these, we have the following expressions for  $u_1v_1, u_2$  and  $v_1$ :  $u_1v_1 = \sigma^2t, u_2 = \sigma u, v_1 = \sigma v$ , whence, by choosing  $\sigma = v, u_1 = t, v_1 = v^2$  and  $u_2 = uv$  follow. For

$p, q, r, s$  then we have, by (8),

$$(17) \quad p = u^4 t, \quad q = v^6, \quad r = uv(u^4 - v^4), \quad s = uv^3 t.$$

Applying the simplest nontrivial solution of (16), i.e.  $(u, v, t) = (3, 2, 7)$ , the result is Solution 6 of the list.

Finally, let us mention the semisolution of Fauquembergue [10], who gave it as an identity. We present it in the following formulation:

$$(18) \quad \begin{aligned} A &= t, \\ B &= 4u^4 + 9uv + 4v^2, \\ C &= 4u^2 + 15uv - 2v^2, \\ D &= -2u^2 + 15uv + 4v^2, \end{aligned}$$

with the quartic equation,

$$(19) \quad t^2 = 4u^4 + 132u^3v + 17u^2v^2 + 132uv^3 + 4v^4.$$

Fauquembergue's example can be easily generalized and developed into a full-scale theory. In the next section, however, we shall give only the main results, owing to the considerable length of calculations.

**7. Fauquembergue Type Semisolutions.** Observing that a sum of two biquadrates in two ways is also a sum of two squares in two ways and that as such it can be represented as a product of two sums of two squares each, we set

$$A^4 + B^4 = C^4 + D^4 = (a^2 + b^2)(c^2 + d^2)$$

and choose

$$(20) \quad A^2 = ac - bd, \quad B^2 = ad + bc, \quad C^2 = ac + bd, \quad D^2 = ad - bc.$$

Then

$$C^2 - B^2 = (a - b)(c - d) \quad \text{and} \quad B^2 - D^2 = 2bc.$$

Hence, without loss of generality, we may assume that

$$(21) \quad C + B = \mu(a - b), \quad C - B = \frac{1}{\mu}(c - d), \quad B + D = 2vc, \quad B - D = \frac{b}{\nu}$$

for some  $\mu$  and  $\nu$ . We shall denote the product  $\mu\nu$  by  $\tau$ . This quantity, which occurs frequently in the formulas, playing the role of an invariant, has the following expressions, derivable from (21) and (20),

$$(22.a) \quad \tau = \frac{B + C}{a - b} \cdot \frac{B + D}{2c} = \frac{(B + C)(B + D)}{A^2 - B^2 + C^2 + D^2},$$

and

$$(22.b) \quad \tau = \frac{c - d}{C - B} \cdot \frac{b}{B - D} = \frac{A^2 + B^2 - C^2 - D^2}{2(C - B)(B - D)}.$$

For  $2B$  we obtain from (21) the expressions,

$$\mu(a - b) - \frac{1}{\mu} \cdot (c - d) \quad \text{and} \quad \frac{b}{\nu} + 2vc,$$

which, when equated, yield a linear relation between the four parameters  $a, b, c, d$ . In addition there exists a quadratic relation, too, namely

$$(23) \quad 4ad = 2B^2 + 2D^2 = (B + D)^2 + (B - D)^2 = (2\nu c)^2 + \left(\frac{b}{\nu}\right)^2.$$

Using the linear relation, we can reduce this last one, (23), to an equation in 3 variables. Introducing 3 new variables,  $\lambda, \varepsilon$  and  $\varphi$ , defined by the linear substitutions,

$$(24) \quad \lambda = d - a\mu^2,$$

$$(25) \quad \varepsilon = \frac{b\mu}{\nu} = \frac{d + a\mu^2 - (2\tau + 1)c}{\tau + 1},$$

$$(26) \quad \varphi = (4\tau + 1)c + (\tau + 1)(2\tau + 1)\varepsilon,$$

the result is a Diophantine equation in which only pure quadratic terms appear, namely

$$(27) \quad \varphi^2 - (4\tau + 1)\lambda^2 = \Delta_1\Delta_2\varepsilon^2.$$

Here we use, for brevity, the notations

$$(28) \quad \Delta_1 = 2\tau^2 + 1, \quad \Delta_2 = 2\tau^2 + 4\tau + 1.$$

The solution of (27) is straightforward (see, e.g., [5, Chapter 4, Section 29]). Using a known solution,  $\varphi_0, \lambda_0, \varepsilon_0$ , and two free parameters,  $u$  and  $v$ , the complete solution of (27) may be written as follows:

$$(29) \quad \begin{aligned} \rho\varphi &= \varphi_0 u^2 + 2\lambda_0(4\tau + 1)uv + \varphi_0 \cdot (4\tau + 1)v^2, \\ \rho\lambda &= \lambda_0 u^2 + 2\varphi_0 uv + \lambda_0 \cdot (4\tau + 1)v^2, \\ \rho\varepsilon &= \varepsilon_0 u^2 - \varepsilon_0 \cdot (4\tau + 1)v^2, \end{aligned}$$

with  $\rho$  denoting a proportionality factor. This can be dropped (or its value set to be equal to 1) since we are interested only in the ratios  $\varphi : \lambda : \varepsilon$ . The initial solution,  $\varphi_0, \lambda_0, \varepsilon_0$ , is returned by the choice  $u = 1, v = 0$ .

An initial solution  $\varphi_0, \lambda_0, \varepsilon_0$  of (27) is readily available from a known solution  $A_0, B_0, C_0, D_0$  of Eq. (1), using the linear relations (21), (24), (25) and (26). As a result, the complete solution of (27) can be expressed in the terms of  $B_0, C_0, D_0$  instead of  $\varphi_0, \lambda_0, \varepsilon_0$ , and so can the parameters  $a, b, c, d$  and the variables  $B, C, D$ . The formulas for the latter are quoted as follows:

$$(30) \quad \begin{aligned} B &= B_0 u^2 - [(2\tau^2 - 1)B_0 + 4\tau C_0 - (2\tau^2 + 1)D_0]uv \\ &\quad + [(4\tau^3 + 6\tau^2 + 2\tau)B_0 - (4\tau^3 + 6\tau^2 - 2\tau - 1)D_0]v^2, \end{aligned}$$

$$(31) \quad \begin{aligned} C &= C_0 u^2 - [(2\tau^2 + 4\tau + 3)B_0 + 2C_0 - (2\tau^2 + 4\tau - 1)D_0]uv \\ &\quad + [(6\tau^2 + 4\tau + 1)(B_0 - D_0) + (4\tau + 1)C_0]v^2, \end{aligned}$$

$$(32) \quad \begin{aligned} D &= D_0 u^2 - [(2\tau^2 - 1)B_0 + 4\tau C_0 - (2\tau^2 + 1)D_0]uv \\ &\quad + [(4\tau^3 + 6\tau^2 + 6\tau + 1)B_0 - (4\tau^3 + 6\tau^2 + 2\tau)D_0]v^2. \end{aligned}$$

These three formulas, together with

$$(33) \quad A = t,$$

express a semisolution whose quartic equation is

$$(34) \quad t^2 = C^2 + D^2 - B^2 + 2\tau(C - B)(B - D).$$

Here on the right-hand side, the expressions given at (30)–(32) are to be substituted for  $B$ ,  $C$  and  $D$ . The relation (34) is an immediate consequence of (22.b) and (33).

The semisolution just derived has a structure similar to the one of Fauquembergue's example (18)–(19). We can obtain infinitely many others from it by subjecting the parameters  $u$ ,  $v$  to linear (nonsingular) transformations. We shall refer to all these as *Fauquembergue type semisolutions*, or briefly *F-solutions*.

The quartic in (34) can be brought into a discriminant form (13) in various ways and thus be solved arithmetically. Following are some discriminant forms of which the last two are symmetric in  $C$  and  $D$ .

$$(35) \quad t^2 = C^2 - (B - D)[B + D + 2\tau(B - C)],$$

$$(36) \quad t^2 = (B - C - D)^2 - 2(\tau + 1)(B - C)(B - D),$$

$$(37) \quad t^2 = [(2\tau + 1)B + C + D]^2 - 4(\tau + 1)[(\tau + 1)B + C - \tau D][(\tau + 1)B + D - \tau C].$$

A further one can be obtained from (35) by interchanging  $C$  and  $D$ .

*Numerical Example.* Starting from  $(A_0, B_0, C_0, D_0) = (292, 193, 256, 257)$ , (a form of Solution 3), we have by (22.a)

$$\tau = \frac{(B_0 + C_0)(B_0 + D_0)}{A_0^2 - B_0^2 + C_0^2 + D_0^2} = \frac{449 \cdot 450}{179600} = \frac{9}{8}.$$

Using this value, we can now compute the right-hand sides of (30)–(32) and (34). The result is the semisolution

$$(38) \quad \begin{aligned} A &= t, \\ B &= 193u^2 - 540uv + 419v^2, \\ C &= 256u^2 - 898uv + 570v^2, \\ D &= 257u^2 - 540uv + 67v^2, \end{aligned}$$

with the quartic equation,

$$(39) \quad t^2 = 85264u^4 - 477344u^3v + 999100u^2v^2 - 927096uv^3 + 273420v^4.$$

The discriminant form (37) of the quartic is as follows:

$$\begin{aligned} t^2 &= \frac{1}{16} \cdot (4561u^2 - 12772uv + 7995v^2)^2 \\ &\quad - \frac{153}{16} \cdot (377u^2 - 1438uv + 1385v^2)(337u^2 - 602uv + 281v^2). \end{aligned}$$

Hence we may set

$$\begin{aligned} \alpha(u, v) &= 377u^2 - 1438uv + 1385v^2, \\ \beta(u, v) &= 4561u^2 - 12772uv + 7995v^2 \end{aligned}$$

and

$$\gamma(u, v) = 153(337u^2 - 602uv + 281v^2)$$

as coefficients of the chain-generating equation, (14), and compute elements of the chain. The initial values  $u_0 = 1$ ,  $v_0 = 0$  (that correspond to the initial solution  $A_0, B_0, C_0, D_0$ ) give for  $x_0/y_0$  the ratio  $9/1$ , and for  $x_1/y_1$  the ratio  $5729/377$ . Using  $x_0 = 9, y_0 = 1$ , we obtain  $u_{-1}/v_{-1} = -313/592$ . On substituting 313 for  $u$  and  $-592$  for  $v$  in (38) and (39), we obtain Solution 41, i.e. (12772, 9153, 13472, 5121), after removing the common factor 29041.

From the formulas (30)–(34) it is clear that F-solutions can be derived from every (nontrivial) numerical solution of Eq. (1). Moreover, since we have 16 different  $\tau$ -invariants for every nontrivial solution of Eq. (1) (these are obtained from formulas (22.a) or (22.b) by applying the elementary transformations to  $A, B, C, D$ ), it is easily seen that every nontrivial solution of Eq. (1) generates 16, essentially different, F-solutions.

In the special case when  $4\tau + 1$  is equal to a rational square, say  $(2n + 1)^2$ , i.e.  $\tau = n(n + 1)$  for some rational  $n$ , the complete solution of (27) can be expressed without the use of a known solution, namely as

$$(40) \quad \rho\varphi = (2n + 1) \cdot (\Delta_1 u^2 + \Delta_2 v^2), \quad \rho\lambda = \Delta_1 u^2 - \Delta_2 v^2, \quad \rho\varepsilon = 2(2n + 1) \cdot uv.$$

Using these, we can derive the following semisolution:

$$(41) \quad \begin{aligned} A &= t, \\ B &= \tau\Delta_1 u^2 - (4\tau^3 + 6\tau^2 - 2\tau - 1)uv + \tau\Delta_2 v^2, \\ C &= -n\Delta_1 u^2 - (6\tau^2 + 4\tau + 1)uv + (n + 1)\Delta_2 v^2, \\ D &= \tau\Delta_1 u^2 - (4\tau^3 + 6\tau^2 + 6\tau + 1)uv + \tau\Delta_2 v^2, \end{aligned}$$

with the quartic equation,

$$(42) \quad \begin{aligned} t^2 &= n^2\Delta_1^2 u^4 - 2n\Delta_1 [2\tau^2 - 2\tau - 1 + 2(n + 1)(\tau + 1)(4\tau + 1)] u^3 v \\ &\quad + [(2\tau + 1)\Delta_1 \Delta_2 + 8\tau^2(\tau + 1)(6\tau^2 + 4\tau + 1)] u^2 v^2 \\ &\quad + 2(n + 1)\Delta_2 [2\tau^2 - 2\tau - 1 - 2n(\tau + 1)(4\tau + 1)] uv^3 \\ &\quad + (n + 1)^2 \Delta_2^2 v^4. \end{aligned}$$

Formulas (41) and (42) become trivial and hence useless when  $n = 0, -1$  or  $-1/2$ . Otherwise  $n$  may take any rational value. When  $n = -1/2$ , (41) and (42) are replaced by the following:

$$(43) \quad \begin{cases} A = t, \\ B = 2u^2 - uv - 2v^2, \\ C = 2u^2 - uv + 4v^2, \\ D = -4u^2 - uv - 2v^2, \end{cases}$$

$$(44) \quad t^2 = 16u^4 + 8u^3v + 23u^2v^2 - 8uv^3 + 16v^4.$$

Fauquembergue's example (18)–(19) corresponds to the case  $\tau = 2$ , i.e.  $n = 1$  or  $-2$ , and can be obtained from (41)–(42) by an appropriate linear transform on  $u$  and  $v$ .

The quartic in (42) has—among others—the following discriminant form:

$$(45) \quad \begin{aligned} t^2 &= [n\Delta_1 u^2 - (2\tau^2 - 2\tau - 1 + 2(n + 1)(\tau + 1)(4\tau + 1))uv \\ &\quad + (n + 1)(6\tau^2 + 6\tau + 1)v^2]^2 \\ &\quad - 4\tau(\tau + 1)(2\tau + 1)(4\tau + 1)[(n + 2)u - (n + 1)v]^2 v^2. \end{aligned}$$

When using this form to generate a chain (15), the resulting new solutions of Eq. (1) are in general different from those obtainable through the use of the other discriminant forms (35)–(37).

**8. Algebraic Reductions.** By making special assumptions, Eq. (1) can be reduced to a linear one and thus solved promptly. One way to do this is by applying *Cauchy's method* of reducing cubic homogeneous Diophantine equations in three unknowns [7]. For this we start from Eq. (5), which is already of 3rd degree in each of its four variables, and observe with Desboves [11] that it can be made a 3-variable homogeneous equation, e.g. by assuming that  $p/r = \mu = \text{const}$  and eliminating  $p$ . However, the same results can be obtained more quickly by direct methods which are possible due to the special symmetric character of Eq. (1). Depending on the assumptions to be made, we arrive at the methods (i) of *Lander* [18], (ii) of *Swinerton-Dyer* [13], (iii) *the perfect cube method* and (iv) *the two-solution method*.

In the case of *Lander's method* we add to the assumption

$$p/r = (C + A)/(B + D) = \mu = \text{const a similar one,}$$

namely  $(A + D)/(C - B) = \nu = \text{const}$ . Then, by denoting the variables of the new solution by primes, we have

$$(46) \quad p' = px, \quad r' = rx,$$

$$(47) \quad A' + D' = (A + D)y, \quad C' - B' = (C - B)y,$$

i.e.

$$(48) \quad q' = qy + (x - y)r, \quad s' = sy + (x - y)p.$$

Substituting the expressions for  $p'$ ,  $q'$ ,  $r'$  and  $s'$  in the equation

$$(49) \quad p'q'(p'^2 + q'^2) - r's'(r'^2 + s'^2) = 0,$$

it is readily seen that this reduces to a linear equation in  $x$  and  $y$ , since it can be simplified by  $xy(x - y)$  for obvious reasons. After simplifications we obtain

$$x(rs^3 - pq^3 + 2rp^3 - 2pr^3 + 3pqr^2 - 3rsp^2) + y[r(s - p)^3 - p(q - r)^3] = 0,$$

Thus

$$(50) \quad \rho x = p(q - r)^3 - r(s - p)^3$$

and

$$(51) \quad \rho y = rs^3 - pq^3 + 2rp^3 - 2pr^3 + 3pqr^2 - 3rsp^2,$$

where  $\rho$  is an appropriate proportionality factor. Substituting (50) and (51) in (46) and (48), we obtain  $p'$ ,  $q'$ ,  $r'$ ,  $s'$  as 5th degree functions of  $p$ ,  $q$ ,  $r$ ,  $s$ .

The relationship between the original and the new solutions is symmetric as is evident from the formulas (46) and (47). That means that the initial variables  $p$ ,  $q$ ,  $r$ ,  $s$ , are also 5th degree functions of  $p'$ ,  $q'$ ,  $r'$ ,  $s'$ . (For this reason the author terms the method as a *dual transform* of the variables.) It is also evident that by changing the signs of  $A$ ,  $B$ ,  $C$ ,  $D$ , each of the eight, essentially different, sign-combinations yield a different solution. Thus any nontrivial solution leads to 8 new solutions. Each of these 8 solutions can of course be employed again to obtain 8 other solutions and so on up to infinity. However, among the 8 solutions obtained from any of the first eight there are only seven new ones, the other one being identical to the initial solution for the symmetry mentioned above.

To illustrate this point, in Tables 1 and 2, respectively, we list the 8 solutions, in normal forms and in increasing order of their peaks, that are obtained from the solutions 2 and 3, i.e. (7, 239, 227, 157) and (256, 257, 292, 193), respectively. These examples serve also to show how widely the order of magnitude of the new solutions varies. Other examples were given in [18] and [19].

TABLE 1

| No. | <i>A</i>    | <i>B</i>    | <i>C</i>    | <i>D</i>   |
|-----|-------------|-------------|-------------|------------|
| 1   | 256         | 257         | 292         | 193        |
| 2   | 3 364       | 4 849       | 4 288       | 4 303      |
| 3   | 94 108      | 378 507     | 333 384     | 301 387    |
| 4   | 219 380     | 858 201     | 840 360     | 463 207    |
| 5   | 840 766     | 518 255     | 869 338     | 161 105    |
| 6   | 1 247 062   | 1 221 659   | 1 466 462   | 381 787    |
| 7   | 154 215 814 | 112 532 691 | 164 145 966 | 6 129 427  |
| 8   | 480 321 046 | 695 642 811 | 732 188 802 | 20 030 203 |

TABLE 2

| No. | <i>A</i>      | <i>B</i>      | <i>C</i>      | <i>D</i>      |
|-----|---------------|---------------|---------------|---------------|
| 1   | 7             | 239           | 227           | 157           |
| 2   | 248           | 2 797         | 2 524         | 2 131         |
| 3   | 3 080         | 39 789        | 30 348        | 35 885        |
| 4   | 21 708        | 1 102 237     | 1 047 672     | 721 699       |
| 5   | 732 965       | 11 610 623    | 11 589 385    | 3 395 261     |
| 6   | 4 925 561     | 37 899 133    | 37 834 817    | 10 984 277    |
| 7   | 524 937 467   | 3 830 530 437 | 3 823 811 431 | 1 121 601 087 |
| 8   | 7 493 624 732 | 2 184 895 107 | 7 507 106 424 | 432 984 899   |

In the case of *Swinnerton-Dyer's method* we keep the transformation formulas (46), but replace (48) by the following ones:

$$(52) \quad q' = qx + vy, \quad s' = sx + uy.$$

Substituting these in (49) and simplifying by  $x$  and  $y$ , we obtain a quadratic equation in  $x$  and  $y$ , i.e.

$$(53) \quad (p^3v + 3pq^2v - r^3u - 3rs^2u)x^2 + 3(pqv^2 - rsu^2)xy + (pv^3 - ru^3)y^2 = 0.$$

This can further be reduced by giving  $u$  and  $v$  values that make the coefficient of  $x^2$  zero, namely

$$(54) \quad u = p(p^2 + 3q^2), \quad v = r(r^2 + 3s^2).$$

Thus, from (53),

$$y/x = -3(pqv^2 - rsu^2) / (pv^3 - ru^3),$$

whence, after substitutions and simplifications,

$$(55) \quad \begin{aligned} \rho x &= r^2(r^2 + 3s^2)^3 - p^2(p^2 + 3q^2)^3, \\ \rho y &= -3qr(r^2 + 3s^2)^2 - ps(p^2 + 3q^2)^2, \end{aligned}$$

with  $\rho$  denoting a proportionality factor.

Substituting the values (55) and (54) in (52), we obtain the variables of the new solution as 9th degree functions of the variables of the initial solution. Unlike Lander's method, here the relationship between the original and the new solution is not symmetric. However, the number of derived solutions obtainable from a nontrivial solution is also eight, since different sign-combinations and/or different orders of  $A, B, C, D$  (see Elementary transformations, Section 2) lead to different solutions.

Despite this prolific character, the method is of little use in practice, unless we do not mind obtaining solutions in big numbers. E.g. starting from  $(A, B, C, D) = (-7, 239, 227, 157)$ , (a form of Solution 2), we obtain

$$\begin{aligned} A' &= 96\,781\,561\,849, & B' &= -22\,312\,231\,691, \\ C' &= -57\,072\,919\,679, & D' &= 93\,787\,787\,597 \end{aligned}$$

as new solution. The sudden increase in the order of magnitude of new solutions is due to the 9th degree expressions mentioned above.

The *perfect cube method* can be applied only in the special case when for the initial solution the ratio  $p/r$  is the cube of a rational number,  $u/v$ . Then the coefficient of  $y^2$  in (53) disappears and thus

$$x/y = -3(pqv^2 - rsu^2)/[pv(p^2 + 3q^2) - ru(r^2 + 3s^2)],$$

whence, after simplifications,

$$(56) \quad \rho x = -3uv(uq - vs), \quad \rho y = u^2(p^2 + 3q^2) - v^2(r^2 + 3s^2).$$

Substituting  $x$  and  $y$  in (46) and (52), we obtain a new solution.

As an example, let us start from the semisolution,

$$(p, q, r, s) = (u^4t, v^6, uv^3t, uv(u^4 - v^4)),$$

where  $t^2 = u^4 - 2v^4$  (a variant of semisolution (17)). Since  $p/r = u^3/v^3$ , the method can be applied. Thus, by (56),  $\rho x = 3u^2v^3t^2$  and  $\rho y = u^2t^2(u^8 - 3u^4v^4 - v^8)$ . Letting  $\rho = u^2t^3$ , from (46) and (52) we obtain for the new variables:

$$(57) \quad p' = 3u^4v^3, \quad q' = vt(u^4 - v^4), \quad r' = 3uv^6, \quad s' = ut(u^4 + 2v^4).$$

This new solution represents, of course, also a semisolution with the same quartic equation. Substituting  $(u, v, t) = (3, 2, 7)$ , the result is Solution 12.

We wish to remark that Euler's well-known 7th degree parametric solution can also be deduced this way. For the initial solution we choose the trivial solution  $(p, q, r, s) = (u^3, v^3, v^3, u^3)$  and have, by (56),

$$\rho x = 3u^2v^2(u^2 - v^2), \quad \rho y = (u^2 - v^2)(u^6 - 2u^4v^2 - 2u^2v^4 + v^6).$$

Letting  $\rho = u^2 - v^2$ , from (46) and (52) we obtain

$$\begin{aligned} p' &= 3u^5v^2, & q' &= v(u^6 - 2u^4v^2 + u^2v^4 + v^6), \\ r' &= 3u^2v^5, & s' &= u(u^6 + u^4v^2 - 2u^2v^4 + v^6), \end{aligned}$$

which is a variant of Euler's solution.

Finally we discuss the *two-solution method*. We denote the known solutions of (5) by  $(p_1, q_1, r_1, s_1)$  and  $(p_2, q_2, r_2, s_2)$ , and assume that

$$(58) \quad p_1/r_1 = p_2/r_2 = \mu = \text{const.}$$

The variables of the new solution are set as

$$(59) \quad p' = p_1x + p_2y, \quad q' = q_1x + q_2y, \quad r' = r_1x + r_2y, \quad s' = s_1x + s_2y.$$

Since by (58)  $p'/r' = p_1/r_1 = p_2/r_2$ , the equation (49) reduces to a cubic equation in  $x$  and  $y$ , namely

$$p_1(q_1x + q_2y) \left[ (p_1x + p_2y)^2 + (q_1x + q_2y)^2 \right] - r_1(s_1x + s_2y) \left[ (r_1x + r_2y)^2 + (s_1x + s_2y)^2 \right] = 0.$$

In this equation the coefficients of  $x^3$  and  $y^3$  are equal to zero (because  $(p_1, q_1, r_1, s_1)$  and  $(p_2, q_2, r_2, s_2)$  satisfy (5)), and the ratio  $x/y$  can thus be readily computed:

$$(60) \quad \begin{aligned} \rho x &= p_1(q_1p_2^2 + 2p_1p_2q_2 + 3q_1q_2^2) - r_1(s_1r_2^2 + 2r_1r_2s_2 + 3s_1s_2^2), \\ \rho y &= -p_1(q_2p_1^2 + 2p_1p_2q_1 + 3q_1^2q_2) + r_1(s_2r_1^2 + 2r_1r_2s_1 + 3s_1^2s_2). \end{aligned}$$

Substituting  $x$  and  $y$  in (59), the variables of the new solution are obtained as 5th degree functions of the initial variables. The relationship between the two initial solutions and the derived solution, however, is quite symmetric, as is evident from (59). That means that by choosing e.g.  $(p_1, q_1, r_1, s_1)$  and  $(p', q', r', s')$  as initial solutions,  $(p_2, q_2, r_2, s_2)$  is obtained as a derived solution.

It should further be noted that from any two solutions satisfying (58) there can actually be obtained one more solution by this method. For this we change the signs of  $p_2$  and  $r_2$  in (59) and (60).

As a numerical example, we mention the computation of Solution 98 by this method, starting from

$$(p_1, q_1, r_1, s_1) = (4563, 1409, 845, 5535)$$

(obtained from a form of Solution 13) and

$$(p_2, q_2, r_2, s_2) = (-1053, 1714, -195, 3342)$$

(obtained from a form of Solution 15), for which  $p_1/r_1 = p_2/r_2 = 27/5$ . The other solution, computed by using

$$(p_2, q_2, r_2, s_2) = (1053, 1714, 195, 3342),$$

is the following:

$$(A, B, C, D) = (13721986, 37753977, 34224263, -28875042).$$

The two-solution method includes, as a special case, the method of Lander. To show this, we choose for initial solutions a known solution,  $(p_1, q_1, r_1, s_1) = (p, q, r, s)$ , and an associated trivial solution,  $(p_2, q_2, r_2, s_2) = (p, r, r, p)$ . Substituting these in (60), for the ratio  $x/(x+y)$  we have

$$(rs^3 - pq^3 + 2rp^3 - 2pr^3 + 3pqr^2 - 3rsp^2) / [p(q-r)^3 - r(s-p)^3].$$

Hence, in view of (50) and (51), we are back at the method of Lander. The two-solution method can thus be regarded as a generalization of Lander's method.

In all four methods discussed in this section we have  $p/r = \text{const}$ , and this same condition is satisfied also by the semisolution (11)–(12). In fact, all derived solutions can also be obtained from the semisolution formulas by using different techniques to solve the quartic equation (12). For instance, when we use the discriminant-form,

$$(61) \quad [g^2u^2 + (f^2 + g^2)uv + f^2v^2]^2 - [g^2(u^2 + uv + v^2) + f^2v^2][g^2u^2 + f^2(u^2 + uv + v^2)],$$

of the quartic in (12) to set up a chain-generating equation, the resulting solutions of Eq. (1) are identical to those obtainable by Lander's method.

**9. Parameter Transformations.** In the context of parameter transformations by parameter we mean *intrinsic parameters*, such as the integral-valued solutions  $u, v, t$  of quartic Diophantine equations,  $t^2 = Q(u, v)$ , with  $u$  and  $v$  having no common prime factors. The term "intrinsic" is used to convey their characteristic behavior and to distinguish them from the "free" parameters, not subject to any constraints.

We arrive at the concept of intrinsic parameters through "homogenization". E.g. instead of considering the problem of finding the rational points  $P(x, y)$  on a quartic (or cubic when  $e_0 = 0$ ) elliptic curve

$$y^2 = e_0x^4 + e_1x^3 + e_2x^2 + e_3x + e_4 \equiv Q(x, 1),$$

we apply the substitutions,  $x = u/v, y = t/v^2$ , and consider the equivalent problem of finding integral-valued solutions of  $t^2 = Q(u, v)$ . The numerators and denominators,  $u$  and  $v$ , thus divorced from each other, that is liberated from being formal parts of a fraction, will then behave as independent entities in the course of further calculations. For example, they will transform independently of each other.

Theoretically, all quartic and cubic Diophantine equations, which can also be looked upon as functional equations satisfied by certain elliptic functions, have their own systems of intrinsic parameters. Due to the large variety of functional equations between elliptic functions, the number of possible parameters is also large. For Eq. (1) alone the number of parameters that are more significant is over 50.

When applying a *parameter transformation*, some parameters take on new values while others remain invariant. Examples of ratios (in the case of Eq. (1)) whose values may remain invariant are as follows:

$$\begin{aligned} I_1 &= (C + A)/(B + D), & I_2 &= (C^2 + A^2)/(B^2 + D^2), \\ I_3 &= (C^2 + A^2)/(B^2 - D^2), \\ I_4 &= (A^2 - B^2 + C^2 - D^2)/(AB - CD), \\ I_5 &= (A^2 + B^2 + C^2 + D^2)/(AB - CD), \\ I_6 &= \tau = (B + C)(B + D)/(A^2 - B^2 + C^2 + D^2), \\ I_7 &= I_4 \cdot (B - D)/(A - B + C + D). \end{aligned}$$

The transformations can be characterized and classified according to which of these and other ratios remain invariant.

We shall call a transformation *dual*, if the transformation formulas are symmetric in terms of the old and new values of transforming parameters. Of the presently

known transformations only two types are dual, but the methods based on these two supply the largest number of new solutions in relatively small integers.

The *Simple Dual Transformation* (SD) is characterized by having two invariants of type  $I_1$  and four invariants of type  $I_6$ . As a method, it is equivalent to the method of Lander, thus in general it results in the same eight new solutions.

The *Composite Dual Transformation* (CD) has  $I_6 = \tau$  (or any of the other 15 similar expressions) as its main invariant. A further invariant is  $I_7$  or a similarly built expression. When using it as method, it is more prolific than the SD, since the number of new solutions obtainable from a nontrivial solution is equal to 32. If the method is applied to one of the derived solutions (and this itself is nontrivial), then of the 32 newer solutions one is identical to the solution used at the outset.

Invariably, all methods of solving Eq. (1), which use one known solution to derive another, imply also a simultaneous transformation of parameters. However, when we talk about parameter transformations as methods, we mean *carrying out the computations in terms of the parameters*. This means a reduction of computational efforts, since the parameters are in general numerically smaller than the original variables  $A, B, C, D$  or  $p, q, r, s$ .

The dual transformations were discovered by the author in the years 1973–1974. Since then they have been used with success, as witnessed by the great number of solutions marked by SD or CD in Table 3. Unfortunately, lack of space does not permit to present here a more detailed account of them.

However, there exist equivalent methods that can readily be defined. For the SD this is Lander's method, already mentioned (see Section 8), or the semisolution method using (11), (12) and discriminant-form (61). The F-solution (30)–(34) provides methods equivalent to SD or to CD. More particularly, when (35) or (36) are used to form chain-generating equations, the results are methods equivalent to SD, and when (37) is used, we have a method equivalent to CD. Accordingly, the initial and the derived solutions, (Solutions 3 and 41), of the numerical example of Section 7 are composite dual transforms of each other.

**10. Parametric Solutions.** Many numerical solutions can be obtained from formulas of two-parameter solutions. The simplest set of formulas, denoted  $E(u, v)$ , is the following:

$$A = A(u, v) = u^7 + u^5v^2 - 2u^3v^4 - 3u^2v^5 + uv^6,$$

or, giving only the coefficients,

$$\begin{aligned} A &= (1, 0, 1, 0, -2, -3, 1, 0), \\ B &= A(v, -u) = (0, 1, 3, -2, 0, 1, 0, 1), \\ C &= A(u, -v) = (1, 0, 1, 0, -2, 3, 1, 0), \\ D &= A(v, u) = (0, 1, -3, -2, 0, 1, 0, 1). \end{aligned} \tag{62}$$

These find their origin in Euler [2], but in their present form are due to G erardin [12].

In contrast to the intrinsic parameters, the parameters  $u, v$  of two-parameter formulas are without any constraints. However, to avoid obvious common factors in the values of  $A, B, C, D$ , we choose for  $u$  and  $v$  integers that are relatively prime.

The modern way of computing sets of two-parameter solutions is by applying one of the existing parameter transformation methods to an already known parametric solution. E.g.  $E(u, v)$  can be derived by applying the simple dual transformation to the trivial solution  $(A, B, C, D) = (u, v, u, -v)$ . Similarly, by applying SD to  $E(u, v)$ , four new sets of parametric solutions are obtained [18], [19]. Including CD in the process will result in further sets of solutions. Of the rich variety of solutions found in this way we cite below the two simplest ones, denoted  $P_1(u, v)$  and  $P_2(u, v)$ . They are of 11th and 13th degree, respectively.

$$(63) \quad \begin{cases} A = (-1, -1, 4, 17, 33, 49, 58, 52, 32, 12, 2, 0), \\ B = (1, 4, 8, 7, 5, 17, 44, 64, 58, 34, 12, 2), \\ C = (1, 3, 8, 13, 9, -13, -44, -64, -58, -34, -12, -2), \\ D = (1, 2, 2, 7, 27, 59, 78, 66, 36, 12, 2, 0). \end{cases}$$

$$(64) \quad \begin{cases} A = (1, 3, 10, 22, 44, 67, 88, 95, 84, 58, 30, 10, 2, 0), \\ B = (0, 0, 3, 9, 24, 45, 72, 91, 94, 80, 54, 28, 10, 2), \\ C = (1, 3, 10, 22, 40, 63, 82, 95, 94, 80, 54, 28, 10, 2), \\ D = (0, 2, 5, 15, 28, 47, 64, 73, 66, 48, 26, 10, 2, 0). \end{cases}$$

Some of the simplest numerical solutions are special instances of these parametric solutions. E.g. Solution 3 can be obtained as  $P_1(1, 1)$  as well as  $P_2(1, 1)$ , Solution 2 as  $P_1(1, -2)$  and Solution 4 as  $P_1(2, -1)$ .

Obviously, the number of numerical solutions obtainable from two-parameter formulas and the number of two-parameter formulas themselves is infinite. However, it is not known whether or not every numerical solution of Eq. (1) can be represented as a special case of a parametric solution.

**11. The List of Numerical Solutions.** In Table 3 we present all known primitive and nontrivial solutions of Eq. (1) whose peaks do not exceed  $10^6$ . Accordingly, the list includes the solutions of the LPS lists, too, these occupying the first 46 entries.

The solutions are listed in their normal forms (see Section 2) and in the increasing order of their peaks. For reference purposes they are numbered with serials 1 through 218. The  $k$ th solution in the list will be denoted by  $S_k$ .

In the "Remark" column the abbreviations PT (Pythagorean triplets), SD (Simple dual transformation), CD (Composite dual transformation), 2S (Two-solution method, see Section 8), SS (Semi-solution method), refer to particular methods by which the solutions were obtained. The notation FS(...), with a numerical value between the parentheses, refers to F-solutions of the type (41)–(42). The inscribed number gives the value of the invariant  $\tau$ .  $E(u, v)$ ,  $P_1(u, v)$  and  $P_2(u, v)$  denote, respectively, solutions computed from sets of two-parameter formulas given by (62), (63) and (64), respectively, with parameter values  $u$  and  $v$ .

At some of the first 46 solutions the Remark box is left blank, indicating that these solutions would not have been discovered yet without the special computer method producing the LPS lists.

Some interesting finds are also among the solutions.  $S_{114}$  has the property that the values of  $A$  and  $B$  have a common factor greater than 1, namely 41. Accordingly, we have a numerical solution of the equation  $41^4 \cdot (a^4 + b^4) = c^4 + d^4$  with values

$a = 1447, b = 3271, c = 123497, d = 100807$ . Further, there are three solutions that are linked together by having their common origin in the triple coincidence

$$401168^4 - 17228^4 = 415137^4 - 248289^4 = 421296^4 - 273588^4,$$

or

$$4^4(100292^4 - 4307^4) = 3^4(138379^4 - 82763^4) = 12^4(35108^4 - 22799^4).$$

Keeping one equation at a time and simplifying by possible common factors, we obtain  $S_{107}$ ,  $S_{118}$  and  $S_{164}$ . The last solution,  $S_{164}$ , was discovered by this observation.

TABLE 3<sup>††</sup>

*A list of primitive nontrivial solutions of the equation*

*$A^4 + B^4 = C^4 + D^4$  in the range  $A, B, C, D < 10^6$*

| NO | A    | B    | C    | D    | REMARK  |
|----|------|------|------|------|---|
| 1  | 134  | 133  | 158  | 59   | E(2, 1), FS(2), FS(6), FS(-1/4), FS(3/4), PT          |
| 2  | 7    | 239  | 227  | 157  | P1(1, -2), FS(2), FS(-4/25), PT                       |
| 3  | 256  | 257  | 292  | 193  | P1(1, 1), P2(1, 1), SD OF S2, FS(2),<br>FS(-4/25), PT |
| 4  | 298  | 497  | 502  | 271  | P1(2, -1), FS(-1/4), FS(-6/25), PT                    |
| 5  | 514  | 359  | 542  | 103  | SS, PT  |
| 6  | 222  | 631  | 558  | 503  | SS, FS(-6/25), PT                                     |
| 7  | 76   | 1203 | 1176 | 653  | E(3, 1), FS(6), FS(12), FS(-2/9), FS(4/9),<br>PT      |
| 8  | 878  | 1331 | 1342 | 997  | PT  |
| 9  | 1324 | 2189 | 1784 | 1997 | PT  |
| 10 | 1042 | 2461 | 2026 | 2141 | SS, PT  |
| 11 | 248  | 2797 | 2524 | 2131 | P2(1, -2), SD OF S3, FS(2), PT                        |
| 12 | 1034 | 2949 | 2854 | 1797 | SS, PT  |
| 13 | 2986 | 2345 | 3190 | 1577 | P2(2, -1), SD OF S4, FS(-1/4), PT                     |
| 14 | 2338 | 3351 | 3494 | 1623 | E(3, 2), FS(3/4)                                      |
| 15 | 661  | 3537 | 3147 | 2767 | PT  |
| 16 | 3364 | 4849 | 4288 | 4303 | SD OF S2, FS(2), PT                                   |
| 17 | 2694 | 4883 | 3966 | 4397 | FS(40/9)  |

†† In the Remark column the notation "S" followed by a number should read with the number in subscript position. Thus e.g. the notation "S22" means "S<sub>22</sub>", etc.

TABLE 3 (continued)

|    |       |       |       |       |                                   |
|----|-------|-------|-------|-------|-----------------------------------|
| 18 | 604   | 5053  | 5048  | 1283  | PT                                |
| 19 | 4840  | 5461  | 6140  | 2027  | PT                                |
| 20 | 274   | 6619  | 5942  | 5093  | PT                                |
| 21 | 3070  | 6701  | 6730  | 2707  | SD OF S4, FS(-1/4), FS(-6/25), PT |
| 22 | 498   | 6761  | 5222  | 6057  | FS(6/25), PT                      |
| 23 | 1259  | 7557  | 7269  | 4661  | PT                                |
| 24 | 6336  | 7037  | 7604  | 5181  | SS                                |
| 25 | 7432  | 7559  | 8912  | 1657  | FS(-4/25), PT                     |
| 26 | 6262  | 8961  | 7234  | 8511  | SS                                |
| 27 | 6842  | 8409  | 9018  | 4903  | P1(2, 1), FS(3/4), PT             |
| 28 | 5098  | 9043  | 6742  | 8531  | P1(2, -3), FS(3/4), PT            |
| 29 | 635   | 9109  | 9065  | 3391  | FS(234/25), PT                    |
| 30 | 1104  | 9253  | 8972  | 5403  | FS(-14/225), PT                   |
| 31 | 1142  | 9289  | 4946  | 9097  | PT                                |
| 32 | 4408  | 9197  | 9316  | 173   |                                   |
| 33 | 5452  | 9733  | 7528  | 9029  | CD OF S22, FS(6/25), PT           |
| 34 | 7054  | 9527  | 10142 | 3401  |                                   |
| 35 | 5277  | 10409 | 9517  | 8103  |                                   |
| 36 | 8332  | 9533  | 10652 | 3779  | SS                                |
| 37 | 3644  | 11515 | 5960  | 11333 | FS(-6/49), FS(-66/1225), PT       |
| 38 | 2903  | 12231 | 10381 | 10203 | SD OF S1                          |
| 39 | 3550  | 12213 | 12234 | 1525  | FS(-9/100), PT                    |
| 40 | 1149  | 12653 | 12167 | 7809  | SD OF S17, FS(40/9)               |
| 41 | 12772 | 9153  | 13472 | 5121  | SD OF S19, CD OF S3               |
| 42 | 5526  | 13751 | 11022 | 12169 |                                   |
| 43 | 6470  | 14421 | 14190 | 8171  | SS                                |
| 44 | 6496  | 14643 | 13268 | 11379 |                                   |
| 45 | 261   | 14861 | 14461 | 8427  | SD OF S6, FS(-6/25)               |
| 46 | 581   | 15109 | 14723 | 8461  | SD OF S36                         |
| 47 | 6101  | 15265 | 13085 | 12743 | SD OF S25, FS(-4/25)              |
| 48 | 15594 | 6485  | 15642 | 5675  | PT                                |
| 49 | 4441  | 15869 | 14767 | 11291 | PT                                |
| 50 | 7168  | 16293 | 15188 | 11877 | FS(10/9), PT                      |
| 51 | 691   | 16377 | 15663 | 10411 | PT                                |
| 52 | 15566 | 13297 | 16886 | 9649  | PT                                |

TABLE 3 (continued)

|    |       |       |       |       |                                   |
|----|-------|-------|-------|-------|-----------------------------------|
| 53 | 17236 | 6673  | 17332 | 529   | E(4,1)                            |
| 54 | 4058  | 20117 | 17554 | 16213 | PT                                |
| 55 | 4091  | 22131 | 21027 | 14539 | SD OF S48,PT                      |
| 56 | 21526 | 19447 | 23702 | 14327 | FS(171/100),FS(138/289),PT        |
| 57 | 6502  | 24207 | 9738  | 24079 | PT                                |
| 58 | 19218 | 25451 | 27294 | 5653  | FS(-30/289),PT                    |
| 59 | 758   | 27407 | 27374 | 7217  | SD OF S39,FS(-9/100)              |
| 60 | 15393 | 27785 | 25355 | 22107 | FS(40/9),PT                       |
| 61 | 2558  | 28061 | 28058 | 4189  | FS(56),PT                         |
| 62 | 12787 | 30411 | 26511 | 24959 | PT                                |
| 63 | 5468  | 31731 | 25596 | 27661 | SD OF S15                         |
| 64 | 6484  | 32187 | 29812 | 23109 | E(4,3)                            |
| 65 | 4535  | 32241 | 32237 | 5565  | SD OF S78                         |
| 66 | 7713  | 36977 | 34107 | 26851 | SD OF S20,PT                      |
| 67 | 13348 | 37721 | 37868 | 167   | PT                                |
| 68 | 25489 | 38281 | 36001 | 30713 | CD OF S3,PT                       |
| 69 | 21676 | 38939 | 39448 | 17701 | FS(-6/25),PT                      |
| 70 | 3080  | 39789 | 30348 | 35885 | SD OF S3,PT                       |
| 71 | 11888 | 40465 | 40540 | 2513  | E(5,1),FS(-4/25)                  |
| 72 | 28544 | 41591 | 43676 | 11447 | SD OF S9,PT                       |
| 73 | 1499  | 44203 | 43007 | 25097 | CD OF S3,FS(-4/25),FS(-72/289),PT |
| 74 | 15052 | 45453 | 41324 | 34419 | PT                                |
| 75 | 18292 | 45883 | 46136 | 10757 | P1(1,-3),FS(6),PT                 |
| 76 | 41524 | 43847 | 49792 | 26887 | PT                                |
| 77 | 31494 | 53935 | 35710 | 52881 | E(5,3)                            |
| 78 | 45942 | 55247 | 53742 | 48271 | PT                                |
| 79 | 28997 | 60369 | 59777 | 33237 | P1(1,2),FS(6),PT                  |
| 80 | 5966  | 61583 | 61478 | 17743 | PT                                |
| 81 | 38078 | 60763 | 62206 | 29531 | P2(2,1),SD OF S28,FS(3/4)         |
| 82 | 23841 | 64369 | 60033 | 46063 | SD OF S8                          |
| 83 | 61528 | 45471 | 65196 | 27103 | FS(6)                             |
| 84 | 60328 | 56941 | 66308 | 45869 | PT                                |
| 85 | 33050 | 68303 | 46130 | 65521 | PT                                |
| 86 | 3698  | 72121 | 70594 | 38599 | PT                                |
| 87 | 1661  | 73059 | 71807 | 37143 | SD OF S10                         |

TABLE 3 (continued)

|     |        |        |        |        |                                   |
|-----|--------|--------|--------|--------|-----------------------------------|
| 88  | 11884  | 73833  | 37404  | 72599  | SD OF S36                         |
| 89  | 5728   | 74253  | 54212  | 68301  | CD OF S8,PT                       |
| 90  | 6464   | 74411  | 54044  | 68587  | CD OF S89,PT                      |
| 91  | 22813  | 78021  | 71089  | 58593  | SD OF S31                         |
| 92  | 8427   | 80399  | 79419  | 37631  | PT                                |
| 93  | 14493  | 81539  | 80623  | 37593  | CD OF S6,FS(-6/25),PT             |
| 94  | 37996  | 81885  | 54520  | 78621  | SD OF S50,FS(10/9),PT             |
| 95  | 23359  | 83771  | 74167  | 66269  | PT                                |
| 96  | 39393  | 87797  | 85173  | 55073  | PT                                |
| 97  | 15322  | 89345  | 59678  | 84545  | E(5,2)                            |
| 98  | 37686  | 90017  | 81622  | 69471  | 2S FROM S13 AND S15               |
| 99  | 27879  | 90829  | 89841  | 43307  | P1(3,-2),FS(-2/9),PT              |
| 100 | 89236  | 58231  | 93032  | 2359   | PT                                |
| 101 | 37879  | 94543  | 92213  | 55733  | SD OF S42                         |
| 102 | 17006  | 97681  | 29882  | 97489  | PT                                |
| 103 | 1788   | 101819 | 60752  | 98427  | SD OF S33,FS(6/25),PT             |
| 104 | 47139  | 103543 | 98049  | 72389  | PT                                |
| 105 | 57832  | 103809 | 83004  | 94529  | P1(3,-1),FS(-2,3),PT              |
| 106 | 13614  | 104909 | 57582  | 102451 | SD OF S42                         |
| 107 | 100292 | 68397  | 105324 | 4307   | PT                                |
| 108 | 5444   | 106931 | 78952  | 97907  | SD OF S89,PT                      |
| 109 | 99978  | 76405  | 107478 | 27275  | PT                                |
| 110 | 29286  | 117473 | 111838 | 76767  | SD OF S56,FS(171/100),FS(138/289) |
| 111 | 12840  | 126253 | 72960  | 122579 | PT                                |
| 112 | 39717  | 126659 | 109213 | 104133 | PT                                |
| 113 | 110758 | 108619 | 127034 | 73547  | PT                                |
| 114 | 59327  | 134111 | 123497 | 100807 | PT                                |
| 115 | 34813  | 134413 | 114613 | 111637 | SD OF S1,FS(2)                    |
| 116 | 122664 | 112507 | 139356 | 55483  | SD OF S205                        |
| 117 | 3800   | 140017 | 49328  | 139505 | PT                                |
| 118 | 91196  | 138379 | 140432 | 82763  | PT                                |
| 119 | 125844 | 135829 | 143844 | 113003 | PT                                |
| 120 | 8052   | 144401 | 135504 | 99409  | CD OF S23,PT                      |
| 121 | 72274  | 144733 | 73766  | 144541 | CD OF S22,PT                      |
| 122 | 91508  | 147941 | 99848  | 145627 | PT                                |

TABLE 3 (continued)

|     |        |        |        |        |                                    |
|-----|--------|--------|--------|--------|------------------------------------|
| 123 | 78804  | 153863 | 129644 | 133383 | PT                                 |
| 124 | 151394 | 92839  | 154522 | 73703  | CD OF S1, SD OF S27, FS(3/4), PT   |
| 125 | 157582 | 85491  | 158642 | 77811  | FS(3/4)                            |
| 126 | 28580  | 160133 | 159544 | 56635  | SD OF S181                         |
| 127 | 126168 | 164705 | 131760 | 161951 | PT                                 |
| 128 | 113690 | 156939 | 166314 | 54155  | CD OF S50, FS(10/9)                |
| 129 | 125516 | 161405 | 174484 | 7805   | E(5, 4)                            |
| 130 | 29259  | 175033 | 156241 | 136131 | PT                                 |
| 131 | 18657  | 178559 | 178509 | 33499  | PT                                 |
| 132 | 171266 | 148247 | 191218 | 50327  | PT                                 |
| 133 | 48478  | 198665 | 168254 | 166135 | FS(-9/100), PT                     |
| 134 | 6758   | 200635 | 36350  | 200584 | FS(56), PT                         |
| 135 | 190444 | 207971 | 191512 | 207139 | PT                                 |
| 136 | 153664 | 203349 | 213672 | 116309 | PT                                 |
| 137 | 219256 | 47769  | 219372 | 23641  | SD OF S147, FS(42)                 |
| 138 | 88198  | 226063 | 138394 | 219121 | CD OF S4, PT                       |
| 139 | 22125  | 228901 | 228825 | 44393  | PT                                 |
| 140 | 81416  | 235201 | 233212 | 109951 | FS(396/625), PT                    |
| 141 | 248034 | 134611 | 252974 | 64851  | FS(-560/7569), PT                  |
| 142 | 53797  | 253163 | 249751 | 122527 | SD OF S10                          |
| 143 | 112304 | 255295 | 253172 | 131455 | P2(1, 2), FS(6), SD OF S75         |
| 144 | 243690 | 196343 | 255718 | 164745 | PT                                 |
| 145 | 32458  | 261143 | 88046  | 260311 | PT                                 |
| 146 | 72489  | 266063 | 230099 | 217443 | FS(10/9)                           |
| 147 | 266116 | 52361  | 266192 | 36553  | FS(42), PT                         |
| 148 | 95248  | 282751 | 277724 | 151361 | SD OF S37, FS(-6/49), FS(-66/1225) |
| 149 | 287178 | 67429  | 287394 | 20773  | E(6, 1)                            |
| 150 | 283546 | 226531 | 308822 | 45683  | PT                                 |
| 151 | 166448 | 331047 | 295116 | 268441 | PT                                 |
| 152 | 30519  | 334883 | 327183 | 182869 | SD OF S83, FS(6)                   |
| 153 | 136321 | 342081 | 328619 | 220803 | CD OF S120, PT                     |
| 154 | 217863 | 348197 | 315957 | 289111 | PT                                 |
| 155 | 240394 | 332259 | 349582 | 155997 | SD OF S8, PT                       |
| 156 | 146514 | 354041 | 350254 | 183033 | P2(2, -3), SD OF S27, FS(3/4)      |
| 157 | 177070 | 356307 | 338310 | 251501 | SD OF S95                          |

TABLE 3 (continued)

|     |        |        |        |        |                                  |
|-----|--------|--------|--------|--------|----------------------------------|
| 158 | 130841 | 357787 | 356663 | 149387 | SD OF S5                         |
| 159 | 143066 | 362975 | 358090 | 191137 | SD OF S35                        |
| 160 | 33058  | 374989 | 338918 | 284813 | SD OF S5                         |
| 161 | 94108  | 378507 | 333384 | 301387 | SD OF S2                         |
| 162 | 238231 | 379915 | 338231 | 323605 | PT                               |
| 163 | 379674 | 157775 | 382090 | 96207  | P2(1,-3),SD OF S79,FS(6)         |
| 164 | 17228  | 415137 | 401168 | 248289 | TRIPLE COINCIDENCE (SEE SEC. 11) |
| 165 | 31238  | 419909 | 419762 | 81659  | E(7,1)                           |
| 166 | 19687  | 421653 | 410253 | 239359 | PT                               |
| 167 | 389242 | 381583 | 441718 | 279311 | SD OF S22,FS(6/25),PT            |
| 168 | 292304 | 454681 | 335108 | 439847 | SD OF S18                        |
| 169 | 348208 | 476025 | 396792 | 450695 | PT                               |
| 170 | 345588 | 444311 | 480032 | 108201 | E(7,3)                           |
| 171 | 482944 | 106163 | 483172 | 70157  | PT                               |
| 172 | 418394 | 405359 | 487906 | 176687 | SD OF S73,FS(-4/25),FS(-72/289)  |
| 173 | 485288 | 378327 | 500508 | 338921 | SD OF S135                       |
| 174 | 59870  | 515353 | 175754 | 513625 | CD OF S31,SD OF S102             |
| 175 | 142934 | 519249 | 487022 | 360303 | P2(3,-1),SD OF S99,FS(-2/9)      |
| 176 | 452420 | 434539 | 525152 | 146565 | CD OF S1,FS(6)                   |
| 177 | 149317 | 533957 | 498473 | 376271 | PT                               |
| 178 | 504474 | 364829 | 535658 | 111459 | CD OF S3,SD OF S68               |
| 179 | 119014 | 539943 | 470878 | 435687 | SD OF S12                        |
| 180 | 490250 | 500971 | 548278 | 417515 | CD OF S13,PT                     |
| 181 | 258176 | 547461 | 554092 | 57669  | PT                               |
| 182 | 227697 | 558305 | 531145 | 377271 | PT                               |
| 183 | 346622 | 565325 | 564730 | 349171 | PT                               |
| 184 | 21103  | 569609 | 569459 | 102653 | SD OF S61,FS(56)                 |
| 185 | 50131  | 571037 | 570971 | 86299  | CD OF S61,SD OF S134,FS(56)      |
| 186 | 317810 | 622241 | 627862 | 261985 | SD OF S6,FS(-6/25)               |
| 187 | 37945  | 631909 | 630563 | 191905 | SD OF S9                         |
| 188 | 358894 | 633457 | 537338 | 554063 | PT                               |
| 189 | 34468  | 634003 | 278128 | 628051 | PT                               |
| 190 | 214349 | 635423 | 623861 | 341849 | CD OF S67,PT                     |
| 191 | 196179 | 639311 | 599511 | 445397 | P1(3,-4),FS(4/9)                 |
| 192 | 426592 | 616049 | 640612 | 305713 | PT                               |

TABLE 3 (continued)

|     |        |        |        |        |                      |
|-----|--------|--------|--------|--------|----------------------|
| 193 | 3119   | 641471 | 567683 | 505829 | FS(2)                |
| 194 | 507934 | 589471 | 657848 | 81249  | P1(3,1),FS(4/9)      |
| 195 | 424494 | 674693 | 535674 | 629819 | FS(-9/100)           |
| 196 | 14586  | 683105 | 635586 | 483295 | E(6,5)               |
| 197 | 558182 | 711809 | 590654 | 694079 | CD UF S2,PT          |
| 198 | 651215 | 727017 | 720115 | 660483 | PT                   |
| 199 | 232434 | 739885 | 520640 | 691859 | E(7,5)               |
| 200 | 465236 | 747633 | 614656 | 682161 | PT                   |
| 201 | 689308 | 564749 | 756424 | 100019 | SD UF S16,FS(2)      |
| 202 | 421689 | 763169 | 726783 | 550489 | SD UF S4,FS(-6/25)   |
| 203 | 751414 | 399679 | 766018 | 38017  | SD UF S49            |
| 204 | 367446 | 774887 | 778382 | 328807 | SD UF S48            |
| 205 | 305123 | 785947 | 766783 | 459407 | PT                   |
| 206 | 16409  | 826669 | 804679 | 467443 | SD UF S5             |
| 207 | 532244 | 827969 | 768896 | 869313 | PT                   |
| 208 | 842204 | 438241 | 850912 | 354271 | PT                   |
| 209 | 219380 | 854201 | 840360 | 403207 | SD UF S2             |
| 210 | 244553 | 864709 | 730471 | 726091 | CD UF S13,SD UF S138 |
| 211 | 329626 | 867849 | 538734 | 838711 | SD UF S60,FS(40/9)   |
| 212 | 840766 | 518255 | 869338 | 161105 | SD UF S2             |
| 213 | 69492  | 875477 | 241352 | 874219 | PT                   |
| 214 | 3106   | 884947 | 400262 | 875539 | E(7,2)               |
| 215 | 505431 | 905509 | 874987 | 623833 | SD UF S4             |
| 216 | 897898 | 465669 | 906222 | 387653 | SD UF S28,FS(3/4)    |
| 217 | 168824 | 909613 | 877004 | 553453 | PT                   |
| 218 | 230394 | 925087 | 769086 | 787873 | FS(-60/361),PT       |

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